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ON SPECIAL RADICALS, SUPERNILPOTENT RADICALS AND WEAKLY HOMOMORPHICALLY CLOSED CLASSES

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Abstract

It is proved that a regular essentially closed and weakly homomorphically closed proper subclass of rings consists of semiprime rings. A regular class **M** defines a supernilpotent upper radical if and only if **M** consists of semiprime rings and the essential cover $\mathbf{M}_k$ of **M** is contained in the semisimple class $\mathbb{S}\,\mathfrak{U}\,\mathbf{M}$. A regular essentially closed class **M** containing all semisimple prime rings, defines a special upper radical if and only if **M** satisfies condition (S): every **M**-ring is a subdirect sum of prime **M**-rings. Thus we obtained a characterization of semisimple classes of special radicals; a subclas **S** of rings is the semisimple class of a special radical if and only if **S** is regular, subdirectly closed, essentially closed, and satisfies condition (S). The results are valid for alternative rings too.

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1. Introduction

Recent investigations have shown that semisimple classes of radicals, hereditary radicals and supernilpotent radicals, respectively, can be characterized by nice algebraic properties (see Sands (1976), van Leeuwen, Roos and Wiegandt (1977), Sands (Preprint), Anderson and Wiegandt (1979), (Preprint a) and (Preprint b)). As far as structure theorems are concerned, the best kind of radical is that of special radicals introduced by Andrunakievich (1958). Though constructions of

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special radicals were treated recently by Leavitt and Watters (1976), no characterization of semi-simple classes of special radicals were known.

The main purpose of this paper is to characterize the semisimple classes of special radicals (Theorem 4). To this end classes defining special upper radicals and those defining supernilpotent upper radicals are characterized in Theorem 3 and Theorem 2, respectively. Since one of the characteristic properties of semisimple classes of supernilpotent radicals is that the class should be weakly homomorphically closed, this condition is discussed in the context of other algebraic properties in Section 1.

In this paper we shall work in the variety of *associative* or *alternative rings*. (That we work in a variety, is used only in Section 2 where the split-null extension is applied; the other results are valid in any homomorphically closed hereditary class of alternative rings). A subclass $\mathbf{M}$ of rings will always mean a non-empty abstract class, that is, a class which is closed under isomorphisms. Further, $\mathbf{O}$ will denote the class of one-element rings and $\mathbf{A}^0$ that of all rings with zero multiplication. For a ring A the symbols A^+ and A^0 will denote the additive group of A and the zero-ring on the abelian group A^+, respectively.

A class $\mathbf{M}$ of rings is said to be *hereditary*, if $I \lhd A \in \mathbf{M}$ implies $I \in \mathbf{M}$. $\mathbf{M}$ is a *regular class*, if every nonzero ideal of an $\mathbf{M}$-ring has a nonzero homomorphic image in $\mathbf{M}$ (W. G. Leavitt (1970) used the term $\mathcal{S}$-complete for the term regular). $\mathbf{M}$ is called *subdirectly closed*, if every subdirect sum of $\mathbf{M}$-rings is again an $\mathbf{M}$-ring. We say that the class $\mathbf{M}$ is *essentially closed*, if $B \in \mathbf{M}$ implies $A \in \mathbf{M}$ whenever B is an essential (that is large) ideal in A. The fact that B is an essential ideal of A, will be denoted by $B \lhd \cdot A$. The *essential cover* $\mathbf{M}_k$ of a class $\mathbf{M}$ is defined as

$$\mathbf{M}_k = \{A\colon \text{there exists a ring } B \in \mathbf{M} \text{ such that } B \lhd \cdot A\}.$$

Obviously, a class $\mathbf{M}$ is essentially closed if and only if $\mathbf{M} = \mathbf{M}_k$. A class $\mathbf{M}$ is said to be *weakly homomorphically closed*, if $B \lhd A \in \mathbf{M}$ and $B \in \mathbf{A}^0$ imply $A/B \in \mathbf{M}$.

