COMMUTATION PROPERTIES OF OPERATOR POLYNOMIALS

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Suppose A and B are continuous linear operators mapping a complex Banach space X into itself. For any polynomial p over C, it is obvious that when A commutes with B, then $p(A)$ commutes with B. To see that the reverse implication is false, let A be nilpotent of order n. Then A^n commutes with all B but A cannot do so. Sufficient conditions for the implication: $p(A)$ commutes with B implies A commutes with B: were given by Embry [2] for the case $p(\lambda) = \lambda^n$ and Finkelstein and Lebow [3] in the general case. The latter authors proved in fact that if f is a function holomorphic on $\sigma(A)$ and if f is univalent with non-vanishing derivative on $\sigma(A)$, then A can be expressed as a function of $f(A)$.

In this paper, similar questions are studied when A and B are closed operators with domain and range in X. Immediately the question of the definition of commutativity arises. Several definitions appear in the literature. A well-known approach is

C_1 : *B commutes with A iff* $D(B)$, *the domain of B is all of X and*

$$AB \text{ is an extension of } BA.$$

See, for example, [5].

More recently, Marti [4] used the condition:

C_2 : *B commutes with A iff* $D(A) \subseteq D(B)$, $BD(A) \subseteq D(A)$ *and*

$$ABx = BAx$$

for all $x \in D(BA)$.

It is a simple exercise to show that C_1 implies C_2. Both C_1 and C_2 suffer from an evident lack of symmetry. A symmetrical definition appears in [1]:

C_3 : *B commutes with A iff* $D(A) \cap D(AB) = D(B) \cap D(BA)$ *and*

$$ABx = BAx$$

for all $x \in D(AB) \cap D(BA)$.

Again, it is straightforward to verify that C_2 implies C_3. Moreover, if $D(B) = X$, then C_3 implies C_1. If A and B are closed operators with non empty resolvent sets, then from [1], we know that C_3 is a necessary and sufficient condition for the commutativity of the resolvent operators.

In that which follows, we obtain a sufficient condition that the C_3-commutativity of $p(A)$ with B should imply the C_3-commutativity of A with B when A and B are closed operators with non empty resolvent sets. Suppose that p is a monic polynomial of degree n and let $\lambda_0 \in \rho(A)$. If $\mu_1, \mu_2, \cdots, \mu_n$ denote the roots of $p(\mu) = p(\lambda_0)$ with $\mu_1 = \lambda_0$ then, since $\rho(A)$ is an open set we can assume without loss of generality that $p'(\mu_k) \neq 0$ for $k = 1, 2, \cdots, n$ and that the μ_k are distinct. In these terms we can state

THEOREM. *Suppose that $p(A)$ commutes with B in the C_3 sense. Suppose also that for some $\lambda_0 \in \rho(A)$ we have*

$$\sum_{k=1}^{n} \frac{1}{p'(\mu_k)(\lambda_1-\mu_k)(\lambda_2-\mu_k)} \neq 0 \tag{1}$$

for all $\lambda_1, \lambda_2 \in \sigma(A)$. Then A commutes with B in the C_3 sense.

PROOF. Since $p(\mu)-p(\lambda_0) = \prod_{k=1}^{n}(\mu-\mu_k)$ and the μ_k are distinct, we can write $[p(\mu)-p(\lambda_0)]^{-1} = \sum_{k=1}^{n} a_k(\mu-\mu_k)^{-1}$ and hence

$$a_k = \lim_{\mu\to\mu_k} \frac{\mu-\mu_k}{p(\mu)-p(\lambda_0)} = \lim_{\mu\to\mu_k} \frac{\mu-\mu_k}{p(\mu)-p(\mu_k)} = \frac{1}{p'(\mu_k)}.$$

Moreover

$$\begin{aligned}(A-\mu_k)^{-1} &= [(A-\mu_1)[I+(A-\mu_1)^{-1}(\mu_1-\mu_k)]]^{-1}\\ &= [I+(A-\lambda_0)^{-1}(\mu_1-\mu_k)]^{-1}(A-\lambda_0)^{-1},\end{aligned}$$

so that

$$[p(A)-p(\lambda_0)]^{-1} = \sum_{k=1}^{n} \frac{[I+(A-\lambda_0)^{-1}(\mu_1-\mu_k)]^{-1}(A-\lambda_0)^{-1}}{p'(\mu_k)}.$$

