7.5 MASER THEORY OF PULSAR RADIATION

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Abstract. In this paper we present an account of a theory of pulsar radio emission. The emission mechanism is via a maser amplification process. This theory avoids the difficulty of coherent plasma emission, that the bandwidth of radiation must be less than $\frac{1}{2}\lambda$. The high brightness radio temperature and the insensitivity of pulsar radio flux to pulsar periods can be easily accounted for.

1. Introduction

In 1968 Hewish and Miss Bell discovered the existence of pulsating celestial radio sources (Hewish *et al.*, 1968). These radio sources emit bursts of radio emission in the 100–1000 MHz frequency region with a $\nu^{-\alpha}$ type spectrum ($\alpha \sim 2$) at regular intervals (periods) ranging from a fraction of a second to a few seconds. Subsequent work by other workers resulted in the discovery of more pulsars (Maran and Cameron, 1969).

The period lies between $\frac{1}{30}$ sec to 3.7 sec. In cases where accurate determination of the period can be made, it is found that the period P lengthens at a rate $\mathrm{d}P/\mathrm{d}t = 10^{-12}$ to 10^{-16}, and in no case does the period shorten regularly. The lengthening of the period shows that the time keeping mechanism is rotation and not oscillation (which is the cause for variability of some stars), for in the case of rotation the period lengthens as a result of dissipation of rotational energy and in the case of oscillation, a linearized theory will give rise to a constant period independent of the amplitude and when nonlinear effects are considered, the period decreases as the amplitude is damped, such as the case of a pendulum.

If the basic time keeping mechanism is rotation then the value of the period gives an upper limit to the size of the pulsars. The centrifugal force per unit mass at the equator of a rotating star is $\omega^2 R$ and the gravitational force is GM/R^2 when R is the radius, ω the rotational angular velocity, M the mass of the star. In order that the rotation be stable, it is necessary that $\omega^2 R < (GM/R^2)$, which gives $R < 100$ km for the case of the Pulsar NP 0532 which has the shortest period of $\frac{1}{30}$ sec.

Individual pulses are detectable in the Arecibo 1000 ft radio telescope. It is found that the signal strength and structure changes from pulse to pulse, and in the case of NP 0532 microtime structure with a time constant as short as 100 μsec has been detected. If the dimension of the emitting region is R, then the time structure of the signal emitted cannot have a time constant shorter than the light transit time t_c which is R/c. For NP 0532, $t_c \sim 100$ μsec and this means that $R < 30$ km (Drake and Craft, 1968).

No known astronomical objects other than asteroids are known to have a physical dimension less than 1000 km (the white dwarf has a minimum theoretical dimension of 10^8 cm $= 10^3$ km) except the neutron star, which was predicted by Landau in 1932

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soon after the discovery of the neutron. At high density the Fermi energy of the electrons will be high enough so that the following reaction will proceed favorably

$$e^- + p \rightarrow n + \nu \tag{1}$$

where p and n may be free or bound to a nucleus. The inverse reaction is forbidden because the available electron states are occupied. As (1) proceeds the elements become richer and richer in neutrons. At a density of 10^{12} g/cm and above, few nuclei can exist and all but a few protons are converted into neutrons. A small fraction of matter remains in the proton and electron state so that the inverse of (1) is prevented. Stable neutron stars have densities between 10^{14} to 10^{16} g/cm^3; the upper limit is dependent on the detailed structure of nuclear interaction, which is poorly known at high densities. The radius of a neutron star is of the order of 10 km.

The neutron stars are results of stellar collapse which also gives rise to the supernova phenomenon. A pulsar (with the shortest period of $\frac{1}{30}$ sec) is found at the center of the Crab Nebula, a well known remnant of a supernova which flared up in 1054 A.D. Many theories predict that the star remnants of supernovae are neutron stars, hence, in all probability, pulsars are neutron stars. This pulsar NP 0532 is the youngest one on record and is the only one which has been found to emit not only in the radio frequency but also in the optical and X-ray spectrum ($10^{13} \rightarrow 10^{18}$ Hz). The division of energy flux in the radio, optical, and X-ray regions are roughly: 10^{31} erg sec^{-1}, 10^{34} erg sec^{-1}, and 10^{37} erg sec^{-1} respectively.

The neutron stars are themselves objects of immense interest. First, during the formation of a neutron star, a normal star suffers a change of radius from 10^{11} to 10^6 m, representing a compression ratio of 10^5. The angular momentum conservation law $I\omega = MR^2\omega = $ constant, requires that the period of rotation be increased as R^{-2}. The rotation period of a normal star such as the sun is 26 days, and after collapse a period of rotation as short as 1 msec may be achieved. (The centrifugal force at the equator gives a limiting rotation period of about 1 msec.) The rotational energy of a neutron star U_R, is roughly:

$$U_R \approx \tfrac{1}{2} I\omega^2 \approx \tfrac{1}{2} MR^2\omega^2 \approx 7 \times 10^{46} (M/\odot)(R/10\text{ km})^2 P^{-2} \text{ ergs} \tag{2}$$

where

$$\odot = \text{solar mass} = 2 \times 10^{33}\text{ g}.$$

Many stars possess magnetic fields of the order of 10^3 G or greater. During compression, the magnetic flux lines of a plasma are conserved and increase as R^{-2}. If the loss of flux lines is small during formation, the magnetic field of a neutron star may be as high as 10^{13} G.

