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Beyond the Wilson action

The imposition of an ultraviolet cutoff is a highly non-unique procedure. Even in the framework of a lattice theory, innumerable variations are possible. Several decades of success with perturbative quantum electrodynamics had led to the lore that the removal of any regulator yields the unique renormalized theory depending only on a small number of physical couplings and masses. Indeed, renormalizability is often regarded as a primary constraint on models for fundamental interactions.

On a non-perturbative level, however, little is rigorously known about even the existence of any four-dimensional theory, let alone its uniqueness. In some cases the theory may depend on even less parameters than suggested in perturbative analysis; for example, as discussed in chapter 13, Yang–Mills theories should undergo dimensional transmutation with dimensionless ratios being determined independent of any coupling constants.

The Monte Carlo technique provides an opportunity for non-perturbative exploration of cutoff dependence. Thus we can begin numerically to address these questions of the uniqueness of the continuum limit. In this chapter we discuss some of the simple variations of the Wilson scheme from this viewpoint of universality.

A simple alternative to the Wilson model places a vector field A_μ on the lattice sites and uses an action obtained by replacing derivatives in the continuum Yang–Mills Lagrangian with nearest-neighbor differences. This would be naively similar to the procedure followed in chapter 4 for scalar fields. This differs from the conventional lattice gauge theory in two important respects. First, the cutoff theory no longer has an exact local symmetry. This should not matter if the gauge breaking terms go away sufficiently rapidly in the continuum limit, but will complicate the renormalization procedure. Second, the integral over gauges is no longer compact. The path integral will not be well-defined until gauge fixing is imposed. Because of its awkwardness, little work has been done with such a scheme, although Patrasciou, Seiler and Stametescu (1981) have done some preliminary Monte Carlo studies. They have not as yet seen the area law for large loops, but this is probably due to a renormalization of the bare charge making the linear potential appreciable only at extremely strong coupling.

Remaining closer in spirit to the Wilson formulation, Edgar (1982) considered replacing the plaquette with the two-by-one Wilson loop as the fundamental term in the action. In two space-time dimensions with the gauge group Z_2 this model is equivalent to the Ising model and therefore must have a phase transition, unlike the two-dimensional Wilson theory, which is trivial. The model possesses some extra global symmetries which can be broken; indeed, Edgar has seen a first-order phase transition in this '*fenêtre*' model with the gauge group $SU(2)$ in four dimensions. The moral of this is that the mere presence or absence of a phase transition is not a universal property of the gauge group. As we will see again later in this chapter, when the lattice spacing is not small, variations on the action can introduce new phenomena as lattice artifacts.

Drawing still closer to the Wilson theory, one can keep the action a class function of the group elements associated with the plaquettes, but change the detailed form of that function. We have already done that to some extent when we discussed duality and the Migdal–Kadanoff recursion relations, and we will pursue such generalizations further here. Manton (1980) presented a particularly simple alternative, taking for the action on a plaquette

$$S_\square(U) = \beta\, d^2(U, I), \tag{20.1}$$

where d is the minimal distance in the group manifold between the element U and the identity I. The concept of a distance in the group manifold is formulated in terms of the metric tensor briefly mentioned in chapter 8. This metric is unique up to an overall normalization. In the case of $SU(2)$ the distance is simply

$$d(U_1, U_2) \propto \arccos\left(\tfrac{1}{2}\operatorname{Tr}(U_1 U_2^{-1})\right). \tag{20.2}$$

The Manton action is convenient for analytic work in the weak coupling limit. It is, however, singular for those elements with maximum distance from the identity, such as $-I$ for $SU(2)$. An amusing technical consequence of this singularity is that the transfer matrix is never positive definite (Grosse and Kuhnelt, 1981).

Another generalization, similar in spirit but different in detail from that of Manton, is the 'heat kernel' or generalized Villain (1975) action (Drouffe, 1978; Menotti and Onofri, 1981). This is based on the desire that the Boltzmann weight or exponentiated action

$$B(U_\square) = \exp\left(-S_\square(\beta, U_\square)\right) \tag{20.3}$$

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should peak strongly near the identity element for weak coupling but should become uniform over the group for a simple strong coupling limit. This is reminiscent of expectation for the evolution of the temperature distribution in a piece of material shaped like the group manifold and initially possessing a spike in temperature at the identity. As time proceeds, the temperature spike should spread and eventually become uniformly distributed over the manifold. These ideas can be made mathematically precise using a group-theoretical generalization of the Laplacian to formulate a heat equation. Recall from chapter 8 the metric tensor

$$M_{ij} = \mathrm{Tr}\,(g^{-1}(\partial_i g)\,g^{-1}(\partial_j g)), \tag{20.4}$$

where the derivatives are with respect to the variables α_i which parameterize the group manifold. In terms of this, the invariant Laplace operator is given by the standard formula of differential geometry

$$\nabla^2 = \det(M)^{-\frac{1}{2}}\,(\partial/\partial\alpha_i)\det(M)^{\frac{1}{2}}\,M_{ij}^{-1}(\partial/\partial\alpha_j). \tag{20.5}$$

