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THE ASYMPTOTIC ANALYSIS OF A CLASS OF SELF-ADJOINT SECOND-ORDER DIFFERENCE EQUATIONS: JORDAN BOX CASE

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Abstract. In this paper, we are computing asymptotic formulas for a base of solutions of the second-order difference equations in the double root case. Two methods are presented.

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1. Introduction. Through most of this paper we are trying to find asymptotic formulas of base solutions of a special recurrence equation. Even though our motivation comes from the spectral theory of self-adjoint Jacobi operators (for the definition see the last section), we think that the specialists working on difference equations may find some results of this paper interesting.

We consider the system of difference equations

$$\lambda_{n-1}u(n-1) + (q_n - \lambda)u(n) + \lambda_n u(n+1) = 0, \qquad n \geq 2, \lambda \in \mathbb{R}, \tag{1}$$

with λ_n and q_n defined by

$$\lambda_n := n^\alpha \left(1 + \sum_{i=1}^{K} \frac{a_i}{n^{\alpha_i}} + V(n)\right), \quad q_n := -2n^\alpha \left(1 + \sum_{i=1}^{K} \frac{b_i}{n^{\alpha_i}} + W(n)\right), \tag{2}$$

where $\alpha \in (0, 1)$, $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}$ and α_i are some positive real numbers, for $i = 1, \ldots, K$. The terms $V(n)$ and $W(n)$ are some real l^1 sequences of the order $\mathcal{O}(n^{-1-\widehat{\alpha}-\delta})$, here δ is a small positive real number and $\widehat{\alpha} := \min\{\alpha_1, \alpha\}$, l^1 stands for the space of summable sequences. We also assume that $\lambda_n > 0$ for all $n \in \mathbb{N}$.

Let us rewrite equation (1) in a matrix form

$$\vec{u}(n+1) = B(n;\lambda)\vec{u}(n), \tag{3}$$

where

$$B(n;\lambda) := \begin{pmatrix} 0 & 1 \\ -\dfrac{\lambda_{n-1}}{\lambda_n} & \dfrac{\lambda - q_n}{\lambda_n} \end{pmatrix} \quad \text{and} \quad \vec{u}(n) := \begin{pmatrix} u(n-1) \\ u(n) \end{pmatrix}. \tag{4}$$

The matrix $B(n;\lambda)$ is called the transfer matrix and in our case it is equal to $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right) + R(n;\lambda)$, where $||R(n;\lambda)||$ decreases to zero for all fixed λ's. The matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right)$ has only one eigenvalue $\mu = 1$ and it is similar to a Jordan box $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Using the terminology from the previous papers (see, for example, [**3**, **6**]) we will call this situation the *Jordan box* or *double root* case. This situation appears always when $\lim \frac{q_n}{\lambda_n} = \pm 2$ and it is quite difficult to study because most of the methods (for example Levinson-type theorems) do not apply. Recently, some efforts have been made to deal with this situation (see, for example, [**4**, **9**]). The methods used in those papers can be applied only in special situations. In [**4**] the reader may find asymptotic analysis based on WKB method of equation (1) with $\lambda_n = n + a$ and $q_n = -2(n + a)$, $a \in \mathbb{R}$. In [**9**] the author used the Birkhoff–Adams theorem to find asymptotics of solutions of (1) with $\lambda_n = c_n n$, $q_n = n$, c_n is a real two-periodic sequence generated by c_1 and c_2. Nevertheless, for example, the results from [**9**] do not extend to a class of recurrences (1) with $\lambda_n = c_n n^\alpha$, $q_n = n^\alpha$, for $\alpha \in (0, 1)$ and $\left|\frac{c_1^2+c_2^2-1}{c_1 c_2}\right| = 2$. This is impossible because the Birkhoff–Adams theorem does not apply.

However, in recent papers [**3**, **5**] two different approaches were proposed to treat the Jordan box case in a larger class of equations. Here, we present the method introduced by Janas in [**3**]. Simply, our results are the generalization of what he has done in his paper. Moreover, we consider a new situation. In [**3**], the recurrence systems (1) were considered with $\lambda_n = n^\alpha(1 + \frac{a}{n} + \frac{D}{n^{2\alpha}} + \frac{V(n)}{n})$ and $q_n = \pm 2n^\alpha(1 + \frac{b}{n} + \frac{E}{n^{2\alpha}} + \frac{W(n)}{n})$, these sequences are like (n^α) or $(\pm 2n^\alpha)$ plus some decreasing to zero perturbation. We allow the situation when the perturbations grow to infinity (see formulas (2)).