Radical and *semisimple classes* are meant in the sense of Kurosh and Amitsur and for details of radical theory we refer to the books of Andrunakievich and Rjabuhin (1979) and Szász (1975). We recall that the upper radical operator $\mathcal{U}$ is defined by

$$\mathcal{U}\mathbf{M} = \{A\colon A \text{ has no nonzero homomorphic image in } \mathbf{M}\}$$

and the semisimple operator $\mathcal{S}$ by

$$\mathcal{S}\mathbf{R} = \{A\colon A \text{ has no nonzero ideal in } \mathbf{R}\}.$$

If $\mathbf{M}$ is a regular class, then $\mathcal{U}\mathbf{M}$ is a radical class and if $\mathbf{R}$ is a radical class, then $\mathcal{S}\mathbf{R}$ is a semisimple class. A semisimple class is always hereditary and subdirectly closed. Recent characterizations of semisimple classes were given by

Sands (1976), (Preprint), van Leeuwen, Roos and Wiegandt (1977) and Anderson and Wiegandt (Preprint a).

A radical class $\mathbf{R}$ is said to be *supernilpotent*, if $\mathbf{A}^0 \subseteq \mathbf{R}$ and $\mathbf{R}$ is hereditary.

PROPOSITION 1 (see Proposition 1 of Anderson and Wiegandt (Preprint b)). *If $B \lhd A$, then either $B \lhd \cdot A$ or $B \cong (B + C)/C \lhd \cdot A/C$ where C denotes an ideal of A being maximal relative to the property $B \cap C = 0$.*

A ring A is called *semiprime*, if A does not contain an ideal $B \neq 0$ in $\mathbf{A}^0$. Clearly, any class of semiprime rings is weakly homomorphically closed. A prime ring is always semiprime. The class of all prime rings will be denoted by $\mathbf{P}$.

PROPOSITION 2. *The class $\mathbf{P}$ is essentially closed.*

PROOF. Let B be an essential ideal of a ring A such that $B \in \mathbf{P}$, and let us consider two ideals C and D of A such that $CD = 0$. Now we have

$$(B \cap C)(B \cap D) \subset CD = 0$$

and since B is prime, it follows $B \cap C = 0$ or $B \cap D = 0$. Taking into consideration that B is essential in A, we get $C = 0$ or $D = 0$ proving that A is a prime ring.

For an ideal B of A the *annihilator* B^* and the right annihilator B^r are defined as

$$B^* = \{x \in A : xB + Bx = 0\}$$

and

$$B^r = \{x \in A : Bx = 0\}.$$

PROPOSITION 3. *$B^* \lhd A$ and $B^r \lhd A$ for every ideal B of a ring A. If $B \lhd A$ and B is a semiprime ring, then $B^r = B^*$ and $B \cap B^* = 0$ hold. The class of all semiprime rings and the class $\mathbf{P}$ of all prime rings are hereditary.*

These assertions are well-known for associative rings and were established for alternative rings by Slater (1968) with the exception of the last statement which can be proved easily by using the techniques of Slater (1968).

PROPOSITION 4 (see Proposition 2 of Anderson and Wiegandt (Preprint b)). *Let $\mathbf{M}$ be a class of semiprime rings. If $C \lhd B \lhd A$ and $B/C \in \mathbf{M}$, then $C \lhd A$.*

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PROPOSITION 5. *Let* $\mathbf{M}$ *be a class of semiprime rings. Further, let* $C \lhd B \lhd A$ *such that* $0 \neq B/C \in \mathbf{M}$. *If* D *denotes the set*

$$D = \{x \in A : xB + Bx \subseteq C\},$$

then $D \lhd A$ *and* $B \cap D = C$ *and the annihilator of* $(B + D)/D$ *in* A/D *is zero.*

This assertion is a part of Proposition 3 of Anderson and Wiegandt (Preprint b).

Let A be a ring and M an A-bimodule. The *split-null extension* (A, M) is defined as the ring on the cartesian product $A \times M$ with the operations

$$(a_1, m_1) + (a_2, m_2) = (a_1 + a_2, m_1 + m_2)$$

and

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2).$$

If A is associative or alternative, then so is (A, M).

PROPOSITION 6. *If* $aM = Ma = 0$ *implies* $a = 0$, *then* $(0, M)$ *is an essential ideal in* (A, M). *Moreover,* $(0, M) \in \mathbf{A}^0$ *and* $A \cong (A, M)/(0, M)$ *hold.*

PROOF. We show that the ideal $(0, M)$ is essential in (A, M). To this end let (a, m) be a nonzero element of an ideal I of (A, M). If $a = 0$, then

$$0 \neq (0, m) \in I \cap (0, M)$$

holds. If $a \neq 0$, then by the assumption there exists an element $m' \in M$ such that $am' \neq 0$ or $m'a \neq 0$. Suppose that $am' \neq 0$. Then

$$0 \neq (0, am') = (a, m)(0, m') \in I \cap (0, M)$$

is valid. Hence $(0, M) \lhd \cdot (A, M)$ holds. The further assertions are straightforward.