We now define the heat equation

$$\nabla^2 K(t,g) = -(\mathrm{d}/\mathrm{d}t)\,K(t,g), \tag{20.6}$$

where for convenience we have set the thermal diffusion coefficient to unity. For an initial condition we take

$$K(0,g) = \delta(g,I). \tag{20.7}$$

The heat kernel action is directly identified with the solution of this equation at a time given by the coupling constant

$$\mathrm{e}^{S_\square(\beta,\,U)} = K(1/\beta, U). \tag{20.8}$$

This action has the technical advantage over the Manton form of being smooth over the entire group manifold and giving rise to a positive definite transfer matrix.

Both the Manton and heat kernel actions have been subjected to Monte Carlo analysis (Lang, Rebbi, Salomonson and Skagerstam, 1981). The string tension was extracted as discussed in the last chapter. For comparison with the Wilson action results, the scheme dependence of the parameters must be calculated perturbatively. The results showed deviations of 20–40% from the theoretical expectations for their ratios, assuming that the physical string tension is universal. This should be regarded as the uncertainty due to the practical fact that the lattice spacing must be kept fairly large and therefore higher terms in the renormalization group function can be important.

Going on to another variant of the action, we note that an interesting change in the qualitative phase structure of the $SU(2)$ theory results from merely changing the trace of a plaquette to the corresponding trace in the adjoint representation (Greensite and Lautrup, 1981; Halliday and Schwimmer, 1981*a*). This amounts to working directly with the group $SO(3)$. In figure 20.1 we show a thermal cycle on this model with a 5^4 site lattice. Figure 20.2 shows the evolution of this system from ordered and

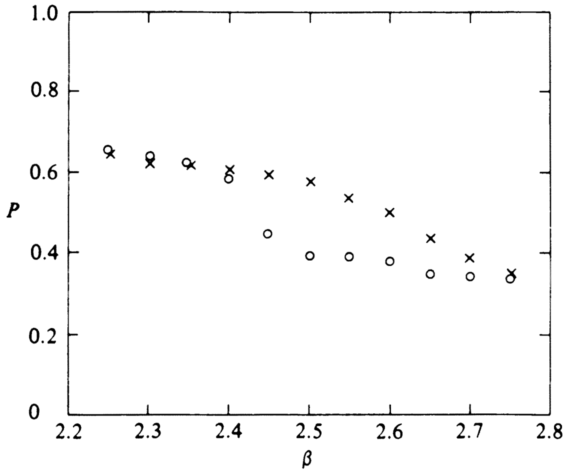

Fig. 20.1. A thermal cycle with $SO(3)$ lattice gauge theory on a 5^4 site lattice. The open circles represent heating; the crosses, cooling. (From Bhanot and Creutz, 1981.)

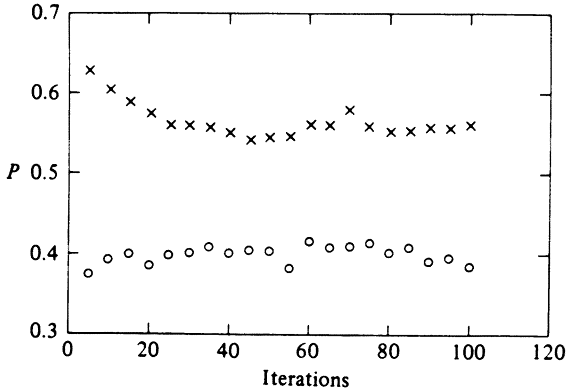

Fig. 20.2. Monte Carlo runs on $SO(3)$ lattice gauge theory at the transition temperature, $\beta = 2.5$. The open circles represent an ordered start, the crosses, random (Bhanot and Creutz, 1981).

disordered starts at the estimated transition temperature. These figures indicate a rather clear first-order transition.