Finally, another piece of observational evidence supports the rotational neutron star hypothesis of pulsars. The nuclei in the surface layer of a neutron star can form crystalline-like lattices (resulting in a smaller Coulomb energy) and the melting temperature of these crystalline structures is of the order of 10^9 K or more for densities between $10^7 \rightarrow 10^{13}$ g/cm^3. The temperature of neutron stars is usually quite low

($\ll 10^9$ K), hence, the outer layer of neutron stars is made of crystalline solid. As the rotation of the star slows down, the equilibrium configuration is changed and strains are present in the crystalline layer. When the strain exceeds the stress limit of the crystalline layer, sudden resettlement (starquake) will take place. This will cause a sudden change in the moment of inertia of the star, resulting in a change of period as well as a change in the phase of the pulse called a glitch. Glitches have been observed in two cases: the case of Vela pulsar and the case of the Crab pulsar. The glitch phenomenon, though easily explained in the rotating neutron star model, is virtually impossible to explain in oscillating models.

2. The Magnetic Field of the Neutron Star

A field as strong as 10^{12} G is hard to contemplate, but there is good evidence that such fields exist. As Pacini (1968) and Ostriker and Gunn (1969) suggested, a rotating neutron star with a magnetic field can emit electromagnetic radiation of frequency the same as that of the rotation. The energy of emission comes from the energy of rotation U_R. The rate of radiation $\mathrm{d}U/\mathrm{d}t$ in the case of a rotating dipole is:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\frac{2\bar{M}\sin^2\theta\omega^4}{3c^2} \tag{3}$$

where $\bar{M}$ is the magnetic dipole moment, which can be approximated by

$$\bar{M} \approx \frac{\phi^2 R^2}{4\pi^2} = \frac{B_p^2 R^6}{4} \tag{4}$$

where ϕ is the flux line through the pole ($\phi \sim \beta_p R^2/\pi^2$ where β_p is the average field at the pole), θ is the angle of inclination of the dipole with respect to the rotational axis.

Equating the rate of loss of rotation of energy to $\mathrm{d}U/\mathrm{d}t$, it is found that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -\frac{\phi^2\sin^2\theta R^2\omega^3}{6\pi^2 Ic^2} = 2\pi\frac{\mathrm{d}P}{\mathrm{d}t}$$
$$\Omega = \Omega_0(1 + 2t/\tau_0)^{-1/2}. \tag{5}$$

For the Crab Nebula,

$$\omega \sim 100\,\mathrm{sec}^{-1}, \qquad \theta \approx \pi/4, \qquad R \approx 10\,\mathrm{km}, \qquad B_p \approx 10^{12}\,\mathrm{G} \tag{6}$$

This gives

$$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} \approx 1.3 \times 10^{-24}\,\mathrm{sec}^{-1}$$

which agrees with observations.

The frequency of the emitted dipole radiation is the same as the pulsar rotation frequency. Such a radiation field is similar to the field in accelerators, and can accelerate particles to high energy. This explains the puzzling fact that the Crab nebula

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appears to be still active in producing high energy particles (which are needed to account for radio, optical, and X-ray emission from the nebula). Without the classical dipole radiation it will be very hard if not impossible to explain the rather large rate of loss of rotational energy of the pulsar. Analysis of other pulsars whose values of $\mathrm{d}P/\mathrm{d}t$ are known all yield a field of the order of 10^{12} G. It thus seems well established that neutron stars may have fields as strong as 10^{12} G.

Goldreich and Julian (1969) pointed out that in the rest frame of the rotating magnetic neutron star there is a component of the electric field parallel to the magnetic H such that the invariant $\mathbf{E}\cdot\mathbf{H}\neq 0$. The field strength at the surface is

$$E \sim H \frac{v}{c} \sim 10^{10} (H/10^{13}\ \mathrm{G})\,(v_r/10\ \mathrm{km/sec}) \quad \mathrm{volt/cm} \tag{7}$$

where v_r is the rotational velocity at the magnetic pole. This field will accelerate electrons and ions to stream out of the surface. As will be seen below, this electric field is also important to account for the radio (and optical and X-ray) emission from pulsars.

The properties of an electron gas in an intense magnetic field have been extensively studied (Chiu, 1970). First, the motion of electrons in a strong field is quantized in the direction perpendicular to the field and the usual expression of the kinetic energy of an electron is replaced by the equation

$$\varepsilon = mc^2 [1 + (p_z/mc)^2 + 2n (H/H_q)]^{1/2} - mc^2 \tag{8}$$

where the field is taken to be in the z direction, n is the quantum number characterizing the size of the classical orbit, $H_q = m^2c^3/e\hbar = 4.4 \times 10^{13}$ G. Therefore, the momentum in the direction perpendicular to the field, $p_\perp$, is quantized by the relation:

$$p_\perp^2/m^2c^2 \to 2n (H/H_q). \tag{9}$$

In the nonrelativistic limit we have

$$\varepsilon = p_z^2/2m + n (H/H_q)\, mc^2. \tag{10}$$

The value of $(H/H_q)mc^2$ is 11.6 keV if $H = 10^{12}$ G. If the electron kinetic energy is less than $(H/H_q)mc^2$, the only allowable state for the electron is the state $n=0$, and then

$$\varepsilon = p_z^2/2m \tag{11}$$

which is the expression for the energy of a one-dimensional particle. The density of state for a one-dimensional electron is

$$\varrho_1(\varepsilon) = \frac{2}{h}\frac{\mathrm{d}p}{\mathrm{d}\varepsilon} = \frac{2m}{h} p_z^{-1}. \tag{12}$$

