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AUTOMORPHISM GROUPS OF SELF-COMPLEMENTARY VERTEX-TRANSITIVE GRAPHS

ZHAOHONG HUANG, JIANGMIN PAN✉, SUYUN DING and ZHE LIU

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Abstract

Li *et al.* ['On finite self-complementary metacirculants', *J. Algebraic Combin.* **40** (2014), 1135–1144] proved that the automorphism group of a self-complementary metacirculant is either soluble or has A_5 as the only insoluble composition factor, and gave a construction of such graphs with insoluble automorphism groups (which are the first examples of self-complementary graphs with this property). In this paper, we will prove that each simple group is a subgroup (so is a section) of the automorphism groups of infinitely many self-complementary vertex-transitive graphs. The proof involves a construction of such graphs. We will also determine all simple sections of the automorphism groups of self-complementary vertex-transitive graphs of 4-power-free order.

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1. Introduction

Throughout the paper, all graphs are assumed to be undirected and simple. For a graph Γ, denote its vertex set by $V\Gamma$. The *complement graph* $\overline{\Gamma}$ of Γ is the graph which has the same vertex set $V\Gamma$ and in which two vertices u and v are adjacent if and only if u and v are not adjacent in Γ. Then Γ is called *vertex transitive* if $\mathrm{Aut}\,\Gamma$ is transitive on $V\Gamma$, Γ is called a *self-complementary graph* if Γ is isomorphic to $\overline{\Gamma}$ and an isomorphism from Γ to $\overline{\Gamma}$ is called a *complementing isomorphism* of Γ.

The study of self-complementary vertex-transitive graphs has a rich history. In 1962, Sachs constructed the first families of self-complementary circulants (that is, where the automorphism group contains a regular cyclic subgroup), and self-complementary circulants were extensively studied (refer to [1, 7, 16, 20, 21]). In 1999, Muzychuk [17] completely determined the orders of general self-complementary vertex-transitive graphs (see Lemma 2.5 below). Self-complementary vertex-transitive graphs have been used as models for finding lower bounds of Ramsey

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numbers (see [4, 5, 10, 19]). More recently, the study of self-complementary vertex-transitive graphs has been significantly developed by Li and Praeger [12] and their joint work with Guralnick and Saxl [11] for the vertex-primitive case. For more results, see the excellent survey of Beezer [2].

For a group G and subgroups $K \lhd H \le G$, the quotient group H/K is called a *section* of G, and is called a *simple section* of G while H/K is simple. In particular, H/K is called a *composition factor* of G if H is subnormal in G and H/K is simple. A subgroup H of G can be viewed as a section of G by identifying H with $H/\{1\}$, but the converse is not true. A simple example is that A_5 is a section of $\mathrm{SL}(2,5)$ but is not a subgroup of $\mathrm{SL}(2,5)$. In a quite recent work, Li *et al.* [15] proved that the only insoluble composition factor of the automorphism groups of self-complementary metacirculants is A_5. (Recall that a graph Γ is called a *metacirculant* if $\mathsf{Aut}\,\Gamma$ contains a metacyclic subgroup which is transitive on $V\Gamma$; and a group R is called *metacyclic* if R has a normal cyclic subgroup N such that R/N is cyclic.) This result naturally motivates the following problem.

Problem 1.1. Which simple groups are sections of the automorphism groups of self-complementary vertex-transitive graphs?

The first main result of this paper solves this problem.

Theorem 1.2. *Let n be an arbitrary positive integer and let $T_1, T_2, \ldots, T_n$ be any simple groups. Then there exist infinitely many self-complementary vertex-transitive graphs Γ such that $T_1 \times T_2 \times \cdots \times T_n \le \mathsf{Aut}\,\Gamma$.*

Theorem 1.2 particularly states that each simple group is a section (actually a subgroup) of the automorphism groups of infinitely many self-complementary vertex-transitive graphs. The proof of Theorem 1.2 also establishes the following results.

Corollary 1.3. *There are infinitely many self-complementary vertex-transitive graphs Γ and Σ, both of prime-cube order, such that $\mathrm{A}_5 \le \mathsf{Aut}\,\Gamma$ and $\mathrm{PSL}(2,7) \le \mathsf{Aut}\,\Sigma$.*

Corollary 1.4.