As far as the classical limit is concerned, $SO(3)$ and $SU(2)$ Yang–Mills theories are identical. They only differ because of global properties which can come into play when quantum fluctuations bring plaquette operators far from the identity. The new transition is a lattice artifact which only shows up when the lattice spacing is not small. This is similar to the situation with the fenêtre action mentioned earlier.

One possible explanation of this $SO(3)$ transition is in terms of Z_2 monopole excitations. These arise because the $SO(3)$ representation of $SU(2)$ does not see the Z_2 center of the group. A plaquette variable near $-I$ in the group $SU(2)$ has the same energy as one near I. This can be used to define a Dirac string as a sequence of plaquettes near $-I$. Several closely related schemes for making this concept precise have been presented (Mack and Petkova, 1979; Tomboulis, 1981; Halliday and Schwimmer, 1981*b*; Brower, Kessler and Levine, 1982). We will follow Halliday and Schwimmer, who consider a slight modification of the theory. To make the action insensitive to the group center, they introduce a new set of variables $\{\sigma_\square\}$, each from the group $Z_2 = \{+1, -1\}$ and located on the lattice plaquettes. The new partition function is

$$Z = \sum_{\{\sigma_\square\}} \int (\mathrm{d}U) \exp\left(\sum_\square \beta\sigma_\square \operatorname{Tr}(U_\square)\right). \tag{20.9}$$

As the action is linear in $\sigma_\square$, that part of the sum can be carried out to give

$$Z = \int (\mathrm{d}U) \exp\left(\sum_\square S_\square(U_\square)\right), \tag{20.10}$$

where

$$S_\square(U) = \log(2\cosh(\beta \operatorname{Tr} U)). \tag{20.11}$$

Being an even function of Tr U, this quantity does not see the group center. Monte Carlo simulation (Halliday and Schwimmer, 1981*b*) has shown that this variation of the $SO(3)$ theory also has a first-order phase transition.

The quantity $\sigma_\square$ is essentially a Dirac string variable; when it is positive, $U_\square$ is weighted towards the identity, and when it is negative, $U_\square$ prefers to be near $-I$. The precise position of the Dirac string is unphysical because it can be moved around by absorbing factors of -1 into the link variables. However, in this process the ends of the string do not move; consequently, a natural definition of a monopole is to count the number of negative string variables entering any given three-dimensional cube and to say that a monopole is in that cube if this number is odd. On a four-dimensional lattice the monopoles will trace out world lines, and the strings sweep out world sheets. Halliday and Schwimmer measured the density of these monopole world lines in their simulation and found a sharp discontinuity at the transition temperature. The monopole density is not an order parameter in the sense of a magnetization for a spin system because thermal fluctuations prevent it from ever being exactly zero at any finite temperature. Nevertheless, it does provide a useful quantity to describe what is physically occurring at the transition.

The monopoles are easily supressed by giving them an ad hoc mass term. This motivates the more general partition function

$$Z = \sum_{\{\sigma_\square\}} \int (\mathrm{d}U) \exp\Big(\sum_\square \beta\sigma_\square \operatorname{Tr}(U_\square) + \lambda \sum_c \prod_{\square \in c} \sigma_\square\Big), \tag{20.12}$$

where the new sum in the exponent is over all three-dimensional cubes of the lattice. The presence of a monopole in any cube is now penalized by a factor of $e^{-2\lambda}$. As λ becomes large, the product of string variables over the surface of any cube must go to unity. An elementary exercise shows that once this has occurred there exists a set of Z_2 variables on the links such that any $\sigma_\square$ is the product of these around the given plaquette. In this event, all Z_2 factors are readily absorbed in the invariant $SU(2)$ measure and the theory goes over into the usual $SU(2)$ theory, which appears not to have any phase transitions. The limit $\beta \to 0$ in eq. (20.12) gives rise to a rather complicated looking Z_2 theory. However, under a duality transformation as discussed in chapter 16, this model turns into the usual four-dimensional Ising model with its second-order phase transition. Halliday and Schwimmer provided Monte Carlo evidence that as λ is increased, the $SO(3)$ transition moves to smaller β and eventually becomes the Ising transition. The place where the transition changes from first to second order is not known.

An alternative means for supressing monopoles is to add to the action of eq. (20.9) an effective potential for the variables $\sigma_\square$. Thus we could consider

$$Z = \sum_{\{\sigma_\square\}} \int (\mathrm{d}U) \exp\Big(\sum_\square \beta\sigma_\square \operatorname{Tr}(U) + \eta \sum_\square \sigma_\square\Big). \tag{20.13}$$

As the parameter η goes to infinity, all $\sigma_\square$ are driven to unity and we again return to the pure $SU(2)$ theory. As $\sigma_\square$ is a Dirac string variable, the new term adds an effective energy per unit length to the strings. With η non-zero the strings become physical because moving them around will now change the total action in proportion to the total change in string length.

