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SUMS OF LAURENT AND BILATERAL HANKEL OPERATORS

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One can define "Laurent" and "Hankel" operators relative to groups more general than the circle group $\mathbb{T}$. We do that here, derive some of their properties, and compute their spectra, using a concrete realization of a crossed product C^*-algebra by the two-element group $\mathbb{Z}_2$.

Definition. Let A be a C^*-algebra and $\gamma: A \to A$ a $*$-isomorphism such that $\gamma^2 = \mathrm{id}_A$. We define

$$C^*(A,\gamma) = \left\{ \begin{pmatrix} a & b \\ \gamma b & \gamma a \end{pmatrix} : a, b \in A \right\}.$$

It is readily verified that $C^*(A,\gamma)$ is a C^*-subalgebra of the 2×2 matrix algebra $M_2(A) = M_2 \otimes A$ over A.

Proposition 1. *Let B be a unital C^*-algebra and A a C^*-subalgebra of B containing the unit 1 of B. Let $s \in B$ be a symmetry (i.e. Hermitian unitary) such that $sAs = A$, and define the $*$-isomorphism $\gamma: A \to A$ by $a \mapsto sas$.*

Then $A + As$ $(= \{a + bs: a, b \in A\})$ is a C^*-subalgebra of B and the map

$$\phi: C^*(A.\gamma) \to B, \begin{pmatrix} a & b \\ sbs & sas \end{pmatrix} \mapsto a + bs,$$

is a $*$-homomorphism with image $A + As$.

Proof. Elementary computations show ϕ is a $*$-homomorphism, and its image is clearly $A + As$, implying $A + As$ is a C^*-subalgebra of B.

Corollary 2. *ϕ is injective if $A \cap As = 0$.*

Now suppose G is a compact abelian group with dual group Γ. Define $S \in B\ (L^2(G))$ by $Sf = \check{f}$ where $\check{f}: x \to f(-x)$. S is clearly a symmetry. If $\phi \in L^\infty(G)$ we define the *Laurent* operator $L_\phi \in B(L^2(G))$ by $f \mapsto \phi f$, and the (bilateral) *Hankel* operator B_ϕ by $B_\phi = L_\phi S$.

Γ forms an orthonormal basis for $L^2(G)$, and when we talk of the matrix of an operator $T \in B(L^2(G))$ we shall mean relative to this basis. If T has matrix $(a_{\gamma,\delta})$ we say it is a *Laurent* (resp. *Hankel*) matrix if $a_{\gamma+\rho,\delta+\rho}=a_{\gamma,\delta}$ (resp. $a_{\gamma+\rho,\delta-\rho}=a_{\gamma,\delta}$) for all $\gamma,\delta,\rho\in\Gamma$. Clearly Laurent operators have Laurent matrices and Hankel operators have Hankel matrices. The proof of the next proposition is the same as the classical case ($G=\mathbb{T}$) and is included for the sake of completeness.

Proposition 3. *Let G be a compact abelian group, and $T\in B(L^2(G))$. If T has Laurent (resp. Hankel) matrix then T is a Laurent (resp. Hankel) operator.*

Proof. Let $\Gamma=\hat{G}$ and $\langle\phi,\psi\rangle=\int_G\phi\bar{\psi}$ if $\phi\bar{\psi}\in L^1(G)$. Let $(a_{\gamma,\delta})$ be the matrix of T. Suppose firstly this is a Laurent matrix and put $\phi=\sum_{\delta\in\Gamma}\alpha_{\delta,0}\delta$ in $L^2(G)$. Let $\gamma\in\Gamma$, and $f\in L^2(G)$. Then

$$\begin{aligned}\langle\phi f,\gamma\rangle&=\langle f,\sum_{\delta\in\Gamma}\bar{a}_{\delta,0}\bar{\delta}\gamma\rangle\\&=\sum_{\delta\in\Gamma}a_{\delta,0}\langle f,\bar{\delta}\gamma\rangle=\sum_{\delta\in\Gamma}\alpha_{\gamma-\delta,0}\langle f,\delta\rangle\\&=\sum_{\delta\in\Gamma}a_{\gamma,\delta}\langle f,\delta\rangle,\quad\text{since}\quad a_{\gamma-\delta,0}=a_{\gamma,\delta},\end{aligned}$$

Thus $\langle\phi f,\gamma\rangle=\langle Tf,\gamma\rangle$ $(\gamma\in\Gamma)$, implying $\phi f=Tf$. Hence $\phi L^2(G)\subseteq L^2(G)$ and so $\phi\in L^\infty(G)$. Thus $T=L_\phi$ is a Laurent operator.