(1) *There are infinitely many self-complementary vertex-transitive graphs Γ of prime-square order such that A_5 is a section of $\mathsf{Aut}\,\Gamma$.*
(2) *There are infinitely many self-complementary vertex-transitive graphs Σ of prime-cube order such that A_6 is a section of $\mathsf{Aut}\,\Sigma$.*

For positive integers n and d, n is called *d-power-free* if there is no prime p such that $p^d \mid n$. Corollaries 1.3 and 1.4 tell us that A_5, A_6 and $\mathrm{PSL}(2,7)$ are simple sections of the automorphism groups of infinitely many self-complementary vertex-transitive graphs of 4-power-free order. The second main result of this paper shows that these three simple groups are the only nonabelian simple sections of the automorphism groups of self-complementary vertex-transitive graphs of 4-power-free order.

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Theorem 1.5.

(1) *The automorphism groups of self-complementary vertex-transitive graphs of squarefree order are soluble.*
(2) A_5 *is the unique nonabelian simple section of the automorphism groups of self-complementary vertex-transitive graphs of cubefree order.*
(3) A_5, A_6 *and* $PSL(2,7)$ *are the only nonabelian simple sections of the automorphism groups of self-complementary vertex-transitive graphs of* 4*-power-free order.*

We note that it was proved in [6] and [14] independently that the automorphism groups of self-complementary vertex-transitive graphs of order a product of two distinct primes are soluble; and it was proved in [15] that the automorphism groups of self-complementary vertex-transitive metacirculants of squarefree order are soluble.

This paper is organised as follows. After this introductory section, we will give some preliminary results in Section 2. Then Theorem 1.2 and Corollaries 1.3 and 1.4 are proved in Section 3 and Theorem 1.5 is proved in Section 4.

2. Preliminaries

In this section, we quote certain preliminary results which will be used later. The first one is a well-known theorem of Dirichlet in number theory.

Theorem 2.1. *Suppose that a, b are coprime positive integers. Then there are infinitely many primes of the form $an + b$, where n is a positive integer.*

For a group G, if it has a normal subgroup N such that the quotient group $G/N \cong M$, then G is called an *extension* of N by M, denoted by $G = N \cdot M$, and, if such an extension is split, then we write $N : M$ instead of $N \cdot M$.

Inspecting subgroups of $GL(3, p)$ with p a prime given in [8, Theorem 2.2], we have the following result; see also [13, Theorem 1.3].

Lemma 2.2. *Let p be a prime. Then the unique nonabelian simple section of* $GL(2, p)$ *is* A_5*, and the only nonabelian simple sections of* $GL(3, p)$ *are* A_5*,* A_6 *and* $PSL(2, 7)$*. Further, the following statements hold:*

(1) $A_5 \le GL(3, p)$ *if* $p \equiv \pm 1 \pmod{10}$*;*
(2) $PSL(2, 7) \le GL(3, p)$ *if* $p^3 \equiv 1 \pmod 7$*;*
(3) $SL(2, 5) \le GL(2, p)$ *(so* A_5 *is a section of* $GL(2, p)$*) if* $p \equiv \pm 1 \pmod{10}$*;*
(4) $\mathbb{Z}_3 \cdot A_6 \le GL(3, p)$ *(so* A_6 *is a section of* $GL(3, p)$*) if* $p \equiv 1$ *or* $19 \pmod{30}$.

Let G be a transitive permutation group on a set Ω. A nonempty subset B of Ω is called a *block* of G if, for each $g \in G$, either $B = B^g := \{b^g \mid b \in B\}$ or $B \cap B^g$ is an empty set; in this case, $\mathcal{B} := \{B^g \mid g \in G\}$ is called a *block system* of G (or *a G-invariant partition*) on Ω. Let $G_B = \{g \in G \mid B^g = B\}$, *the stabiliser of G on B* setwise, and let G_B^B be the induced permutation group of G_B acting on B. Let $G^{\mathcal{B}}$ be the induced permutation group of G on $\mathcal{B}$. Clearly, a single-element set and Ω are blocks, called

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trivial blocks, and other blocks are called *nontrivial*. If G has no nontrivial block, then G is called *primitive*.