For an electron in the absence of field

$$\varrho_3(\varepsilon) = \frac{2}{h^3}\frac{\mathrm{d}^3p}{\mathrm{d}\varepsilon} = \frac{2m}{h^3} p \int \mathrm{d}\Omega \tag{13}$$

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where $d\Omega$ is the solid angle element in the direction of the emerging particle. Therefore for a one-dimensional particle $\varrho_1(\varepsilon)\to\infty$ as $p\to 0$, and for a three-dimensional particle $\varrho_3(\varepsilon)\to 0$ as $p\to 0$.

This one-dimensional behavior has been observed in low temperature experiments and this gives rise to the so-called de Hass–Van Alphen effect. As a result of Equation (9) there are only two radiation processes for 'free' electrons in a magnetic field: (a) the spontaneous transition of an electron from a state n to $n'\neq n$; (b) the Coulomb de-excitation of an electron from a state n to n' (no restriction on n' but the energy must be conserved). (a) is the quantized version of the classical synchrotron radiation, the emission gives rise to lines and the minimum energy of emission is $H/H_q mc^2$ (nonrelativistic case) or $(H/H_q)mc^2\cdot mc^2/\varepsilon$ (relativistic case). The life time of electrons is generally very short; it is

$$\tau_s = \left[\frac{2}{3}\frac{e^2}{\hbar c}\frac{mc^2}{\hbar}\frac{\varepsilon}{mc^2}\left(\frac{H}{H_q}\right)^2\right]^{-1}. \tag{14}$$

For $H\sim 10^{13}$ G, $\varepsilon\sim 1$ MeV, $\tau_s \ll 10^{-15}$ sec.

The bremsstrahlung radiation has been calculated by Goldman and Oster (1964), but their result is very difficult to apply. We have recently calculated the nonrelativistic bremsstrahlung radiation rate in a magnetic field (Canuto and Chiu, 1970). In this calculation the electron wave function in a magnetic field in conjunction with a Green's function appropriate for the field is used. This calculation is therefore expected to be valid for nonrelativistic electrons with energies up to, say, one or two hundred keV, in fields up to a fraction of $H_q = 4.414\times 10^{13}$ G, for frequencies not in the vicinity of the plasma frequency of the Larmor frequency of the electrons or ions.

The expression for the transition probability for the transition at forward angle is (Canuto and Chiu, 1970)

$$W = W_0 C_1(\lambda)(p\omega)^{-1} \quad (\mathrm{cm^3\,sec^{-1}}) \tag{15}$$

where $(\varepsilon = E/mc^2)$

$$W_0 = Z^2\alpha^3\pi^2 N_i \lambda_c^3 \hbar c^2 (H/H_q)^{-2} = 2.09\times 10^{-37} Z^2 N_i (H/H_q)^{-2} \quad (\mathrm{erg\,cm^2\,sec^{-1}})$$

N_i is the ion density and Z the average atomic number, and

$$C_1(\lambda) = e^{\lambda}(1+\lambda)E(\lambda) - 1, \qquad \lambda = (\lambda_c^2/2\omega_H)\left[\left(\frac{\omega}{v}\right)^2 + \mathrm{d}^{-2}\right] \tag{16}$$

$$E(\lambda) = \int_{\lambda}^{\infty} x^{-1}\exp(-x)\,\mathrm{d}x \xrightarrow{\lambda\to 0} -\ln(\gamma\lambda)^{-1} \qquad \gamma = 1.78102\ldots \tag{17}$$

where ω_H is the Larmor frequency, v the electron velocity, α is the Debye length. This term d^2 arises from electron screening. (We assume the validity of classical electron screening theory.) In the regime of interest $\lambda \ll 1$ we can write

$$W = W_0(p\omega)^{-1}\ln(\gamma\lambda)^{-1}. \tag{18}$$

The effect of a dense plasma is to alter the relation between the photon frequency ω and the wave vector $\mathbf{k}$ of the photon.

The dielectric constant is the same for a classical as well as a quantum electron gas, provided $\omega \ll \omega_H$ where ω_H is the Larmor frequency. The dielectric constant in a magnetic field is a tensor quantity. In the direction along the magnetic field the radiation can be split circularly into two circular polarization components, the ordinary component (denoted by a subscript o) and the extraordinary component (denoted by a subscript x):

$$\begin{aligned} n_x &= 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega}{\omega - \omega_H} - \frac{\omega_I^2}{\omega^2}\frac{\omega}{\omega + \Omega_H} \quad \text{(right hand polarization)} \\ n_o &= 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega}{\omega + \omega_H} - \frac{\omega_I^2}{\omega^2}\frac{\omega}{\omega - \Omega_H} \quad \text{(left hand polarization)}. \end{aligned} \tag{19}$$

ω_p and ω_I are the plasma frequencies of the electron gas and ion respectively, ω_H and Ω_H the Larmor frequency for electrons and ions. For $H = 10^{12}$ G, $\omega_H = 10^{19}$ and $\Omega_H = 10^{17}$, $\omega_p = 10^{16}\sqrt{\varrho}$, $\omega_I \approx 10^{14}\sqrt{\varrho}$, where ϱ is the density. Hence, for $\varrho \lesssim 10^6$ g/cm^3, n_x and n_o are close to unity for $\omega \sim 10^{10}$.

