

## PRESENTATIONS FOR SOME DIRECT PRODUCTS OF GROUPS

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We give efficient presentations for the direct product of two copies of the alternating group of degree five and the direct product of the alternating group of degree five and the binary icosahedral group.

A finite group  $G$  is said to be *efficient* if it has a presentation on  $d$  generators and  $d + \text{rank}(M(G))$  relations ( $M(G)$  is the Schur multiplier of  $G$ , see [1]). It is well known that  $A_5$ , the alternating group of degree five, is efficient because

(i)  $M(A_5)$  has order 2 and

(ii)  $A_5$  has a presentation  $\langle a, b \mid a^5 = b^3 = (ab)^2 = 1 \rangle$ .

Several authors ([1, p. 191], [4], [6, p. 149]) have recently asked whether the direct product  $A_5 \times A_5$  of two copies of  $A_5$  is efficient. The answer is yes. We show this by first giving an efficient presentation for  $A_5 \times B$ , where  $B$  is the binary icosahedral group. The method for obtaining this presentation is outlined in the last paragraph.

Let  $H$  be defined by

$$(*) \quad \langle x, y \mid y^6 = x^4 y x^{-1} y^{-3} x^{-1} y^{-1} = (xy^2)^2 x^{-1} y^{-1} (xy^{-1})^2 = 1 \rangle .$$

It is easy to see that  $H$  is perfect. Coset enumeration shows that  $H$

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has order 7200 . It is known that  $B$  has a presentation

$$\langle c, d \mid c^5 = d^3 = (cd)^2 \rangle ,$$

and the mapping  $x \mapsto cd$  ,  $y \mapsto d$  of  $H$  onto  $B$  extends to a homomorphism since the relations of  $H$  map to the identity of  $B$  ; thus  $B$  is a quotient of  $H$  . From the tables given in [5], it follows that  $H$  is  $A_5 \times B$  . Since  $M(A_5 \times B)$  has order 2 (using [1, Theorem 4.11, p. 108]), we have shown that  $A_5 \times B$  is efficient.

Coset enumeration shows that the group obtained by adding the relation  $x^{10} = 1$  to (\*) has order 3600 and is therefore  $A_5 \times A_5$  . The rank of  $M(A_5 \times A_5)$  is two, and so  $A_5 \times A_5$  is also efficient.

These presentations do not shed any light on the problem of whether  $B \times B$  is efficient; note that Wiegold [6, p. 149] has conjectured that  $B \times B$  is not efficient.

It is clear that  $A_5 \times B$  has a presentation

$$\langle a, b, c, d \mid a^5 = b^3 = (ab)^2 = 1, c^5 = d^3 = (cd)^2, \\ [a, c] = [a, d] = [b, c] = [b, d] = 1 \rangle$$

and is generated by the elements  $x = acd$  and  $y = abd$  . Defining relations for  $A_5 \times B$  in terms of  $x$  and  $y$  were constructed by using a method due to Cannon [2] as implemented in Cayley [3]. This resulted in a two-generator four-relation presentation. The coset enumeration above showed the fourth relation,  $[x^2, y^2] = 1$  , to be redundant.

ADDED IN PROOF. B.H. Neumann has asked (private communication) whether the group  $K$  defined by

$$\langle x, y \mid x^{10} = y^6, x^4 y x^{-1} y^{-3} x^{-1} y^{-1} = (xy^2)^2 x^{-1} y^{-1} (xy^{-1})^2 = 1 \rangle$$

is the other perfect group of order 7200 (the direct product of two copies of  $B$  with amalgamated central subgroups, denoted by  $P_4$  in [5]). Coset enumeration once again shows that  $K$  has order 7200 . The normal subgroup lattice of  $K$  was calculated using Cayley [3], and it was found that  $K$  has no normal subgroup of order 60 . This shows that  $K$  is  $P_4$  .

## References

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