Let Γ be a self-complementary vertex-transitive graph. Set $G = \mathsf{Aut}\,\Gamma$ and $X = \langle G, \sigma\rangle$, where σ is a complementing isomorphism of Γ. Suppose further that X is imprimitive on $V\Gamma$. Then there is a nontrivial block system $\mathcal{B}$ of X on $V\Gamma$. Let $B \in \mathcal{B}$ and let B_Γ denote the induced subgraph of Γ on B.

The following nice result provides an induction method for studying vertex-transitive self-complementary graphs.

Lemma 2.3 [12]. *With the notation above, the following statements hold.*

(i) *For each $B \in \mathcal{B}$, the induced subgraph B_Γ is a self-complementary vertex-transitive graph, $G_B^B \le \mathsf{Aut}(B_\Gamma)$ and σ naturally induces a complementing isomorphism of B_Γ.*

(ii) *There exists a self-complementary vertex-transitive graph Σ with vertex set $\mathcal{B}$ such that $G^{\mathcal{B}} \le \mathsf{Aut}\,\Sigma$ and each element in $X^{\mathcal{B}} \setminus G^{\mathcal{B}}$ is a complementary isomorphism of Σ.*

A graph Γ is called a *Cayley graph* of a group G if there is a subset $S \subseteq G \setminus \{1\}$, with $S = S^{-1} := \{g^{-1} \mid g \in S\}$, such that $V\Gamma = G$, and two vertices g and h are adjacent if and only if $hg^{-1} \in S$. This Cayley graph is denoted by $\mathsf{Cay}(G, S)$. It is well known that a graph Σ is isomorphic to a Cayley graph of a group G if and only if $\mathsf{Aut}\,\Sigma$ contains a subgroup which is isomorphic to G and acts regularly on $V\Sigma$ (see [3, Proposition 16.3]).

Let $\Gamma = \mathsf{Cay}(G, S)$ and let

$$\hat{G} = \{\hat{g} \mid \hat{g} : x \mapsto xg \text{ for all } g, x \in G\},$$
$$\mathsf{Aut}(G, S) = \{\sigma \in \mathsf{Aut}(G) \mid S^\sigma = S\}.$$

Then both $\hat{G}$ and $\mathsf{Aut}(G, S)$ are subgroups of $\mathsf{Aut}\,\Gamma$. Further, the following nice property holds.

Lemma 2.4 [9, Lemma 2.1]. *Let $\Gamma = \mathsf{Cay}(G, S)$ be a Cayley graph. Then the normaliser $\mathbf{N}_{\mathsf{Aut}\,\Gamma}(\hat{G}) = \hat{G} : \mathsf{Aut}(G, S)$.*

The next lemma, proved by Muzychuk [17], completely determines the order of general self-complementary vertex-transitive graphs.

Lemma 2.5. *There exists a self-complementary vertex-transitive graph of order n if and only $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$, where $p_1, p_2, \ldots, p_s$ are distinct primes such that $p_i^{r_i} \equiv 1 \pmod 4$ for each $i \in \{1, 2, \ldots, s\}$.*

Let Γ_1 and Γ_2 be graphs. The *lexicographic product* of Γ_1 and Γ_2, denoted by $\Gamma_1[\Gamma_2]$, is defined as the graph with the vertex set $V\Gamma_1 \times V\Gamma_2$ (Cartesian product) and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either u_1 is adjacent to v_1, or $u_1 = v_1$ and u_2 is adjacent to v_2.

The following nice property, given in [2], provides a method to construct 'bigger' self-complementary vertex-transitive graphs from small ones.

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Lemma 2.6. *If both* Γ_1 *and* Γ_2 *are vertex-transitive or self-complementary graphs, then so is* $\Gamma_1[\Gamma_2]$.

3. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2 and Corollaries 1.3 and 1.4.