3. Radiation Mechanisms for Pulsars (Chiu and Canuto, 1970)

From the upper limit of the radiating region of pulsars ($< 3 \times 10^6$ cm) and the energy radio flux of emission (10^{31} ergs/sec) the flux of radiation I_ν at the emitting region is about 10^{18} ergs/cm^2 sec. The frequency of emission is at $\sim 10^3$ MHz. Assuming a bandwidth $\Delta f = f$, we find that the temperature T_B of a black body with an equivalent emissivity in the same frequency regime is given by:

$$I_\omega \sim 2kT_B \left(\frac{\omega}{2\pi}\right)^2 c^{-2} \tag{20}$$

or:

$$T_B = \frac{I_\omega c^2}{2k(\omega/2\pi)^2}.$$

Therefore, for pulsars the brightness temperature is about 10^{24} K. If this radiation is emitted by random processes then T_B is the minimum temperature of the emitter. The unduly large value of T_B requires that the process of emission cannot be an incoherent process.

Several laboratory devices can give rise to large brightness temperature in a relatively cool medium. These are lasers or Klystrons. In a Klystron an energetic electron beam bunched in space mingles with an electromagnetic wave with such a phase relationship that the electrons are decelerated by the electric field of the electromagnetic wave, and thereby the electrons give their kinetic energy to the electromagnetic wave. In this process the electrons must have such a spatial distribution that they are always decelerated, otherwise the electromagnetic wave will give energy to the electrons. For this reason, only electromagnetic waves of wavelengths within a bandwidth $\frac{1}{2}\lambda$ are amplified.

This emission mechanism is also referred to as coherent plasma emission and has been considered as a possible emission mechanism for pulsars. There are a number of reasons why this mechanism may not work. First, the coherence of a plasma extends only over a spatial region of $\frac{1}{2}\lambda$; this also implies that the bandwidth of emission is of the order of $\frac{1}{2}\lambda$. Observation shows that the bandwidth extends from 50–1000 MHz, much beyond this bandwidth. Second, the intensity of radiation depends critically on the plasma density. Since the density of plasma surrounding the star depends on the structure of the magnetosphere which depends critically on the period P, it is expected that the intensity of radiation should also depend on the period of rotation. However, observation shows that there is no such correlation. Third, observation shows that the optical and X-ray emission come from the same emission region. Therefore, the same plasma must be also responsible for optical and X-ray emission.

It is not possible to explain the (so far) unique case of optical and X-ray emission of NP 0532 except by imposing very artificial conditions. The other mechanism that we shall discuss here avoids all these difficulties. In the case of a laser, emission takes place between two atomic states whose energies are $\varepsilon_1, \varepsilon_2$ $(\varepsilon_1 < \varepsilon_2)$. Coherence is achieved via a stimulated emission process, and the condition for coherent stimulated emission is such that the system has a negative temperature. Let the population of these two states be N_2 and N_1. Then an excitation temperature T_e may be defined by the equation

$$\frac{N_2}{N_1} = e^{-(\varepsilon_2 - \varepsilon_1)/kT_e}. \tag{21}$$

The condition for $T_e < 0$ is that $N_2 > N_1$. This case is referred to as population inversion.

The mass absorption coefficient $K(\omega)$ with stimulated emission is given by

$$K(\omega) = N_1 \frac{\sigma c}{\varrho} (1 - N_2/N_1) \tag{22}$$

where σ is the absorption cross-section, ϱ is the density. $K(\omega)$ therefore becomes negative when $N_2 > N_1$. The radiative transfer equation

$$\frac{dI}{ds} = - K(\omega) \varrho I \tag{23}$$

will have an exponentially growing solution

$$I_\omega = I_0 \exp \int [- K(\omega)] \varrho \, ds. \tag{24}$$

Radiation thus can be amplified. However, a laser operated between two atomic states again only gives rise to line emission. In order that a broad band emission be

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produced, it is necessary that the negative absorption coefficient extends over a finite frequency range. To see how this condition can be fulfilled, we can rewrite Equation (21) as follows:

$$\frac{N_2 - N_1}{N_1} = e^{-(\varepsilon_2 - \varepsilon_1)/kT_e} - 1\,. \tag{25}$$