Let A and B be abelian groups and $H = \mathrm{Hom}(A, B)$. On the cartesian product $A \times H \times B$ an addition and a multiplication can be defined by

$$(a_1, k_1, b_1) + (a_2, k_2, b_2) = (a_1 + a_2, k_1 + k_2, b_1 + b_2)$$

and

$$(a_1, k_1, b_1)(a_2, k_2, b_2) = (0, 0, a_1 k_2 + a_2 k_1).$$

Thus we get an algebraic structure $K = (A, H, B)$. Obviously K is an abelian group with respect to the addition. One can easily check the distributivity, further, the multiplication is commutative and in view of $(K^2)K = K(K^2) = 0$, it is also associative.

PROPOSITION 7. *K is a commutative associative ring. If for every nonzero element $\alpha \in A$ there is an element $k \in H$ such that $ak \neq 0$, then $B^0 = (0, 0, B)$ is an essential ideal in K. Moreover, $(0, H, B) \in \mathbf{A}^0$ and $A^0 \cong K/(0, H, B)$ hold.*

PROOF. To prove that B^0 is an essential ideal in K, let us consider an arbitrary element $(a, k, b) \neq 0$ of an ideal I of K. If $(a, k, b) \in B^0$, then $I \cap B^0 \neq 0$. If $k \neq 0$, then there exists an element $a' \in A$ such that $a'k \neq 0$ and therefore

$$0 \neq (0, 0, a'k) = (a, k, b)(a', 0, 0) \in (a, k, b)K \subseteq I \cap B^0.$$

If $a \neq 0$, then by the hypothesis there is a $k' \in H$ with $ak' \neq 0$. Hence we have

$$0 \neq (0, 0, ak') = (a, k, b)(0, k', 0) \in (a, k, b)K \subseteq I \cap B^0.$$

Thus $I \neq 0$ implies $I \cap B^0 \neq 0$ proving that $B^0 \lhd \cdot K$.

The other assertions are straightforward.

2. On weakly homomorphically closed classes

Without assuming that $\mathbf{M}$ is a regular class we can prove

PROPOSITION 8. *Let $\mathbf{M}$ be an essentially closed and weakly homomorphically closed class of rings. If $\mathbf{M} \cap \mathbf{A}^0 \neq \mathbf{O}$ then $\mathbf{A}^0 \subseteq \mathbf{M}$.*

PROOF. Let A be a ring such that $0 \neq A \in \mathbf{M} \cap \mathbf{A}^0$. As is well-known, the abelian group A^+ is an essential subgroup of its injective envelope $E(A^+)$ and so $A \lhd \cdot E(A^+)^0$ holds. Since $A \in \mathbf{M}$ and $\mathbf{M}$ is essentially closed we conclude that $E(A^+)^0 \in \mathbf{M} \cap \mathbf{A}^0$. In addition $E(A^+)^0$ is a divisible abelian group, hence $E(A^+)^0$ is a direct sum of copies of the additive group R^+ of the rationals and of the quasi-cyclic groups $Z(p^\infty)$. Since $\mathbf{M}$ is weakly homomorphically closed, the class $\mathbf{M} \cap \mathbf{A}^0$ is homomorphically closed implying $R^0 \in \mathbf{M}$ or $Z(p^\infty)^0 \in \mathbf{M}$.

Let F be an arbitrary free abelian group. Obviously $\mathrm{Hom}(F, R^+)$ and $\mathrm{Hom}(F, Z(p^\infty))$ satisfy the assumption of Proposition 7. Hence Proposition 7 yields that

$$R^0 \cong (0, 0, R^+) \lhd \cdot K_1 = (F, \mathrm{Hom}(F, R^+), R^+)$$

and

$$Z(p^\infty)^0 \cong (0, 0, Z(p^\infty)) \lhd \cdot K_2 = (F, \mathrm{Hom}(F, Z(p^\infty)), Z(p^\infty)).$$

Moreover

$$F^0 \cong K_1/ (0, \mathrm{Hom}(F, R^+), R^+)$$

and

$$F^0 \cong K_2/\,(0, \operatorname{Hom}(F, Z(p^\infty)), Z(p^\infty))$$

hold. Taking into account that $\mathbf{M}$ is essentially closed, we get $K_1 \in \mathbf{M}$ or $K_2 \in \mathbf{M}$. Since $\mathbf{M}$ is weakly homomorphically closed, it follows $F^0 \in \mathbf{M}$. Thus the class $\mathbf{M} \cap \mathbf{A}^0$ contains the rings F^0 over all free abelian groups F. Since $\mathbf{M} \cap \mathbf{A}^0$ is homomorphically closed, it follows $\mathbf{A}^0 \subseteq \mathbf{M}$.