3.1. Simple sections. Let T be a nonabelian simple section of $\mathrm{GL}(d, p)$, where $d \geq 2$ and p is a prime. Then there exist groups $K \lhd H \leq \mathrm{GL}(d, p)$ such that $H/K \cong T$. Suppose further that p and H satisfy the following condition.

Condition $(*)$: $p \equiv 1 \pmod{2^{m+1}}$, where 2^m is the largest order of 2-elements of H.

Let $N = \mathbb{Z}_p^d$. Then $H \leq \mathsf{Aut}(N)$ and the centre $Z := \mathbf{Z}(\mathrm{GL}(d, p)) \cong \mathbb{Z}_{p-1}$ is of order divisible by 2^{m+1}. Let $\sigma \in Z$ be such that $2^{m+1} \mid o(\sigma)$, and let $U = \langle H, \sigma\rangle$ and $V = \langle H, \sigma^2\rangle$. Then U, V are subgroups of $\mathrm{GL}(d, p)$ and can act naturally on $N^\# := N \setminus \{1\}$.

Lemma 3.1. *Let* Δ *be an orbit of* U *on* $N^\#$. *Then* V *has exactly two orbits of equal size on* Δ.

Proof. Set $o(\sigma) = 2^{m+1}l$, for some positive integer l, and let $w \in \Delta$.

Suppose that, on the contrary, V is transitive on Δ. Then $w^\sigma = w^x$ for some $x \in V$, so $w^{x\sigma^{-1}} = w$. Since σ centralises H, we may write $x = h\sigma^{2s}$ for some $h \in H$ and positive integer s. Then, as 2^{m+1} divides $o(\sigma)$ and H has no element with order divisible by 2^{m+1}, we conclude that $(x\sigma^{-1})^{o(h)} = (h\sigma^{2s-1})^{o(h)} = \sigma^{(2s-1)o(h)} \neq 1$. However, as $x\sigma^{-1}$ fixes $w \neq 1$, so does $\sigma^{(2s-1)o(h)} = (x\sigma^{-1})^{o(h)}$, which is a contradiction as $\sigma^{(2s-1)o(h)}$ is a nonidentity scalar matrix of $\mathrm{GL}(d, p)$.

Thus, V is intransitive on Δ. Noting that V is normal in U with index 2, V has exactly two orbits of equal size on Δ. □

With Lemma 3.1, we can make the following construction.

Construction 3.2. *Let* $\Delta_1, \Delta_2, \ldots, \Delta_k$ *be all the orbits of* U *on* $N^\#$, *and let* Δ_i^+ *and* Δ_i^- *be the orbits of* V *on* Δ_i *for each* $i \in \{1, 2, \ldots, k\}$.

Set

$$S = \bigcup_{i=1}^{k} \Delta_i^{\epsilon_i} \quad \text{where } \epsilon_i = + \text{ or } -,$$
$$\Gamma = \mathsf{Cay}(N, S).$$

A group X is called a *central product* of two subgroups L and S, denoted by $X = L \circ S$, if $X = LS$, the commutator subgroup $[L, S] = 1$ and $L \cap S$ coincides with the centre of X (refer to [18, page 141]).

Lemma 3.3. *Using the notation above, the graph* Γ *in Construction 3.2 is a self-complementary vertex-transitive graph of order* p^d, σ *is a complementary isomorphism of* Γ *and* $\mathsf{Aut}\,\Gamma \geq \mathbb{Z}_p^d : (H \circ \langle\sigma^2\rangle)$. *In particular,* T *is a section of* $\mathsf{Aut}\,\Gamma$.