The critical case of Jordan box was also studied for $\lambda_n = n^\alpha(1 + r(n))$, $q_n = -2n^\alpha(1 + s(n))$ in [**5**] but under additional assumption on the perturbations: $(n^{\alpha/2} r(n))$, $(n^{\alpha/2} s(n)) \in l^1$. However, $r(n)$ and $s(n)$ are not necessary decaying in the power scale. In [**5**] the authors used an *ansatz* approach to find asymptotic formulas for a base of solutions of (1).

System (1) can be transformed into

$$u(n + 2) + p_1(n)u(n + 1) + p_2(n)u(n) = 0, \qquad n \in \mathbb{N}, \tag{5}$$

with coefficients $p_1(n)$ and $p_2(n)$ being the finite sums of fractional powers $n^{-\beta_i}$ of n plus some perturbations which are in l^1. If $\beta_i \in \mathbb{N}$, then we are in a double root case analysed by Birkhoff (see, for example, [**1**] p. 354). This type of recurrences with $\beta_i \geq 0$ was studied by Kooman in [**8**]. In section 10 of his work Kooman described behaviour in infinity of the quotients $u_n v_n^{-1}$ of two linearly independent solutions of (5). In our work we are going to prove more, namely, the asymptotic formulas of u_n and v_n.

We say that (1) is non-oscillatory if for every real solution $(u(n))_{n \in \mathbb{N}}$ there exists N_0 such that $u(n)u(n + 1) > 0$ for all $n \geq N_0$, otherwise (1) is oscillatory. Depending on the case whether equation (1) is non-oscillatory or oscillatory we use two different approaches. In the first case, we proceed like Kelley did in [**6**]. In the second case, we use an ansatz approach given in [**5**].

The order of this work is as follows: In Section 2 the basic facts and definitions are given. Sections 3 and 4 contain the detailed analysis of the case when (1) is non-oscillatory or oscillatory, respectively, and asymptotic formulas of solutions of (1) are given. In Section 5, the reader finds a brief sketch of some applications of our main results (Theorems 3 and 4) in the spectral theory of Jacobi operators.

2. Preparation. If we define

$$r(n) := \sum_{j=1}^{K} \frac{a_j}{n^{\alpha_j}} + V(n) \quad \text{and} \quad s(n) := \sum_{j=1}^{K} \frac{b_j}{n^{\alpha_j}} + W(n),$$

then (2) reads as $\lambda_n = n^\alpha(1 + r(n))$ and $q_n = -2n^\alpha(1 + s(n))$. Throughout the paper, we will use the following notations: Let $i_0 \in \{0, \dots, K-1\}$ be such that $\alpha_{i_0+1} > \alpha$. If for all $i = 1, \dots, K$, $\alpha_i \leq \alpha$, then we set $i_0 := K$.

Let

$$r'(n) := \sum_{j=1}^{i_0} \frac{a_j}{n^{\alpha_j}}, \qquad s'(n) := \sum_{j=1}^{i_0} \frac{b_j}{n^{\alpha_j}},$$

and

$$r''(n) := r(n) - r'(n), \qquad s''(n) := s(n) - s'(n).$$

If in the above formulas $i_0 = 0$, then $r'(n) = s'(n) := 0$.

In what follows we assume that the numbers α and α_i in (2) satisfy

$$(i) \;\; \alpha \in \left(\frac{1}{2}, \frac{2}{3}\right), \tag{6}$$

$$(ii) \;\; \frac{\alpha}{2} \leq \alpha_1 < \alpha_2 < \cdots < \alpha_K, \tag{7}$$

$$(iii) \;\; \widehat{\alpha} - \alpha_{i_0+1} < -\frac{\alpha}{2}, \tag{8}$$

here as before $\widehat{\alpha} := \min\{\alpha_1, \alpha\}$. The method presented in this paper works for all $\alpha \in (0, 1)$ and $\alpha_i > 0$ $(i = 1, \dots, K)$ but to avoid tedious calculations we restrict ourselves to the case when (6), (7) and (8) are fulfilled. This restriction is only technical. For $\alpha \leq \frac{1}{2}$ (or $\alpha \geq \frac{2}{3}$) and $\alpha_1 \leq \frac{1}{4}$ we would have to deal with more elements in the Taylor expansion of $\sqrt{-\beta(n)}$ (see expression (15)). Most calculations which are presented in this work are based on the Taylor expansion. We will use it several times without recalling it.

