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A Fast Explicit Scheme for Solving MHD Equations with Ambipolar Diffusion

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Abstract. We developed a fast numerical scheme for solving ambipolar diffusion MHD equations with the strong coupling approximation, which can be written as the ideal MHD equations with an additional ambipolar diffusion term in the induction equation. The mass, momentum, magnetic fluxes due to the ideal MHD equations can be easily calculated by any Godunov-type schemes. Additional magnetic fluxes due to the ambipolar diffusion term are added in the magnetic fluxes, because of two same spatial gradients operated on the advection fluxes and the ambipolar diffusion term. In this way, we easily kept divergence-free magnetic fields using the constraint transport scheme. In order to overcome a small time step imposed by ambipolar diffusion, we used the super time stepping method. The resultant scheme is fast and robust enough to do the long term evolution of star formation simulations. We also proposed that the decay of alfv́en by ambipolar diffusion be a good test problem for our codes.

Keywords. methods:numerical, MHD, stars: formation

1. Introduction

Ambipolar diffusion (AD), which arises in partially ionized plasmas, causes the relative drift of ions coupled to magnetic fields with respect to neutrals. AD enables molecular cloud cores to collapse gravitationally, so is one of important processes for star formation (e.g., Mestel & Spitzer 1956; Mouschovias 1987; Shu *et al.* 1987).

Several numerical methods have been proposed in the studies of the dynamics of partially ionized plasmas within the frame of single or two fluid formulations. An incomplete list of them is Tóth (1994), Mac Low *et al.* (1995), Mac Low & Smith (1997), Stone (1997), Li *et al.* (2006), and Tilley & Balsara (2008). Implicit schemes for the multifluid treatment of the Hall term and ambipolar diffusion have also been suggested by Falle (2003) and O'Sullivan & Downes (2006, 2007). In this work, we describe a fully explicit method for incorporating ambipolar diffusion with the strong coupling approximation into a multidimensional MHD code based on the total variation diminishing scheme. The divergence-free condition of magnetic fields is ensured by a flux-interpolated constrained transport scheme, and a super time stepping method is used in order to considerably accelerate the otherwise painfully short diffusion-driven time steps. More detailed information on this work can be found in Choi, Kim, & Wita (2009).

2. AD MHD Equations and Numerical Methods

The isothermal MHD equations including ambipolar diffusion with the strong coupling approximation can be written as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0, \tag{2.1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + \frac{a^2}{\rho}\boldsymbol{\nabla}\rho - \frac{1}{\rho}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} = 0, \tag{2.2}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}) = \boldsymbol{\nabla}\times\left\{\left[\frac{1}{\gamma\rho_i\rho}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}\right]\times\boldsymbol{B}\right\}, \tag{2.3}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \tag{2.4}$$

where a is an isothermal sound speed, γ is the collisional coupling constant between ions and neutrals, and ρ_i is the ion density. The other variables ρ, $\boldsymbol{v}$, and $\boldsymbol{B}$ denote neutral density, neutral velocity, and magnetic field, respectively. We further assume that the ion density is constant in this work.

The above equations except the term in the right hand side of equation (2.3) are same as the ideal MHD equations. So our strategy of solving the above AD MHD equations is first to update the fluxes of ρ, $\boldsymbol{v}$, and $\boldsymbol{B}$ using any Godunov type schemes for solving the isothermal MHDs, then to make a correction to the fluxes of $\boldsymbol{B}$ due to AD with the divergence free condition. We take a total variation diminishing scheme in Kim *et al.* (1999) for the flux calculations due to the idea MHD.