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PROOF. Since $\sigma \in Z$ is of order $2^{m+1}l$, $\sigma^{o(\sigma)/2} = \sigma^{2^m l}$ is a scalar involution. Set $\tau = \sigma^{o(\sigma)/2}$. Then there exists $n \in \mathbb{Z}_p$ such that $x^\tau = x^n$ for each $x \in N$. Choose $x \neq 1$. Then $o(x) = p$ and, as $x = x^{\tau^2} = x^{n^2}$, we have $n^2 \equiv 1 \pmod p$. It follows that $n \equiv -1 \pmod p$ as $\tau \neq 1$, that is, τ maps each element $x \in N$ to its inverse x^{-1}. Hence, $(\Delta_i^{\epsilon_i})^{-1} = (\Delta_i^{\epsilon_i})^\tau$ for each $i \in \{1, 2, \ldots, k\}$. Noting that $\tau = \sigma^{2^m l} \in \langle \sigma^2 \rangle \subseteq V$, and each $\Delta_i^{\epsilon_i}$ is an orbit of V, we have $\Delta_i^{\epsilon_i} = (\Delta_i^{\epsilon_i})^\tau = (\Delta_i^{\epsilon_i})^{-1}$. Consequently, $S = S^{-1}$; thus, Γ is undirected. Since each $\Delta_i^{\epsilon_i}$ is an orbit of V, V fixes S setwise. By Lemma 2.4, $V \leq \mathsf{Aut}(N, S) \leq \mathsf{Aut}\,\Gamma$ and $\mathsf{Aut}\,\Gamma \geq \hat{N} : V \cong \mathbb{Z}_p^d : (H \circ \langle \sigma^2 \rangle)$.

Further, since $\sigma \in U \setminus V$, σ interchanges Δ_i^+ and Δ_i^- for each $i \in \{1, 2, \ldots, k\}$. Thus,

$$S^\sigma = \bigcup_{i=1}^{k} (\Delta_i^{\epsilon_i})^\sigma = \bigcup_{i=1}^{k} (\Delta_i^{-\epsilon_i}) = N^\# \setminus S$$

and $\Gamma = \mathsf{Cay}(N, S) \cong \mathsf{Cay}(N, S^\sigma) = \overline{\Gamma}$. Hence, Γ is a self-complementary vertex-transitive graph and σ is a complementary isomorphism of Γ.

As $T \cong H/K$, the last statement of Lemma 3.3 is now obviously true. □

3.2. Proof of Theorem 1.2.

We first give a simple observation.

LEMMA 3.4.

(1) *For each self-complementary vertex-transitive graph* Γ, $\mathbb{Z}_2 \leq \mathsf{Aut}\,\Gamma$.

(2) *For each odd prime* p, *there exist infinitely many self-complementary vertex-transitive graphs* Σ *such that* $\mathbb{Z}_p \leq \mathsf{Aut}\,\Sigma$.

PROOF. For each self-complementary vertex-transitive graph Γ, let σ be a complementary isomorphism of Γ. Set $o(\sigma) = 2^k n$ with n odd. Clearly, σ^n is also a complementary isomorphism of Γ, so it does not interchange any two distinct vertices of Γ; hence, $k \geq 2$. It follows that $\mathbb{Z}_2 \cong \langle \sigma^{2^{k-1} n} \rangle \leq \mathsf{Aut}\,\Gamma$.

For each odd prime p, $p^{2m} \equiv 1 \pmod 4$ for any positive integer m. By Lemma 2.5, there exists a self-complementary vertex-transitive graph Σ of order p^{2m}. Thus, $\mathsf{Aut}\,\Sigma$ has an element of order p and $\mathbb{Z}_p \leq \mathsf{Aut}\,\Sigma$. □

Now suppose that T is a nonabelian simple group and that the largest order of 2-elements of T is 2^m. By Lemma 2.1, there exist infinitely many primes p of the form

$$p = 2^{m+1}k + 1,$$

where k is a positive integer. Since each group is isomorphic to a permutation group, we may view T as a permutation group of degree d. Let F_p^d denote the d-dimensional vector space over the p-element field F_p, with a basis $v_1, v_2, \ldots, v_d$. Then T has the following group representation (usually called *the permutation representation*) on F_p^d:

$$v_i^t = v_{i^t} \quad \text{for } t \in T \text{ and } i = 1, 2, \ldots, d.$$

Clearly, this representation is faithful and so we have $T \leq \mathrm{GL}(d, p)$.

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Lemma 3.5. *For each nonabelian simple group T, there exist an integer $d \geq 2$ and infinitely many primes p of the form $p = 2^{m+1}k + 1$ such that $T \leq \mathrm{GL}(d, p)$, where 2^m is the largest order of 2-elements of T and k is a positive integer.*

We are ready to prove Theorem 1.2 now.