The next result generalizes Theorem 2 of Anderson Wiegandt (1979) where it was assumed that the considered class was a semisimple class.

THEOREM 1. *If* $\mathbf{M}$ *is an essentially closed and weakly homomorphically closed class of rings such that* $\mathbf{M} \cap \mathbf{A}^0 \neq \mathbf{O}$, *then* $\mathbf{M}$ *is the class of all rings.*

PROOF. Let $A \neq 0$ be an arbitrary ring and let Z denote the ring of integers. Further, let us consider the Dorroh extension B of A and the split-null extension (A, B). Since $1 \in B$, we have $a \in aB + Ba$ for every $a \in A$. Thus Proposition 6 is applicable and so $(0, B)$ is an essential ideal in (A, B), moreover, $(0, B) \in \mathbf{A}^0$ holds. In view of Proposition 8 it follows that $(0, B) \in \mathbf{M}$ and since $\mathbf{M}$ is essentially closed, we get $(A, B) \in \mathbf{M}$. Taking into account that $\mathbf{M}$ is weakly homomorphically closed, by Proposition 6 it follows

$$A \cong (A, B)/\,(0, B) \in \mathbf{M}.$$

COROLLARY. *A regular essentially closed and weakly homomorphically closed class* $\mathbf{M}$ *of rings either consists of semiprime rings, or coincides with the class of all rings.*

PROOF. If $\mathbf{M}$ is not the class of all rings, then by Theorem 1 it follows that $\mathbf{M} \cap \mathbf{A}^0 = \mathbf{O}$. Suppose that $\mathbf{M}$ contains a ring A which is not semiprime. Then A has a nonzero ideal in $\mathbf{A}^0$. This, however, contradicts the regularity of $\mathbf{M}$.

3. Classes defining supernilpotent radicals

It has been first noted by Armendariz (1968) that a radical is hereditary if and only if its semisimple class is essentially closed. The following criterion for a radical to be hereditary, seems to be useful.

PROPOSITION 9. *A radical class* $\mathbf{R}$ *is hereditary if and only if it satisfies condition*

$$\text{(H)} \quad B \lhd \cdot A \in \mathbf{R} \text{ implies } B \in \mathbf{R}.$$

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PROOF. The necessity is trivial. The sufficiency follows by a straightforward application of Proposition 1.

We say that a radical $\mathbf{R}$ has the *intersection property relative to the class* $\mathbf{M}$ if

$$\mathbf{R}(A) = \bigcap_{\alpha} (I_\alpha \lhd A : A/I_\alpha \in \mathbf{M})$$

holds for every ring A.

An essentially closed class $\mathbf{M}$ of rings satisfies trivially condition

(E) $0 \neq B \lhd \cdot A$ and $B \in \mathbf{M}$ imply $A \notin \mathfrak{U}\mathbf{M}$.

We shall use the following

PROPOSITION 10. *Let* $\mathbf{M}$ *be a regular class of semiprime rings. The following conditions are equivalent.*

(i) *The radical* $\mathfrak{U}\mathbf{M}$ *is hereditary,*

(ii) $\mathbf{M}_k \subseteq \mathbb{S}\mathfrak{U}\mathbf{M}$,

(iii) $\mathfrak{U}\mathbf{M} = \mathfrak{U}\mathbf{M}_k$,

(iv) $\mathfrak{U}\mathbf{M}$ *has the intersection property relative to the class* $\mathbf{M}_k$,

(v) $\mathfrak{U}\mathbf{M} \cap \mathbf{M}_k = \mathbf{O}$,

(vi) $\mathbf{M}$ *satisfies condition* (E).