Let us consider a continuum and let $N_2 \to N_1$. Then we can write $N_2 - N_1 = \mathrm{d}N$, $\varepsilon_2 - \varepsilon_1 = \mathrm{d}\varepsilon$, and expand the exponential. We find

$$\frac{\mathrm{d}N}{N} = -\frac{\mathrm{d}\varepsilon}{kT_e} \tag{26}$$

or:

$$\frac{\mathrm{d}N}{\mathrm{d}\varepsilon} = -\frac{N}{kT_e}\,. \tag{27}$$

The requirement for a negative temperature is then $\mathrm{d}N/\mathrm{d}E > 0$. (Note: N refers to the occupation number per state and not the number of particles per unit energy range.) However, since the condition $\mathrm{d}N/\mathrm{d}E > 0$ cannot be maintained throughout the entire energy range (otherwise the total number of particles will necessarily be infinite) somewhere $\mathrm{d}N/\mathrm{d}E$ will become less than zero. The condition for a negative absorption coefficient is more complicated than that for discrete states. The expression for the absorption coefficient $\mathrm{K}(\omega)$ for an electron gas in the lowest Landau level is

$$K(\omega) = \frac{2m\hbar\omega}{\varrho} \int_0^\infty g h^{-1}\, \mathrm{d}p \left[f(p) \frac{\partial}{\partial p} \frac{\mathscr{W}(p)}{p + p'} + f(-p) \frac{\partial}{\partial p} \frac{\mathscr{W}(p)}{p - p'} \right] \tag{28}$$

where $\mathscr{W}(p)$ is the transition probability for the absorption of a photon of energy ω by an electron of momentum p, p' is the electron momentum after absorption of a photon, $f(p)$ is the electron distribution function, g is a statistical weight factor of the electron (g is a function of the magnetic field) ϱ is the density. $\mathscr{W}(p)$ is related to $W(p)$ [Equation (15)] via the detailed balance theorem which for a one-dimensional particle is simply

$$g\mathscr{W}(p)\,\mathrm{d}p = gW(p)\,\mathrm{d}p', \qquad \mathrm{d}p'/\mathrm{d}p = p/p'. \tag{29}$$

Amplification of radiation (maser effect) is possible if $\mathrm{K}(\omega) < 0$. The condition for this is not straightforward, as is clear from the form of Equation (28). However, if the electron distribution function is asymmetrical, e.g., $f(p) \gg f(-p)$ (this corresponds to a population inversion in terms of a coherent streaming motion in the $+$p direction with respect to the medium) then the condition for amplification of radiation is just:

$$\frac{\partial}{\partial p} \frac{\mathscr{W}(p)}{p + p'} < 0 \tag{30}$$

or $\mathscr{W}(p)$ must not rise faster than the first power of p. As is clear from the form of W for the $0\to0$ transition, Equation (15), this condition is completely fulfilled. Hence, it is possible to have a negative emission coefficient.

4. A Simple Model (Chiu and Canuto, 1970)

We choose $f(p)$ a delta function at p_m $[f(p)=A\delta(p-p_m)]$ where A is a normalization constant. $(A=g^{-1}N_e$ where N_e is the electron density.) We then have

$$\varrho K(\omega) = -\,2m\hbar c^{-1}N_e W_0 p_m^{-3}\ln(\gamma\lambda)^{-1}\,. \tag{31}$$

The absorption coefficient is therefore negative and amplification of radiation takes place. The amplification is frequency independent, and covers the entire range of radio spectrum.

The radiation transfer equation is:

$$\frac{\mathrm{d}I(\omega, s)}{\mathrm{d}s} = -\,I(\omega, s)\,\varrho K(\omega) \tag{32}$$

whereas the spontaneous emission term is neglected. Equation (32) then gives

$$I(\omega, s) = I(\omega, 0)\exp\left\{\int_0^s p_m^{-3}\,\delta\ln[\gamma\lambda]\right\} \qquad \delta = 2m\hbar c^{-1}N_e W_0 \tag{33}$$

and s is the path length over which amplification takes place.

Before we proceed further, we will discuss a very important effect in the theory of light and microwave amplification by stimulated emission. This is the effect of saturation of amplification. As it turns out, the amplification process is so efficient that the saturation phenomena takes place in all cases. The saturation phenomenon gives rise to non-linear effects and in effect places the *average* brightness temperature within a range compatible with observations.

If a small signal is fed into an amplifier (such as an audio amplifier), the output power is proportional to the input signal. However, when the output power reaches the inherent power limit of the amplifier, further increase in the input signal level will not increase the output power. When this happens, the amplifier no longer amplifies the input signal linearly, and this amplifier is said to have reached saturation.

The same saturation effect also exists in amplification by stimulated emission.* Let us consider a two-level system. The reasoning can be easily extended to continuum energy level systems. Let there be n_2 particles in the upper energy state 2 and let n_1 be the number of particles in the lower state 1. In the absence of stimulated radiation, the lifetime of the state 2 is τ_2. Let $N=n_2-n_1$ be the difference between particles in the upper and in the lower states, and population inversion is achieved when $N>0$.

* This is very extensively discussed in literature; see, for example, Troup (1963).