On the other hand, if T has Hankel matrix then TS has Laurent matrix so $TS=L_\phi$ i.e. $T=B_\phi$, $\exists\phi\in L^\infty(G)$. □

Remark. It is easy to calculate the spectra for L_ϕ and B_ϕ. The map $L^\infty(G)\to B(L^2(G))$, $\phi\to L_\phi$, is clearly an isometric $*$-homomorphism, so $\|L_\phi\|=\|\phi\|_\infty$ and $\sigma(L_\phi)=\sigma(\phi)$ (the spectrum relative to $L^\infty(G)$). Also $B_\phi^2=L_\phi(SL_\phi S)=L_\phi L_{\tilde{\phi}}=L_{\phi\tilde{\phi}}$, so $\sigma(B_\phi)^2=\sigma(\phi\tilde{\phi})$. We need to do a little more to calculate $\sigma(L_\sigma+B_\psi)$ for $\phi,\psi\in L^\infty(G)$:

Proposition 4. *If G is a nontrivial connected compact abelian group, then $C=\{L_\phi+B_\psi:\phi\psi\in L^\infty(G)\}$ is a C^*-subalgebra of $B(L^2(G))$. If $\phi,\psi\in L^\infty(G)$ then*

$$\max\{\|\phi\|_\infty,\|\psi\|_\infty\}\leqq\|L_\phi+B_\psi\|\leqq 2\max\{\|\phi\|_\infty,\|\psi\|_\infty\}\quad\textit{and}$$

$$\sigma(L_\phi+B_\psi)=\{\lambda\in\mathbb{C}:(\lambda-\phi)(\lambda-\tilde{\phi})-\psi\tilde{\psi}\textit{ is noninvertible in }L^\infty(G)\}.$$

Proof. We can replace the condition G is connected by the weaker one that $\Gamma=\hat{G}$ has an element of infinite order. (That G connected implies this condition on Γ is an elementary exercise.)

Let $A=\{L_\phi:\phi\in L^\infty(G)\}$. If $T\in A\cap AS$ and has matrix $(a_{\gamma,\delta})$ and ρ is an element of infinite order in Γ then $(\gamma+2n\rho,\delta)_{n=1}^\infty$ is an infinite sequence in $\Gamma\times\Gamma$ and since $a_{\gamma,\delta}=a_{\gamma+\rho,\delta+\rho}=a_{\gamma+2\rho,\delta}$, so $a_{\gamma,\delta}=a_{\gamma+2n\rho,\delta}$ $(n=1,2,3,\ldots)$. We have $\sum_{n=1}^\infty|a_{\gamma+2n\rho,\delta}|^2<\infty$, and so $a_{\gamma,\delta}=0$. Thus $A\cap AS=0$.

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We now deduce from Proposition 1 and Corollary 2 that the map

$$C^*(A,\psi)\to A+AS,\begin{pmatrix} L_\phi & L_\psi \\ L_{\tilde\psi} & L_{\tilde\phi}\end{pmatrix}\mapsto L_\phi+B_\psi,$$

is a $*$-isomorphism onto the C^*-subalgebra $C=A+AS$ of $B(L^2(G))$ ($\mu\colon A\to A$ is defined by $L_\phi\to L_{\tilde\phi}$).

Let $\phi,\ \psi\in L^\infty(G)$. Clearly

$$\|L_\phi+B_\psi\|=\left\|\begin{pmatrix} L_\phi & L_\psi \\ L_{\tilde\psi} & L_{\tilde\phi}\end{pmatrix}\right\|\geqq\|L_\phi\|=\|\phi\|_\infty$$

and since

$$\|L_\phi S+B_\psi S\|=\|L_\psi+B_\phi\|\geqq\|\psi\|_\infty$$

we have

$$\|L_\phi+B_\psi\|\geqq\max\left(\|\phi\|_\infty,\|\psi\|_\infty\right).$$

Also

$$\|L_\phi+B_\psi\|\leqq\|\phi\|_\infty+\|\psi\|_\infty\leqq 2\max\{\|\phi\|_\infty,\|\psi\|_\infty\}.$$

Finally $L_\phi+B_\psi-\lambda$ is invertible iff

$$\begin{pmatrix} L_{\phi-\lambda} & L_\psi \\ L_{\tilde\psi} & L_{\tilde\phi-\lambda}\end{pmatrix}$$

is invertible iff the determinant $L_{\phi-\lambda}L_{\tilde\phi-\lambda}-L_{\psi\tilde\psi}$ is invertible. (This is an elementary computation which works because all the matrix entries commute.) Thus $L_\phi+B_\psi-\lambda$ is invertible iff $(\phi-\lambda)(\tilde\phi-\lambda)-\psi\tilde\psi$ is invertible in $L^\infty(G)$. □

Remark. The results of Proposition 4 were obtained by Walsh ([4], unpublished) in the case of $G=\mathbb{T}$ by methods quite different from ours. He analyses these operators in greater detail in this classical case.