In particular, if $\mathfrak{U}\mathbf{M}$ *is hereditary, then every ring of the semisimple class* $\mathbb{S}\mathfrak{U}\mathbf{M}$ *is a subdirect sum of* $\mathbf{M}_k$*-rings.*

PROOF. For associative rings the equivalence of (i)–(v) can be found in Theorem 7 of Le Roux, Heyman and Jenkins (Preprint). The crucial point is the proof of the implication (iii) $\to$ (iv) and for alternative rings one can prove it similarly as in Theorem 2 of Anderson and Wiegandt (Preprint b). The equivalence of (v) and (vi) is straightforward.

REMARK. For associative rings the essential cover of a hereditary class of semiprime rings is essentially closed (see Theorem 4 of Heyman and Roos (1977)), but in the case of a regular class of semiprime rings it need not be so (see Watters (Preprint)). For a hereditary class of semiprime rings the equivalence of (i)–(v) was first shown by Heyman and Roos (1977) Theorem 7.

The next Theorem characterizes the regular classes determing supernilpotent radicals.

THEOREM 2. *Let* $\mathbf{M}$ *be a regular class of rings. The radical class* $\mathbf{R} = \mathfrak{U}\mathbf{M}$ *is supernilpotent if and only if* $\mathbf{M}$ *consists of semiprime rings and satisfies condition* (E).

PROOF. Let $\mathbf{R} = \mathcal{U}\mathbf{M}$ be supernilpotent. Then $\mathbf{M}$ consists of semiprime rings, moreover, by Proposition 10 $\mathbf{M}$ satisfies condition (E).

Assume that the class $\mathbf{M}$ consists of semiprime rings and satisfies condition (E). Obviously $\mathbf{A}^0 \subseteq \mathcal{U}\mathbf{M}$ holds and again by Proposition 10 $\mathbf{R} = \mathcal{U}\mathbf{M}$ is hereditary.

REMARK. Let us remind the reader that a class $\mathbf{M}$ of rings is called a *weakly special class*, if $\mathbf{M}$ is hereditary, consists of semiprime rings and satisfies condition

(A) $B \lhd A$ and $B \in \mathbf{M}$ imply $A/B^* \in \mathbf{M}$ where B^* denotes the annihilator of B in A.

As it was proved by Rjabuhin (1965), the upper radical of a weakly special class is supernilpotent; further the semisimple class of a supernilpotent radical is weakly special (see also Andrunakievich and Rjabuhin (1979) and Szász (1975)). If $\mathbf{M}$ consists of semiprime rings then condition (A) is equivalent to the assumption that $\mathbf{M}$ is essentially closed (see Heyman and Roos (1977) and also Anderson and Wiegandt (Preprint b)). Though by Proposition 10 in Theorem 2 condition (E) can be replaced by demanding $\mathbf{M}_k \subseteq \mathcal{S}\mathcal{U}\mathbf{M}$, the class $\mathbf{M}$ need not be essentially closed. For instance, let Z denote the ring of integers and let $\mathbf{P}^*$ denote the class of all prime rings but Z. The class $\mathbf{P}^*$ is hereditary, for $1 \in Z$ and therefore Z cannot be an ideal of a prime ring. Z is a subdirect sum of the prime fields $Z/(p) \in \mathbf{P}^*$ (p ranges through the primes) hence $Z \in \mathcal{S}\mathcal{U}\mathbf{P}^*$ holds. The essential cover of $\mathbf{P}^*$ is $\mathbf{P}$, because $\mathbf{P}$ is essentially closed and $(p) \in \mathbf{P}^*$, $(p) \lhd \cdot Z$ are valid. Thus from $\mathbf{P}^* \subset \mathbf{P} \subset \mathcal{S}\mathcal{U}\mathbf{P}^*$ it follows $\mathcal{U}\mathbf{P}^* \subseteq \mathcal{U}\mathbf{P} \subseteq \mathcal{U}\mathbf{P}^*$. Nevertheless the class $\mathbf{P}^*$ is not essentially closed because $Z \notin \mathbf{P}^*$, but it satisfies condition (E) in view of Theorem 2. Thus there are hereditary but not essentially closed classes $\mathbf{M}$ of semiprime rings such that $\mathcal{U}\mathbf{M}$ is supernilpotent.

4. Classes defining special radicals

Let us recall that a class $\mathbf{M}$ of rings is said to be a *special class*, if $\mathbf{M}$ is a weakly special class consisting of prime rings. A *special radical* is defined as the upper radical of a special class of rings. In this last section our aim is to characterize the semisimple classes of special radicals and to this end we introduce the notion of subdirectly indecomposable rings. A ring A is said to be *subdirectly indecomposable*, if A cannot be decomposed as a nontrivial subdirect sum of two rings, that is, for any two ideals B and C of A the relation $B \cap C = 0$ implies $B = 0$ or $C = 0$. Clearly every subdirectly irreducible ring is subdirectly indecomposable but not conversely.