The action in eq. (20.13) is linear in $\sigma_\square$. These variables can be summed out to give an action dependent on the $U_\square$ only, as in eq. (20.10), but now

$$S_\square(U) = \log(2\cosh(\beta \operatorname{Tr} U + \eta)). \tag{20.14}$$

Unlike in eq. (20.11), this is no longer insensitive to the group center. Expanding this action in characters

$$S_\square(U) = \sum_R \beta_R \chi_R(U) \tag{20.15}$$

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will give rise to terms with both integer and half-integer spin representations of $SU(2)$. Only the half-integer terms distinguish the group center. The action in eq. (20.14) has not been simulated directly, but it motivates a simpler form obtained by taking just the spin one-half and spin one terms in eq. (20.15) (Bhanot and Creutz, 1981).

$$S_\square(U) = \tfrac{1}{2}\beta \operatorname{Tr}(U) + \tfrac{1}{3}\beta_A \operatorname{Tr}_A(U). \tag{20.16}$$

Here Tr_A denotes the trace or character in the adjoint or spin one representation. The factors in front of the couplings β and β_A are inserted for normalization convenience.

The theory defined by eq. (20.16) has several interesting limits. For vanishing β_A it reduces to the ordinary Wilson $SU(2)$ model, which we believe exhibits no phase transitions. In contrast, the limit of vanishing β gives the $SO(3)$ model, which we saw in figures 20.1 and 20.2 to have a first-order transition. The third interesting limit occurs as β_A goes to infinity. In this case all plaquettes are forced to lie in the center of the gauge group. This means that up to a gauge transformation all links are themselves driven to the center. Thus for $SU(2)$ the model becomes a Z_2 gauge theory with coupling β. As discussed in chapter 16, this model has a strong first-order phase transition at the self-dual point. At the outset, therefore, we know that the model of eq. (20.16) must have non-trivial phase structure, with two first-order lines entering the phase diagram.

Monte Carlo simulations have explored the evolution of these transitions into the two coupling plane (Bhanot and Creutz, 1981). The resulting phase diagram is shown in figure 20.3. Note that the $Z(2)$ and $SO(3)$ transitions are stable and meet at a triple point located at

$$(\beta, \beta_A) = (0.55 \pm 0.03, 2.34 \pm 0.03). \tag{20.17}$$

A third first-order line extends from this point and aims toward the Wilson axis but terminates before reaching it at a critical point located at

$$(\beta, \beta_A) = (1.57 \pm 0.05, 0.78 \pm 0.05). \tag{20.18}$$

This line points directly at the position of the peak in the specific heat of the ordinary $SU(2)$ model (Lautrup and Nauenberg, 1980*b*). That peak may be interpreted as a remnant of this transition, a shadow of its critical endpoint.

We can use this system to test the uniqueness of the continuum limit. The connection between the bare charge and the parameters is

$$g_0^{-2} = \beta/4 + 2\beta_A/3. \tag{20.19}$$

A continuum limit requires taking g_0^2 to zero; however, this can be done along many paths in the (β, β_A) plane. Conventionally concentration is placed on the Wilson trajectory $\beta_A = 0$, $\beta \to \infty$. Along that line no singularities are encountered. Thus we have the usual claim that confinement, which is present in strong coupling, should persist into the weak coupling domain. However, an equally justified path would be, for example, $\beta = \beta_A \to \infty$. In this case we cross a first-order transition. Because one can continue around it in our larger coupling constant space, the transition is not deconfining and is simply an artifact of the lattice action.