Proof of Theorem 1.2. We complete our proof by induction on the number n of the simple groups.

Assume first that $n = 1$. If T_1 is soluble, Theorem 1.2 is true by Lemma 3.4. If T_1 is nonabelian, suppose that the largest order of 2-elements of T_1 is 2^m. By Lemma 3.5, there exist infinitely many primes p such that $T_1 \leq \mathrm{GL}(d, p)$ and $p \equiv 1 \pmod{2^{m+1}}$, that is, p and T_1 (as H there) satisfy Condition $(*)$. By Lemma 3.3, for each prime $p \equiv 1 \pmod{2^{m+1}}$, there is a self-complementary vertex-transitive graph Γ such that $\mathsf{Aut}\,\Gamma \geq \mathbb{Z}_p^d : (T_1 \circ \langle\sigma^2\rangle) = \mathbb{Z}_p^d : (T_1 \times \langle\sigma^2\rangle)$ and so $T_1 \leq \mathsf{Aut}\,\Gamma$. Theorem 1.2 is true in this case.

Assume now that Theorem 1.2 is true for fewer than n simple groups. Then there are infinitely many self-complementary vertex-transitive graphs Σ and Ω such that $T_1 \times T_2 \times \cdots \times T_{n-1} \leq \mathsf{Aut}\,\Sigma$ and $T_n \leq \mathsf{Aut}\,\Omega$. Let $\Gamma = \Omega[\Sigma]$, the lexicographic product of Ω and Σ. Then $\mathsf{Aut}\,\Sigma \wr \mathsf{Aut}\,\Omega \leq \mathsf{Aut}\,\Gamma$. Note that $\mathsf{Aut}\,\Sigma \wr \mathsf{Aut}\,\Omega \cong (\mathsf{Aut}\,\Sigma)^m : \mathsf{Aut}\,\Omega$, where $m = |V\Omega|$, and $\mathsf{Aut}\,\Omega$ acts on $(\mathsf{Aut}\,\Sigma)^m$ in the wreath product. Let

$$L = \{(a, a, \ldots, a) \mid a \in T_1 \times T_2 \times \cdots \times T_{n-1}\} \leq (\mathsf{Aut}\,\Sigma)^m.$$

Then $L \cong T_1 \times T_2 \times \cdots \times T_{n-1}$ and $T_n \leq \mathsf{Aut}\,\Omega$ centralises L. So, $T_1 \times T_2 \times \cdots \times T_n = \langle L, T_n\rangle \leq \mathsf{Aut}\,\Gamma$. This completes the proof of Theorem 1.2. □

Proof of Corollary 1.3. There are infinitely many primes $p \equiv 1$ or $9 \pmod{20}$, by Theorem 2.1. Since $p \equiv \pm 1 \pmod{10}$, by Lemma 2.2(1), $\mathrm{A}_5 \leq \mathrm{GL}(3, p)$; since $p \equiv 1 \pmod 4$ and 2-elements of A_5 are involutions, p and A_5 (as H there) satisfy Condition $(*)$. By Lemma 3.3, for each prime $p \equiv 1$ or $9 \pmod{20}$, there is a self-complementary vertex-transitive graph Γ of order p^3 such that $\mathrm{A}_5 \leq \mathsf{Aut}\,\Gamma$.

Again, by Theorem 2.1, there are infinitely many primes $p \equiv 1 \pmod{56}$. Since $p \equiv 1 \pmod 7$, Lemma 2.2(2) implies that $\mathrm{PSL}(2, 7) \leq \mathrm{GL}(3, p)$. As the largest order of 2-elements of $\mathrm{PSL}(2, 7)$ equals 4, and $p \equiv 1 \pmod 8$, p and $\mathrm{PSL}(2, 7)$ (as H there) satisfy Condition $(*)$. It then follows from Lemma 3.3 that there are infinitely many self-complementary vertex-transitive graphs Σ of order p^3 such that $\mathrm{PSL}(2, 7) \leq \mathsf{Aut}\,\Sigma$. □

Proof of Corollary 1.4. (The proof is similar to the proof of Corollary 1.3.)