Let's now look at the induction equation in details. The B_x component, for example, of the equation can be written as

$$\frac{\partial B_x}{\partial t} + \frac{\partial}{\partial y}(B_x v_y - B_y v_x) - \frac{\partial}{\partial z}(B_z v_x - B_x v_z) = \frac{\partial S_z}{\partial y} - \frac{\partial S_y}{\partial z}, \tag{2.5}$$

where S_y and S_z are defined by

$$\begin{aligned} S_y &= \frac{1}{\gamma\rho_i\rho}\left[\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)B_y B_x + \left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right)(B_x^2 + B_z^2)\right. \\ &\quad \left. + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)B_y B_z\right], \end{aligned} \tag{2.6}$$

$$\begin{aligned} S_z &= \frac{1}{\gamma\rho_i\rho}\left[\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)B_z B_y + \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right)(B_y^2 + B_x^2)\right. \\ &\quad \left. + \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)B_z B_x\right]. \end{aligned} \tag{2.7}$$

Note that the first and second terms on the right-hand side of equation (2.5) have the same gradients as the second and third terms do on the left-hand side, respectively. Applying the second-order finite difference operators to equations (2.6) and (2.7), we calculate their values at every grid center, $S_{y,i,j,k}$, and $S_{z,i,j,k}$. And their values at every face center can be simply calculated as the half of the sum of two nearby grid centered values. Then we can define a new flux for B_x along y- and z-directions at face centers as

$$f^{(5)}_{y,i,j+1/2,k} = \bar{f}^{(5)}_{y,i,j+1/2,k} - \frac{1}{2}(S_{z,i,j,k} + S_{z,i,j+1,k}), \tag{2.8}$$

$$f^{(5)}_{z,i,j,k+1/2} = \bar{f}^{(5)}_{z,i,j,k+1/2} + \frac{1}{2}(S_{y,i,j,k} + S_{y,i,j,k+1}), \tag{2.9}$$

respectively, where the first term in right hand side is the conventional flux due to the advection and the second term is due to ambipolar diffusion. Similarly, we can define $f^{(6)}_{x,i+1/2,j,k}$, $f^{(6)}_{z,i,j,k+1/2}$ $f^{(7)}_{x,i+1/2,j,k}$, and $f^{(7)}_{y,i,j+1/2,k}$, where the first two are fluxes for B_y

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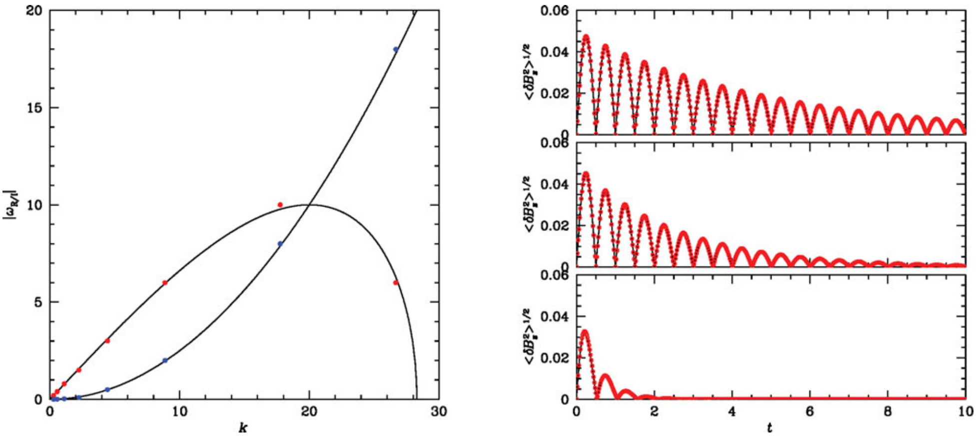

Figure 1. Left panel: A dispersion relation of the Alfv́en waves in a partially ionized medium. The solid lines represent the real and imaginary (monotonically increasing one) parts of the complex angular frequency. Circles are measurements from numerical experiments. Right panel: Time evolution of the root-mean-square of the B_z component for the cases of $\gamma\rho_i = 100$ (top), 50 (middle), 10 (bottom). Solid lines are from theoretical predictions and open circles are from numerical simulations.

along x- and z-directions, and the latter two are fluxes for B_z along x- and y-directions. These combined fluxes at face centers, $f^{(n)}_{x,i+1/2,j,k}$, $f^{(n)}_{y,i,j+1/2,k}$, and $f^{(n)}_{z,i,j,k+1/2}$, are used to enforce $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ as well as to update the magnetic field components to the next time step, as the advection only fluxes for the ideal MHD case do. In fact, we used the flux-interpolated CT (Constraint Transport) scheme developed in Balsara & Spicer(1999).