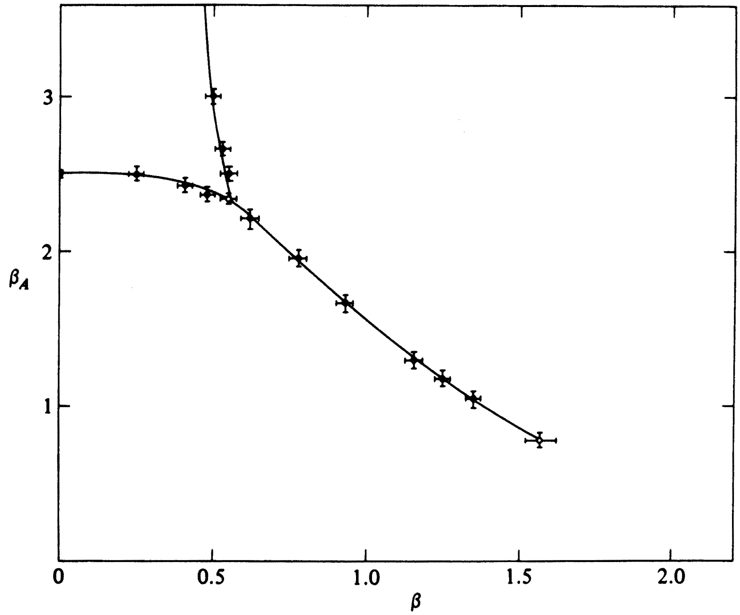

Fig. 20.3. Phase diagram for $SU(2)$ lattice gauge theory with fundamental and adjoint couplings (Bhanot and Creutz, 1981).

To test whether physical observables are indeed independent of direction in this plane, we can consider Wilson loops in the weak coupling regime. The loop by itself is not an observable because of self-energy divergences (Dotsenko and Vergeles, 1980; also recall problem 4 of chapter 6). These divergences should cancel in ratios of loops with the same perimeters and numbers of sharp corners. This leads us to consider the ratios

$$R(I, J, K, L) = \frac{W(I, J)\, W(K, L)}{W(I, L)\, W(J, K)}, \tag{20.20}$$

where $W(I, J)$ denotes the rectangular Wilson loop of dimensions I-by-J in lattice units. Wishing to compare points which give similar physics, we can consider for each value of β_A the value of β for which some R ratio

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has a particular value. In figure 20.4 we show points from Monte Carlo simulation for $R(2,2,3,3)$ having the values 0.87 and 0.93. The dashed lines in the figure represent constant bare charge from eq. (20.19). This particular simulation was performed with a 120-element subgroup approximating $SU(2)$. This is a good approximation where we are working, but does give rise to an extra transition to a highly ordered state at large

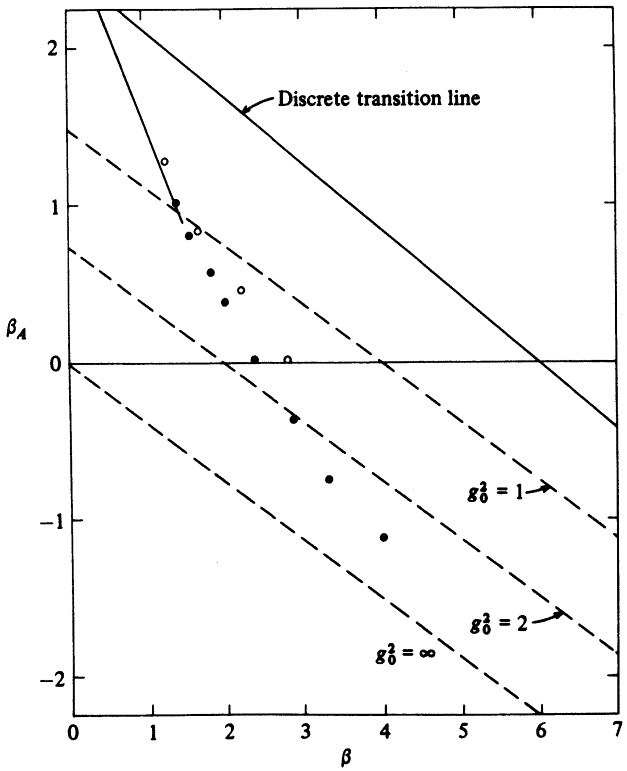

Fig. 20.4. Points of constant 'physics' as obtained from $R(2,2,3,3) = 0.87$ (solid circles) and 0.93 (open circles) (Bhanot and Creutz, 1981).

values of inverse coupling. The location of this 'discreteness' transition line is also indicated in figure 20.4.