By Theorem 2.1, there are infinitely many primes $p \equiv 1$ or $9 \pmod{40}$. As $p \equiv \pm 1 \pmod{10}$, $\mathrm{SL}(2, 5) \leq \mathrm{GL}(2, p)$, by Lemma 2.2(3). Let $H = \mathrm{SL}(2, 5)$ and $K \cong \mathbb{Z}_2$ be the centre of H. Then $\mathrm{A}_5 \cong H/K$. Since the largest order of 2-elements of H equals 4, and $p \equiv 1 \pmod 8$, p and H satisfy Condition $(*)$. Thus, for each prime $p \equiv 1$ or $9 \pmod{40}$, by Lemma 3.3, there is a self-complementary vertex-transitive graph Γ of order p^2 such that $\mathsf{Aut}\,\Gamma \geq \mathbb{Z}_p^2 : (\mathrm{SL}(2, 5) \circ \langle\sigma^2\rangle)$. Thus, A_5 is a section of $\mathsf{Aut}\,\Gamma$.

Similarly, there are infinitely many primes $p \equiv 1$ or $49 \pmod{120}$. As $p \equiv 1$ or $19 \pmod{30}$, Lemma 2.2(4) implies that $\mathbb{Z}_3 \cdot \mathrm{A}_6 \le \mathrm{GL}(3, p)$. Let $H_1 = \mathbb{Z}_3 \cdot \mathrm{A}_6$ and $K_1 \cong \mathbb{Z}_3$ be the normal subgroup of H_1. Then $\mathrm{A}_6 \cong H_1/K_1$. Since the largest order of 2-elements of H_1 equals 4, and $p \equiv 1 \pmod 8$, p and H_1 (as H there) satisfy Condition (∗). By Lemma 3.3, there are infinitely many self-complementary vertex-transitive graphs Σ of order p^3 such that $\mathsf{Aut}\,\Sigma \ge \mathbb{Z}_p^3 : (\mathbb{Z}_3 \cdot \mathrm{A}_6 \circ \langle \sigma^2 \rangle)$. Hence, A_6 is a section of $\mathsf{Aut}\,\Sigma$. □

4. Proof of Theorem 1.5

To prove Theorem 1.5, we first give a result regarding simple sections of groups.

Lemma 4.1. *Suppose that $G = N \cdot M$ and T is a simple section of G. Then T is a section of either N or M.*

Proof. By definition, there exist subgroups $K \lhd H \le G$ such that $H/K \cong T$. Since $(H \cap N)K/K \lhd H/K$, either $(H \cap N)K/K \cong T$ or 1. For the former case, $(H \cap N)K = H$; it follows that $T \cong (H \cap N)K/K \cong (H \cap N)/(K \cap N)$ and, noting that $H \cap N \le N$, T is a section of N. For the latter case, $H \cap N \subseteq K$, so $H \cap N = K \cap N$ and hence

$$T \cong (H/(H \cap N))/(K/(K \cap N)) \cong (HN/N)/(KN/N),$$

as $HN/N \le G/N \cong M$. We conclude that T is a section of M. □

For a group G, its socle, denoted by $\mathsf{soc}(G)$, is the product of all minimal normal subgroups of G.

Proof of Theorem 1.5. Let Γ be a self-complementary vertex-transitive graph of d-power-free order and let $G = \mathsf{Aut}\,\Gamma$, where $2 \le d \le 4$. Let T be a simple section of G. By Corollaries 1.3 and 1.4, we only need to prove that T satisfies the following assertion.

Assertion (∗)

(1) T is soluble if $|V\Gamma|$ is squarefree.
(2) T is soluble or $T = \mathrm{A}_5$ if $|V\Gamma|$ is cubefree.
(3) T is soluble or $T \in \{\mathrm{A}_5, \mathrm{A}_6, \mathrm{PSL}(2, 7)\}$ if $|V\Gamma|$ is 4-power-free.