Let N_0 be the steady state value of N. (N_0 depends on the pumping rate and other properties of the system.) The rate equation for N in the absence of stimulated radiation is:

$$\frac{dN}{dt} = \frac{N_0 - N}{\tau_2}. \tag{34}$$

Now let us introduce stimulated radiation and let the transition probability via stimulated emission be P_{21}. The rate of change of N due to stimulated emission is simply $-2NP_{21}$. (The factor 2 arises from the fact that each transition from the state 2 to the state 1 changes the value of N by 2). The rate Equation (34) now becomes

$$\frac{dN}{dt} = \frac{N_0 - N}{\tau_2} - 2NP_{21} \tag{35}$$

A steady state obtains when N_s satisfies

$$\frac{N_0 - N_s}{\tau_2} - 2N_sP_{21} = 0 \tag{36}$$

or when

$$\frac{N_s}{N_0} = [1 + 2P_{21}\tau_2]^{-1}. \tag{37}$$

P_{21} is proportional to the intensity of stimulated radiation, $I(\omega, s)$. When $I(\omega, s)$ becomes large such that $2P_{21}\tau_2 \sim 1$, the value of N_s begins to decrease appreciably from the original value N_0 and this laser amplifier no longer amplifies exponentially; the 'gain' decreases with increased output. Further increase in $I(\omega, s)$ will only cause a further decrease in the gain and, finally, as N_s approaches zero, the absorption coefficient being proportional to N, also becomes zero and the medium becomes transparent to radiation.

We can obtain the saturated intensity from the condition for the onset of saturation:

$$2P_{21}\tau_2 \approx 1. \tag{38}$$

P_{21} is the transition probability for emission per photon state and is easily seen from the Boltzmann equation for the photon distribution function to be just

$$P_{21} = F(\omega)\,\varrho K(\omega) \tag{39}$$

where $F(\omega)$ is the number of photons per state.

τ_2^{-1} is the transition probability for spontaneous emission, or collision, whichever is greater. The collision rate is about 10^{10} per sec. The transition probability for spontaneous emission is much greater. The expression for τ_2^{-1} for spontaneous emission is

$$\tau_2^{-1} = N_e W(p) = 2.09 \times 10^{-37}\, Z^2 N_e N_i (p\omega)^{-1} (H/H_q)^{-2} C_1(\lambda). \tag{40}$$

Numerically, if we set $N_e = 10^{24}\,\text{cm}^{-3}$, $N_i = N_e/26$ (ion composition at the surface),

$p = p_m = 0.01\, mc$ (see Section 8), $H/H_q = 0.02\, (H \sim 10^{12}$ G) $\omega \approx 10^9$ Hz, we find

$$\tau_2^{-1} \approx 10^{27}. \tag{41}$$

The value of $\varrho K(\omega)$ which will give rise to saturation is about 40. Hence, from Equation (39) we obtain the lower limit for the saturated value of $F(\omega)$:

$$F(\omega) \sim 2 \times 10^{25}. \tag{42}$$

The brightness temperature T_B is: (cf. Equation (20))

$$F(\omega) = \frac{kT_B}{\hbar\omega} \cong 10^2 T_B \quad (\omega \sim 10^9). \tag{43}$$

Comparing Equation (42) with Equation (43), we find that at saturation the lower limit of the brightness temperature is about 10^{23} K. However, this value of T_B is an underestimate, for Equation (39) gives the condition for the onset of saturation, and when saturation fully takes place, due to non-linear effects, the value of T_B may be several orders of magnitude greater than its value at the onset of saturation. The intensity of pulsar radiation is therefore independent on the period of pulsars, being a function of other parameters (such as the magnetic field, etc.).

5. Polarization of Radiation

In previous papers (Chiu and Canuto, 1969c; Chiu, Canuto and Fassio-Canuto, 1969; Chiu and Canuto, 1969d; Chiu, 1969) we have suggested that the propagation of radiation is strongly affected by the magnetic field resulting in a beamed emission. In this section we will discuss the dielectric properties of a plasma in a magnetic field.

As the frequency of radiation is small compared with the Larmor frequency of the ions and the electrons, as well as the plasma frequency of the electrons and the ions, we can use the dielectric constant computed for a cold ionic neutral plasma in a magnetic field. It turns out that quantum mechanical calculations (Kelly, 1964) and classical calculations coincide in this limit:

The dielectric tensor is (assuming that the magnetic field is in the z-direction) (Stix, 1969):

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \tag{44}$$

where

$$R = 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega}{\omega - \omega_H} - \frac{\omega_I^2}{\omega^2}\frac{\omega}{\omega + \Omega_H} \tag{45}$$

$$L = 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega}{\omega + \omega_H} - \frac{\omega_I^2}{\omega^2}\frac{\omega}{\omega - \Omega_H}$$

$$P = 1 - \left(\frac{\omega_p}{\omega}\right)^2 - \left(\frac{\omega_I}{\omega}\right)^2, \qquad 2D = R - L, \qquad 2S = R + L \tag{46}$$

$$\omega_p^2 = 4\pi e^2 N_e/m \qquad \omega_I^2 = 4\pi e^2 Z N_z/m$$
$$\omega_H = eH/mc \qquad \Omega_H = Z\omega_H m/M_i \tag{47}$$

ω_p and ω_I are plasma frequencies of the electron gas and the ion gas, respectively, and ω_H and Ω_H the corresponding Larmor frequencies.