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PROPOSITION 11. *A subdirectly indecomposable semiprime ring is a prime ring. Conversely, every prime ring is subdirectly indecomposable.*

PROOF. Let B and C be ideals of a subdirectly indecomposable semiprime ring A such that $BC = 0$. Now C is contained in the right annihilator B^r of B in A. Since being semiprime is a hereditary property by Proposition 3, also B is a semiprime ring and therefore $B^r = B^*$ holds in view of Proposition 3. Moreover,

$$B \cap C \subseteq B \cap B^r = B \cap B^* = 0$$

holds for B is semiprime. Since A is subdirectly indecomposable, we get $B = 0$ or $C = 0$ proving that A is a prime ring.

The converse statement is straightforward.

PROPOSITION 12. *Let* $\mathbf{M}$ *be a regular class of rings. If the upper radical* $\mathbf{R} = \mathfrak{U}\mathbf{M}$ *is special, then* $\mathbf{M}$ *consists of semiprime rings and satisfies condition* (E) *and the following condition*

(Q) *Every* $\mathfrak{S}\mathfrak{U}\mathbf{M}$*-ring is a subdirect sum of prime rings of the class* $\mathfrak{S}\mathfrak{U}\mathbf{M}$.

PROOF. Since $\mathbf{R}$ is supernilpotent, by Theorem 2 the class consists of semiprime rings and satisfies (E). Moreover, the semisimple class $\mathfrak{S}\mathbf{R}$ is hereditary and by Armendariz (1968) also essentially closed. Hence by Propositions 2 and 3 the class $\mathbf{P} \cap \mathfrak{S}\mathbf{R}$ is hereditary, essentially closed and consists of prime rings. In view of Heyman and Roos (1977) (see also Anderson and Wiegandt (Preprint b)) $\mathbf{P} \cap \mathfrak{S}\mathbf{R}$ satisfies condition (A), so $\mathbf{P} \cap \mathfrak{S}\mathbf{R}$ is a special class. Since $\mathbf{R}$ is a special radical, it is the upper radical of a special class $\mathbf{Q}$ and in addition $\mathbf{Q} \subseteq \mathbf{P} \cap \mathfrak{S}\mathbf{R} \subseteq \mathfrak{S}\mathbf{R}$ holds. Thus we get

$$\mathbf{R} \supseteq \mathfrak{U}\mathbf{Q} \supseteq \mathfrak{U}(\mathbf{P} \cap \mathfrak{S}\mathbf{R}) \supseteq \mathbf{R}$$

and Proposition 10 is applicable yielding the validity of condition (Q).

PROPOSITION 13. *Let* $\mathbf{M}$ *be a regular class of semiprime rings satisfying condition* (E). *Then the upper radical* $\mathbf{R} = \mathfrak{U}\mathbf{M}$ *is special, provided that* $\mathbf{P} \cap \mathfrak{S}\mathbf{R} \subseteq \mathbf{M}_k$ *holds and the following condition* (R) *is satisfied.*

(R) *Every* $\mathbf{M}$*-ring is a subdirect sum of prime* $\mathbf{M}_k$*-rings.*

PROOF. Theorem 2 yields that $\mathbf{R}$ is supernilpotent. As we have seen in the proof of Proposition 13, the class $\mathbf{P} \cap \mathfrak{S}\mathbf{R}$ is a special class satisfying $\mathbf{P} \cap \mathfrak{S}\mathbf{R} \subseteq \mathbf{M}_k$. Thus in view of Proposition 10 (iii) we have

$$\mathbf{R} = \mathfrak{U}\mathbf{M} = \mathfrak{U}\mathbf{M}_k \subseteq \mathfrak{U}(\mathbf{P} \cap \mathfrak{S}\mathbf{R}).$$

Assume that there is a ring A which is in $\mathfrak{U}(\mathbf{P} \cap \mathfrak{S}\mathbf{R})$ but not in $\mathbf{R}$. Now A can be mapped homomorphically onto a nonzero $\mathbf{M}$-ring B and by condition (R) the