If physics is indeed similar at all points along one of these contours of constant R ratio, then it should not matter which ratio we chose. In figure 20.5 we show several such ratios as functions of β_A along the $R(2,2,3,3) = 0.87$ contour. The comparison is quite good considering that finite cutoff corrections are ignored. Note that in this comparison the bare charge is far from being a constant. Along the 0.87 contour of 'constant

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physics', g_0^2 varied from less than unity to nearly 4 in the measured region. Such variation is permissible and perhaps even expected since the bare charge is unobservable and should depend on the cutoff prescription. The dependence can be characterized by a β_A dependent renormalization scale $\Lambda_0(\beta_A)$. The expected dependence of the renormalization scale on the new coupling β_A is calculable in perturbation theory (Gonzales-Arroyo and Korthals-Altes, 1982; Bhanot and Dashen, 1982). In the vicinity of the

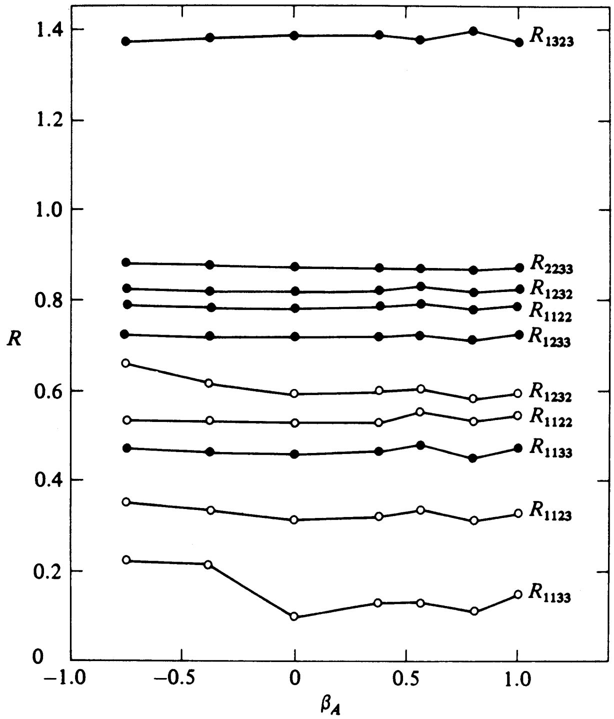

Fig. 20.5. Various R ratios along the $R(2, 2, 3, 3) = 0.87$ contour (Bhanot and Creutz, 1981).

Wilson action, that is when $|\beta_A a| < 0.5$, the prediction works reasonably well. However, as we approach the critical endpoint at positive β_A, large deviations from the points in figure 20.4 are found. This indicates that important additional physics is affecting the Monte Carlo results. We are seeing lattice artifacts near the new critical point. For negative β_A where the bare coupling becomes large, the agreement with the perturbative result is again poor. This can presumably be understood because of higher terms in the renormalization group function coming into play as the coupling increases (Grossman and Samuel, 1983).

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This analysis indicates the privileged role played by the Wilson action. It appears to lie in a middle region where the scaling of asymptotic freedom appears on the modest lattices available to Monte Carlo simulation.

As the parameter β_A increases relative to β, the extremum of the action at $U = -I$ changes from a maximum to a minimum. This occurs along the line

$$\beta_A = 3\beta/8. \tag{20.21}$$

Finally, along the β_A axis the two minima are degenerate. Note that the critical endpoint lies slightly above the line in eq. (20.21). Bhanot (1982) has studied a similar two-coupling $SU(3)$ theory and finds a critical endpoint near the appearance of new minima of the plaquette action for group elements lying in the group center. As the n of $SU(n)$ increases beyond four, those elements of the group center near the identity become minima of the action even for the conventional Wilson action (Bachas and Dashen, 1982). This observation correlates well with the Monte Carlo results that the Wilson $SU(4)$, $SU(5)$ and $SU(6)$ theories all display first-order phase transitions (Creutz, 1981*b*; Moriarty, 1981; Creutz and Moriarty, 1982*a*). Presumably a negative β_A removes the extraneous action minima and will permit continuation around these transitions, which would therefore not be deconfining.

In the last few chapters we have seen that Monte Carlo simulation indeed provides a powerful tool. The technique not only permits calculation of observables, but also opens a way to investigate questions of existence and uniqueness. These investigations of the solutions of non-trivial quantum field theories indicate that we are truly at an exciting time in the development of elementary particle physics.

Problems

1. Show that if the product of the $\sigma_\Box$ variables in eq. 12 is unity for every three-dimensional cube, then these parameters can be written as the product of Z_2 variables on the links surrounding the corresponding plaquettes.

2. Verify the assertion that the $\beta_A \to \infty$ limit of the theory defined by eq. (20.16) is indeed a Z_2 gauge theory.

3. Consider a three-parameter generalized $SU(2)$–$SO(3)$ action with both the λ term of eq. (20.12) and the η term of eq. (20.13). Discuss the various two parameter limits of this model.