Corollary 5. *If $\phi,\ \psi\in L^\infty(G)$ and $\phi(x)=\phi(-x)$, $\psi(x)=\psi(-x)$ $(x\in G)$ then $\sigma(L_\phi+B_\psi)=\sigma(\phi+\psi)\cup\sigma(\phi-\psi)$.*

Proof. In this case $(\lambda-\phi)(\lambda-\tilde\phi)-\psi\tilde\psi$

$$=(\lambda-\phi)^2-\psi^2=(\lambda-\phi-\psi)(\lambda-\phi+\psi).$$ □

These results are not true for arbitrary G. If G is the Cantor group $(\mathbb{Z}_2)^{\mathbb{N}}$, then $x=-x$ $(x\in G)$, implying $S=\mathrm{id}$ and so $A\cap AS\neq 0$. Hence $\sigma(L_\phi+B_\psi)=\sigma(\phi+\psi)\neq\sigma(\phi+\psi)\cup\sigma(\phi-\psi)$ in general (Take $\phi=\psi$ invertible).

Here is a trivial but interesting remark:

Take $G=\mathbb{T}$ and E any subset of $\mathbb{T}$ of positive measure not intersecting $E^{-1}=\{x^{-1}:x\in E\}$. Let $\phi=\chi_E$. Then $\phi\in L^\infty(\mathbb{T})$, $\phi\tilde{\phi}=0$, and so $B_\phi^2=L_{\phi\tilde{\phi}}=0$, but $B_\phi\neq 0$.

This is quite different from the behaviour of unilateral Hankel operators on the Hardy space H^2, none of which (0 excluded) are nilpotent. (See [3], [4]).

Now let us take a brief closer look at the C^*-algebra $C^*(A,\gamma)$ where A is a unital C^*-algebra and $\gamma: A\to A$ is a $*$-isomorphism such that $\gamma^2=\mathrm{id}$. Define an action of $\mathbb{Z}_2$ on A by $\tilde{\gamma}:\mathbb{Z}_2\to\mathrm{Aut}\,A$, $\tilde{\gamma}(0)=\mathrm{id}$, $\tilde{\gamma}(1)=\gamma$. Then the crossed product [1] $B=Ax_{\tilde{\gamma}}\mathbb{Z}_2$ has a symmetry s such that $sAs=A, B=A+As$, and A contains the unit of B. Moreover $A\cap As=0$. Thus we have a $*$-isomorphism

$$C^*(A,\gamma)\to Ax_{\tilde{\gamma}}\mathbb{Z}_2, \begin{pmatrix} a & b \\ \gamma b & \gamma a\end{pmatrix}\to a+bs.$$

(This follows from Proposition 1 and Corollary 2.)

In particular, if G is a connected compact abelian group then the C^*-algebra $\{L_\phi+B_\psi:\phi,\psi\in L^\infty(G)\}$ is just the crossed product $L^\infty(G)x_\mu\mathbb{Z}_2$ for a suitable action μ.

A final remark. In the study of C^*-algebras of the related class of Toeplitz operators, much use is made of commutator ideals and their identification. That this approach is not appropriate in the context of the algebra $C=\{L_\phi+B_\psi:\phi,\psi\in L^\infty(G)\}$ is indicated by the following. Suppose there exists a Borel set E in G with $-E=G\backslash E$. (This is obviously true for $G=\mathbb{T}$.) Let $\phi=\chi_E$. Then $\tilde{\phi}=\chi_{-E}=1-\chi_E=1-\phi$. Let $\psi=2\phi-1$. Then $\psi^2=1$. But

$$\psi=\frac{\psi-\tilde{\psi}}{2} \quad (\text{since } \tilde{\psi}=2\tilde{\phi}-1=2(1-\phi)-1=1-2\phi=\psi),$$

and so

$$L_\psi=\tfrac{1}{2}(L_\psi-SL_\psi S)=\tfrac{1}{2}(L_\psi S-SL_\psi)S.$$

Thus if I is the commutator ideal of C (i.e. the smallest closed ideal containing all commutators $xy-yx$, $x,y\in C$) then $L_\psi\in I$ and so $1=L_\psi^2\in I$. Thus $I=C$.

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