The time step for the ambipolar diffusion term is proportional to the square of the grid size, so the explicit treatment of ambipolar diffusion terms leads to very small time steps (Mac Low *et al.* 1995). In this work we adopt the “super time stepping” approach (Alexiades *et al.* 1996) to increase the effective time interval and allow much faster computations for ambipolar diffusion. O’Sullivan & Downes (2006,2007) also used this strategy in their multifluid MHD models. The super time stepping technique considerably accelerates the explicit schemes for parabolic problems (Alexiades *et al.* 1996). The key advantage of this approach is that it demands stability over large compound time steps, rather than over each of the constituent substeps. In addition to allowing larger effective time steps, the super time stepping approach offers relatively simple implementation. Readers who are interested in the technical details may look up Alexiades *et al.* (1996) and Choi *et al.*(2009).

3. A New Test Problem

Probably, the most popular test problem for AD or two fluid codes is oblique C (continuous) shocks, whose steady state solutions can be easily obtained (Mac Low *et al.* 1995). In fact, we also tested our code with this problem and presented its results in Figure 1 in Choi *et al.* (2009). One drawback of this test problem is the steady-state nature, which doesn’t enable us to check any states in between from an initial state to the final steady state.

We proposed a new test problem that is standing Alfv́en waves in a weakly ionized plasma. In the strong coupling approximation, a dispersion relation for the Alfv́en waves

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can be simply written as

$$\omega^2 - i\frac{c_A^2 k^2}{\gamma\rho_i}\omega - c_A^2 k^2 = 0, \tag{3.1}$$

where $\omega = \omega_R + i\omega_I$ is the complex angular frequency of a wave and k is a real wavenumber parallel to the direction of magnetic fields. The first and third terms give the well-known Alfv́en waves of the ideal MHDs, and the additional second term gives the damping of the Alfv́en waves by AD. The real part and imaginary parts of the angular frequency of the solution of equation (3.1) as a function of the wavenumber are plotted with two solid lines (The monotonically increasing one is for the imaginary part, which gives the damping rate of the Alfv́en waves.) in the left panel of Figure 1, where $\gamma\rho_i = 10$ and $c_A = 1/\sqrt{2}$. We setup up a standing Alfv́en wave in a computational domain and measured the period and damping rate of the wave, which correspond to the real and imaginary parts of ω, respectively. In the right panel of Figure 1, the root-mean-square values of the B_z component as a function of time are plotted for the cases of $\gamma\rho_i = 100$ (top), 50 (middle), 10 (bottom). As the couple of neutrals and ions becomes weaken, the damping of the Alfv́en waves becomes stronger. We did several experiments with different wavenumbers for the case of $\gamma\rho_i = 10$, measured the periods and decay rates, and put circles in the left panel of Figure 1. It shows good agreements between the numerical measurements and the theoretical prediction.

We also did decay and forced turbulence simulations with AD. The successful simulation results show the flexibility of our method as well as its ability to follow complex MHD flows in the presence of ambipolar diffusion. Readers who are interested in these results may look up Choi *et al.* (2009).

4. Conclusion

We described a method for incorporating ambipolar diffusion in the strong coupling approximation into a multidimensional magnetohydrodynamics code based on the total variation diminishing scheme. Contributions from ambipolar diffusion terms are included by explicit finite difference operators in a fully unsplit way, maintaining second order accuracy. The divergence-free condition of magnetic fields is exactly ensured at all times by a flux-interpolated constrained transport scheme. The super time stepping method is used to accelerate the timestep in high resolution calculations and/or in strong ambipolar diffusion. The test results of the decay of Alfvén waves in this paper and the steady-state oblique C-type shocks in Choi *et al.* (2009) showed the accuracy and robustness of our numerical approach.

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Maite Beltrán and Josep Miquel Girart at the banquet