Generally speaking, the photon propagation in a doubly refractive medium such as Equation (44) can be analyzed into two modes, the ordinary and the extraordinary mode of propagation. The propagation of these two modes can be easily studied for the direction $\theta=0$ and $\theta=\frac{1}{2}\pi$, and can be studied numerically at other angles. Here we will describe the cases $\theta=0$ and $\theta=\frac{1}{2}\pi$, respectively.

A. ALONG THE MAGNETIC FIELD $\theta = 0$

In this direction these two modes are right-handed (R) and left-handed (L) circularly polarized, and the refractive indices for these two modes are: (o = ordinary, x = extraordinary)

$$n_x^2 = R \tag{48}$$

$$n_o^2 = L. \tag{49}$$

The emissivity of the x- and the o-modes in the $0\to 0$ bremsstrahlung transition is:

$$\text{Ordinary mode: } W^{(o)} = L^{1/2} W_0 (p\omega)^{-1}\, C_1(\lambda)\, E_1^{-2} \tag{50}$$

$$\text{Extraordinary mode: } W^{(x)} = R^{1/2} W_0 (p\omega)^{-1}\, C_1(\lambda)\, E_2^{-2} \tag{51}$$

where

$$E_1^{-1} = (H/H_q)\left[\hbar\omega/mc^2 - H/H_q - \tfrac{1}{2}L(\hbar\omega/mc^2) - L^{1/2}(\hbar\omega/mc^2)(p/mc)\right]^{-1} \tag{52}$$

$$E_2^{-1} = (H/H_q)\left[\hbar\omega/mc^2 + H/H_q + \tfrac{1}{2}R(\hbar\omega/mc^2) - R^{1/2}(\hbar\omega/mc^2)(p/mc)\right]^{-1}. \tag{53}$$

Now we will substitute into these equations quantities pertinent to a neutron star: $H=10^{12}$ G, $N_e=10^{24}$. We find $\omega_p \simeq 10^{16}$, $\omega_I \simeq 10^{14}$, $\omega_H \simeq 10^{19}$, $\Omega_H \simeq 10^{16}$. If we choose $\omega=10^9$ Hz, we find that $E_1^{-1}=E_2^{-1}=1$. For R and L we can expand equations (45) and (46) into power series in ω/ω_H and ω/Ω_H. For a charge neutral plasma, for which Equation (47) is valid, we find that the first nonvanishing term in the expansion gives:

$$R = 1 + \frac{\omega_p^2}{\omega_H^2} + \frac{\omega_I^2}{\Omega_H^2} \simeq 1 \tag{54}$$

$$L = 1 - \frac{\omega_p^2}{\omega_H^2} - \frac{\omega_I^2}{\Omega_H^2} \simeq 1 \tag{55}$$

and hence

$$W^{(o)}(p) = W^{(x)}(p) = W_0 (p\omega)^{-1}\, C_1(\lambda) \tag{56}$$

Therefore these two modes have the same emissivity as that in vacuum.

We therefore conclude that the radiation emerging from the dense plasma of the star along the magnetic field has both right- and left-hand polarization and has the same intensity. It is not known whether these two modes can combine into a linearly polarized beam or these two modes will propagate separately to yield an unpolarized beam.

B. PROPAGATION PERPENDICULAR TO THE FIELD, $\theta = \frac{1}{2}\pi$.

In the perpendicular direction there is only one mode of propagation; this is the extraordinary mode. The refractive index for the ordinary mode is simply $1 < (\omega_I/\omega)^2 + (\omega_p/\omega)^2$, and propagation of this mode is forbidden. For the extraordinary mode the refractive index is:

$$n_x^2 = RL/S \approx 1 \tag{57}$$

and propagation is the same as in vacuum. This mode has one hundred percent linear polarization whose direction is perpendicular to both the wave vector **k** and the magnetic field H. There is another mode of emission whose electric field is along the magnetic field. Our computation shows that this mode is important in vacuum. However, since this mode cannot propagate in a dense plasma, it is not considered here.

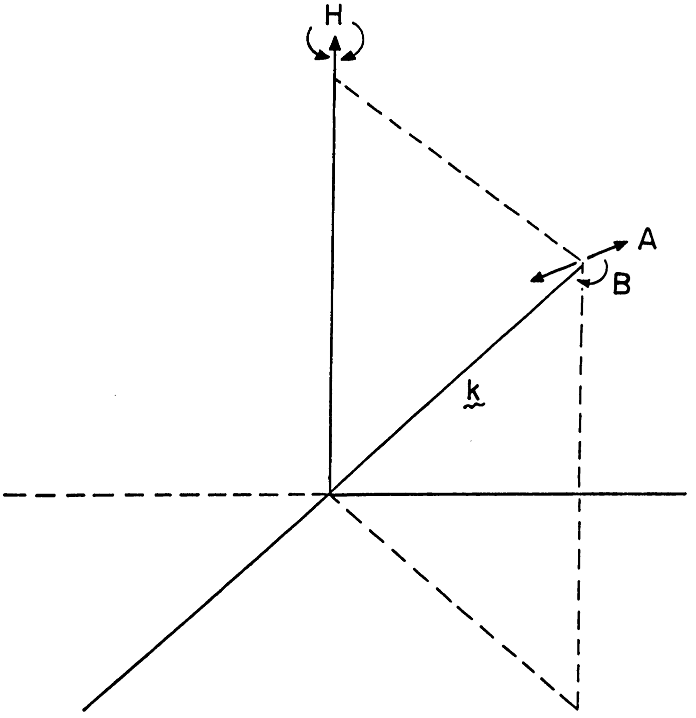

Fig. 1. Polarization of the bremsstrahlung radiation in a strongly magnetized medium. **k** = propagation vector, H = magnetic field. A is the linearly polarized component perpendicular to the field; B is the circularly polarized component of the extraordinary component.