If we define

$$f(\lambda) = \sum_{k=1}^{n} \frac{\lambda}{p'(\mu_k)[1-(\mu_1-\mu_k)\lambda]}$$

then $[p(A)-p(\lambda_0)]^{-1} = f[(A-\lambda_0)^{-1}]$. If f fulfils the requirements of the result of Finkelstein and Lebow, then we can conclude that $(A-\lambda_0)^{-1}$ is a function of $[p(A)-p(\lambda_0)]^{-1}$ and hence the result follows.

Consider now the properties of f. Evidently f is analytic except when $\lambda = (\mu_k-\lambda_0)^{-1}$. Now since $p(A)-p(\lambda_0) = \prod_{k=1}^{n} (A-\mu_k)$ it is evident that all μ_k belong to $\rho(A)$. Hence $(\mu_k-\lambda_0)^{-1} \in \rho[(A-\lambda_0)^{-1}]$ so that f is analytic on $\sigma[(A-\lambda_0)^{-1}]$. It remains to show that the restriction of f to $\sigma[(A-\lambda_0)^{-1}]$ is univalent with non-vanishing derivative. Straightforward calculations show that this requirement is precisely the assumed property (1). For example, if $\theta_1, \theta_2 \in \sigma[(A-\lambda_0)^{-1}]$ then there exists $\lambda_1, \lambda_2 \in \sigma(A)$ such that $(\lambda_i-\lambda_0)^{-1} = \theta_i$, $i = 1, 2$. Suppose $\theta_1 \neq \theta_2$ but $f(\theta_1) = f(\theta_2)$; then

$$\sum_{k=1}^{n}\left\{(\lambda_1-\lambda_0)p'(\mu_k)\left[1+\frac{\mu_1-\mu_k}{\lambda_1-\lambda_0}\right]\right\}^{-1}=\sum_{k=1}^{n}\left\{(\lambda_2-\lambda_0)p'(\mu_k)\left[1+\frac{\mu_1-\mu_k}{\lambda_1-\lambda_0}\right]\right\}^{-1}$$

which reduces to

$$\sum_{k=1}^{n}[p'(\mu_k)(\lambda_1-\mu_k)]^{-1}=\sum_{k=1}^{n}[p'(\mu_k)(\lambda_2-\mu_k)]^{-1}\quad\text{i.e.}$$

$$\sum_{k=1}^{n}[p'(\mu_k)(\lambda_1-\mu_k)(\lambda_2-\mu_k)]^{-1}=0. \tag{2}$$

Since this contradicts (1), we know that f is univalent on $\sigma[(A-\lambda_0)^{-1}]$. In a similar way, the assumption that $f'(\theta_1)=0$ leads to equation (1) with $\lambda_1=\lambda_2$. This concludes the proof.

REMARK. The relation of the result of our theorem and the results of [2] and [3] seems obscure. Even when A and B are in $B(X)$ and $p(\lambda)=\lambda^n$, (1) reduces to

$$\sum_{k=1}^{n}\frac{\omega^k}{(\omega^k\lambda_1-\lambda_0)(\omega^k\lambda_2-\lambda_0)}\neq 0\qquad\text{for } \lambda_1,\lambda_2\in\sigma(A) \tag{3}$$

where $\omega=\exp(2i\pi/n)$. It is not obvious that this condition is related in any simple way to that of [2]: $\sigma(A)\cap\sigma(\omega^kA)=\phi$, $k=2,3,\cdots,n$. However when $n=2$, (3) reduces to $\lambda_0(\lambda_1+\lambda_2)\neq 0$ so that (1) is equivalent to the condition of [2].

COROLLARY. *If $\sigma(A)=\phi$ and $p(A)$ commutes with B in the C_3 sense, then A commutes with B in the C_3 sense.*

References

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