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ring B can be mapped homomorphically onto a nonzero prime ring C being in $\mathbf{M}_k$. Hence by Proposition 10 we get that $C \in \mathbf{P} \cap \mathbf{M}_k \subseteq \mathbf{P} \cap \mathfrak{S}\mathbf{R}$. Thus A can be mapped homomorphically onto a nonzero ring in $\mathbf{P} \cap \mathfrak{S}\mathbf{R}$ contradicting the hypothesis that $A \in \mathfrak{U}(\mathbf{P} \cap \mathfrak{S}\mathbf{R})$. Hence $\mathbf{R} = \mathfrak{U}(\mathbf{P} \cap \mathfrak{S}\mathbf{R})$ holds proving that $\mathbf{R}$ is a special radical.

Propositions 12 and 13 yield immediately

THEOREM 3. *Let* $\mathbf{M}$ *be a regular and essentially closed class of rings such that* $\mathbf{P} \cap \mathfrak{S}\mathfrak{U}\mathbf{M} \subseteq \mathbf{M}$. *The upper radical* $\mathbf{R} = \mathfrak{U}\mathbf{M}$ *is special if and only if* $\mathbf{M}$ *satisfies the following condition*

(S) *every* $\mathbf{M}$-*ring is a subdirect sum of prime* $\mathbf{M}$-*rings.*

Now we can easily get the following characterizations of semisimple classes of special radicals.

THEOREM 14. *The following two conditions are equivalent for a class* $\mathbf{S}$ *of rings*:
(i) $\mathbf{S}$ *is the semisimple class of a special radical*;
(ii) $\mathbf{S}$ *is regular, subdirectly closed, essentially closed, and satisfies condition* (S).

PROOF. (i) implies (ii) trivially by Theorem 3.

(ii) $\to$ (i) By Corollary 2 of Anderson and Wiegandt (Preprint a) S is the semisimple class of a hereditary radical, if S is regular, subdirectly closed and essentially closed. Taking into account that S is regular, condition (S) implies that S is a class of semiprime rings, moreover S satisfies trivially $P \cap S \subseteq S$. Thus by Theorem 3 the upper radical of S is special.

Finally we show that in Theorem 4 the condition that $\mathbf{S}$ is subdirectly closed cannot be weakened. A class $\mathbf{M}$ is said to be *coinductive*, if for any descending chain $I_1 \supseteq \cdots \supseteq I_\alpha \supseteq \ldots$ of ideals of a ring A satisfying $A/I_\alpha \in \mathbf{M}$ for each α, also the factor ring $A/\bigcap I_\alpha$ is in $\mathbf{M}$.

PROPOSITION 14. *The class* $\mathbf{P}$ *of all prime rings is coinductive.*

PROOF. Let $I_1 \supseteq \cdots \supseteq I_\alpha \supseteq \ldots$ be a descending chain of ideals of a ring A such that $A/I_\alpha \in \mathbf{M}$ for every α. Without loss of generality we may assume that $\bigcap I_\alpha = 0$. Let B_1 and B_2 be ideals of A such that $B_1 B_2 = 0$. For every α and for $i = 1, 2$ we have

$$B_i/(B_i \cap I_\alpha) \cong (B_i + I_\alpha)/I_\alpha \lhd A/I_\alpha \in \mathbf{P},$$

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and also

$$(B_1 + I_\alpha)(B_2 + I\alpha) \subseteq B_1B_2 + I_\alpha = I_\alpha.$$

Since every I_α is a prime ideal of A, we get $B_i \subseteq I_\alpha$ for all α and for $i = 1$ or $i = 2$. If $B_1 \neq 0$, then there is an index α_0 such that $B_1 \subseteq I_\beta$ for every $\beta > \alpha_0$. Hence $B_2 \subseteq I_\beta$ for every $\beta \geqslant \alpha_0$ which implies $B_2 = 0$. Thus A is a prime ring and $\mathbf{P}$ is a coinductive class.

Let $\mathbf{R}$ be a special radical. Then the class $\mathbf{P} \cap \mathfrak{S}\mathbf{R}$ is a special class which satisfies trivially condition (S). Moreover, by Proposition 17 the class $\mathbf{P} \cap \mathfrak{S}\mathbf{R}$ is coinductive (because $\mathfrak{S}\mathbf{R}$ is always coinductive). Nevertheless $\mathbf{P} \cap \mathfrak{S}\mathbf{R}$ is not subdirectly closed and hence not a semisimple class.

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