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C. ALONG AN ARBITRARY DIRECTION

As it turns out the refractive index for the extraordinary mode is always close to unity for the field strength and plasma frequency we have quoted. The emergent radiation has an elliptical polarization which can be decomposed into a circular polarization plus a linear polarization. The polarization of the radiation from bremsstrahlung process that can emerge from the medium can either be circularly polarized, or linearly polarized, depending on the emissivity associated with these two polarization states. However, from simple arguments it seems that linear polarization is favored over circular polarization.

Figure 1 shows the directions of polarization of the emerging radiation. It is clear that if we look into the direction of the magnetic field, the radiation will appear unpolarized, assuming that these two modes (*o* and *x*) propagate and become amplified independently of each other. However, if we look at the medium at an oblique angle, the radiation will appear partially linearly polarized and the degree and inclination of polarization with respect to a fixed plane of rotation will depend on the relative orientation of the emitting surface with respect to the observer. This is illustrated in Figure 2. Thus, in general, we expect that pulsar radiation will be

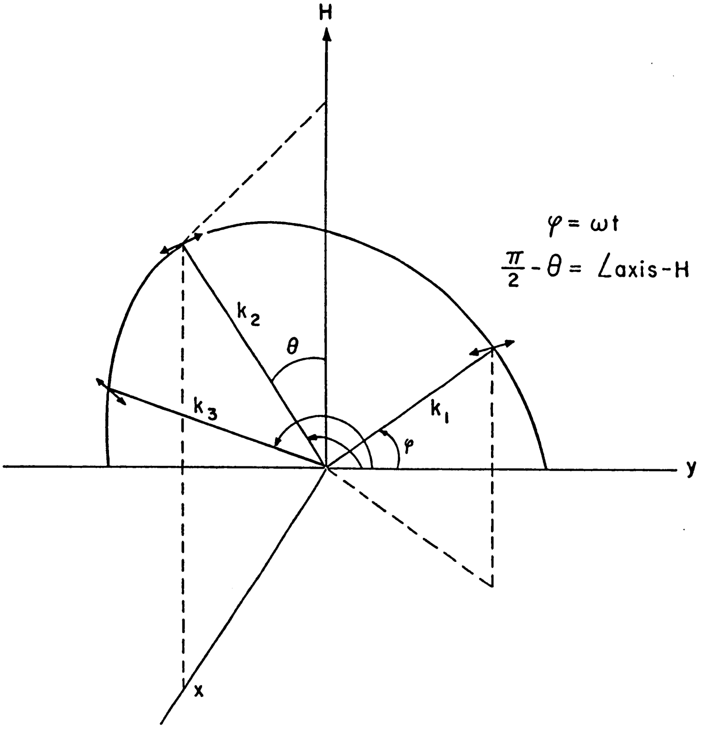

Fig. 2. The relative orientation of polarization in the observer's plane for a pulsar whose rotational axis is perpendicular to the line of sight and the magnetic field is inclined at an angle $\pi/2 - \theta$ with respect to the rotational axis.

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linearly polarized and the polarization strength and orientation will change through the cycle. Observation shows that in some cases pulsar radiation is found to be polarized and the orientation and the degree of polarization change with the phase of the pulse (Radhakrishnan and Cooke, 1969). We are now in the process of extending our computation of the emissivity of radiation at an arbitrary angle so that a comparison of our theory with observation can be made.

6. Conclusion

In this paper we have presented a model for pulsars. This model is based on the behavior of electrons in an intense magnetic field. While the model has not developed into a full quantitative treatment, it successfully arrived at a mechanism for emissions of intense radio emission. This mechanism avoids the usual difficulty of coherent plasma emission, that is the emission be limited to a bandwidth of $\frac{1}{2}\lambda$. A full account of this model is being published (Chiu and Canuto, 1971).

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(For a more complete bibliography see Chiu (1970))

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Discussion

C. Heiles: At 430 MHz, T_B for NP 0532 is $\sim 10^{28}$ K in the average and $\gtrsim 10^{31}$ K for strong pulses. At 100 MHz these numbers would be up by more than two orders of magnitude.

H. Y. Chiu: The brightness temperature I gave represents the typical value for a rotation at 1 sec period. When saturation occurs the brightness temperature depends on other factors, and can go as high as 10^{31} K. I would expect that the brightness temperature should increase at lower frequencies.

G. Chanmugam: Does the quantisation depend on the assumption that the magnetic energy goes as the scalar product of $\boldsymbol{\mu}$ and H? If so is this justified for fields $\sim 10^{12}$ G?

H. Y. Chiu: The theory we developed is based on the solution of the Dirac equation of a free electron in a magnetic field, and is valid even when the field is greater than 10^{13} G.