P 132. For subsets A and B of a metric space X, let $\alpha(A, L)$ be the set of points of X which are equidistant from A and B. Show that X is not connected if and only if there exist non-void subsets A and B of X for which $\alpha(A, B)$ is void.

J. Wilker, University of Toronto

Solution by B.L.D. Thorp, University College, Cardiff, Wales.

For each non-empty subset S of X, the function $f_S(x) = \inf \{d(x, s): s \in S\}$ is continuous on X. If X is not connected there exist disjoint, non-empty closed sets A, B such that $X = A \cup B$. If $x \in A$ then $f_A(x) = 0$, $f_B(x) \neq 0$, and similarly if $x \in B$. Thus

$$\alpha(A, B) = \{x: f_A(x) = f_B(x)\} = \phi .$$

Conversely, if A, B satisfy $\alpha(A, B) = \phi$, then $f_A(x) \neq f_B(x) \; \forall \; x \in X$, so the disjoint sets

$$A' = \{x: f_A(x) - f_B(x) < 0\} ,$$

$$B' = \{x: f_A(x) - f_B(x) > 0\}$$

satisfy $X = A' \cup B'$.

Finally, it follows from the continuity of $f_A - f_B$ that A' and B' are open; hence X is not connected.

Also solved by E.M. Roberts, J. Marsden and the proposer.

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