Assume first that G is primitive on $V\Gamma$. Since Γ is undirected, by [11, Theorem 1.3], either G is affine or G is of product action with socle $\mathrm{PSL}(2, q^2)$ and order $|V\Gamma| = (\frac{1}{2}q^2(q^2 + 1))^l$, where q is an odd prime power and $l \ge 2$. The latter case is impossible, as $(\frac{1}{2}q^2(q^2 + 1))^l$ is not 4-power-free. Thus, G is affine, so $N := \mathsf{soc}(G) = \mathbb{Z}_p^e$ is regular on $V\Gamma$ and Γ is a Cayley graph of N, where p is a prime and $e \le d - 1$. As $N \lhd G$, Lemma 2.4 implies that $G \le N : \mathsf{Aut}(N) \cong \mathbb{Z}_p^e : \mathrm{GL}(e, p)$.

If $d = 2$, then $e = 1$ and $G \le \mathbb{Z}_p : \mathbb{Z}_{p-1}$ is soluble. Hence so is T. If $d = 3$, then $G \le \mathbb{Z}_p^2 : \mathrm{GL}(2, p)$. By Lemma 4.1, either T is soluble or T is a nonabelian simple section of $\mathrm{GL}(2, p)$ and $T = \mathrm{A}_5$, by Lemma 2.2. If $d = 4$, then $G \le \mathbb{Z}_p^3 : \mathrm{GL}(3, p)$. If T is insoluble, then T is a section of $\mathrm{GL}(3, p)$, by Lemma 4.1; it then follows from

Lemma 2.2 that $T \in \{A_5, A_6, PSL(2,7)\}$. Therefore, T satisfies Assertion $(*)$ in the case where G is primitive on $V\Gamma$.

Assume now that G is imprimitive on $V\Gamma$. Assume by induction that Assertion $(*)$ holds for each simple section of the automorphism groups of self-complementary vertex-transitive graphs with d-power-free order less than $|V\Gamma|$.

Let B be a nontrivial block of G on $V\Gamma$ and let $\mathcal{B} = \{B^g \mid g \in G\} = \{B_1, B_2, \ldots, B_m\}$ be the corresponding block system of G on $V\Gamma$. As $|V\Gamma| = |B_i||\mathcal{B}|$, both $|\mathcal{B}|$ and $|B_i|$ are d-power-free and less than $|V\Gamma|$. Let K be the kernel of G acting on $\mathcal{B}$. Then $G = K.G^{\mathcal{B}}$ and, by Lemma 4.1, T is a section of either K or $G^{\mathcal{B}}$.

Suppose first that T is a section of K. Since K fixes each B_i in $\mathcal{B}$,

$$K \leq K^{B_1} \times K^{B_2} \times \cdots \times K^{B_m},$$

so T is a section of $K^{B_1} \times K^{B_2} \times \cdots \times K^{B_m}$ and, in turn, T is a section of K^{B_i} for some i, by Lemma 4.1. Further, by Lemma 2.3(i), the induced graph $(B_i)_\Gamma$ is a self-complementary vertex-transitive graph of order $|B_i|$ and $K^{B_i} \lhd G_{B_i}^{B_i} \leq \mathsf{Aut}((B_i)_\Gamma)$ Since $|B_i| < |V\Gamma|$ is d-power-free, by assumption, T satisfies Assertion $(*)$.

Suppose now that T is a section of $G^{\mathcal{B}}$. By Lemma 2.3(ii), there is a self-complementary vertex-transitive graph Σ with order $|\mathcal{B}|$ such that $G^{\mathcal{B}} \leq \mathsf{Aut}\,\Sigma$. Since $|\mathcal{B}| < |V\Gamma|$ is d-power-free, by assumption, T also satisfies Assertion $(*)$. This completes the proof of Theorem 1.5. □

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ZHAOHONG HUANG, School of Mathematics and Statistics,
Yunnan University, Kunming, PR China

JIANGMIN PAN, School of Statistics and Mathematics,
Yunnan University of Finance and Economics, Kunming, PR China
e-mail: jmpan@ynu.edu.cn

SUYUN DING, School of Mathematics and Statistics,
Yunnan University, Kunming, PR China

ZHE LIU, College of Science,
Zhejiang University of Agriculture and Forestry, Hangzhou, PR China