By Lemma 1 in **[3]** we know that (1) is non-oscillatory if and only if

$$\lambda_{n+1} + \lambda_n + q_n - \lambda \leq 0, \qquad n > N_0, \tag{9}$$

for some N_0 large enough. In our case, for large n, we have

$$\begin{aligned} \lambda_{n+1} + \lambda_n + q_n - \lambda &= (n+1)^\alpha \left(1 + \sum_{i=1}^{K} \frac{a_i}{(n+1)^{\alpha_i}}\right) \\ &\quad + n^\alpha \left(1 + \sum_{i=1}^{K} \frac{a_i}{n^{\alpha_i}}\right) - 2n^\alpha \left(1 + \sum_{i=1}^{K} \frac{b_i}{n^{\alpha_i}}\right) - \lambda \\ &= n^\alpha \left(\frac{2(a_1 - b_1)}{n^{\alpha_1}} + \cdots + \frac{2(a_{i_0} - b_{i_0})}{n^{\alpha_{i_0}}} - \frac{\lambda}{n^\alpha}\right) \\ &\quad + n^\alpha \left(\frac{2(a_{i_0+1} - b_{i_0+1})}{n^{\alpha_{i_0+1}}} + \cdots + \frac{2(a_K - b_K)}{n^{\alpha_K}}\right) + o(1). \end{aligned} \tag{10}$$

Using assumptions on α_{i_0+1} we can see that the second term in (10) goes to zero if n goes to infinity. We conclude now that (9) follows from

$$n^{\alpha}\left(\frac{2(a_1-b_1)}{n^{\alpha_1}}+\cdots+\frac{2(a_{i_0}-b_{i_0})}{n^{\alpha_{i_0}}}\right)-\lambda<0, \qquad n\geq N_0, \tag{11}$$

here $N_0 = N_0(\lambda)$. If we define

$$\rho := \begin{cases} 2(a_1-b_1) & \text{for} \quad \alpha_1<\alpha, \\ 2(a_1-b_1)-\lambda & \text{for} \quad \alpha_1=\alpha, \\ -\lambda & \text{for} \quad \alpha_1>\alpha, \end{cases} \tag{12}$$

then for $\alpha_1 < \alpha$ (11) is valid if $\rho < 0$ and λ belongs to any compact interval of the real line. For $\alpha_1 = \alpha$ condition (11) is true if λ is a real number greater then $2(a_1 - b_1)$. In the last case ($\alpha_1 > \alpha$) all positive real λ's fulfill (11).

To apply Kelley's method we need to transform our equation (1) into a more suitable form. Dividing (1) by λ_n and making the change of variable

$$w(n) := u(n)\prod_{k=1}^{n-1}\frac{2\lambda_k}{\lambda-q_k} \tag{13}$$

we obtain

$$w(n+1)-2w(n)+(1+\beta(n))w(n-1)=0, \tag{14}$$

where

$$1+\beta(n)=\frac{4\lambda_{n-1}^2}{(q_n-\lambda)(q_{n-1}-\lambda)}. \tag{15}$$

If we assume that (11) holds, then equation (1) is non-oscillatory which is of course equivalent that (14) is non-oscillatory as well. From [**2**] we know that equation (14) is non-oscillatory if and only if its every real non-trivial solution satisfies

$$\lim_{n\to+\infty}\frac{w(n+1)}{w(n)}=1.$$

Which we can rewrite as

$$\frac{w(n+1)}{w(n)}=1+X(n), \tag{16}$$

where $X(n)$ tends to zero if n goes to infinity. Dividing (14) by $w(n)$ and using (16) we obtain

$$X(n)=(1+\beta(n))\frac{X(n-1)}{1+X(n-1)}-\beta(n). \tag{17}$$

Because for n large enough $|X(n-1)| < 1$, we may rewrite $\frac{X(n-1)}{1+X(n-1)}$ as a sum of a geometric sequence. After some calculations we can transform (17) into

$$X(n)-X(n-1)+\beta(n)+X^2(n-1)+(\beta(n)+X^2(n-1))\sum_{k=1}^{+\infty}(-X(n-1))^k=0. \tag{18}$$

Here, we recall some theorems and one lemma which the reader may find in [**6**] and also in [**3**]. We will use those results in the next section.

LEMMA 1. *Assume that* $f(n)$ *is given by*

$$f(n) = \prod_{s=1}^{n-1}(1+\beta(s)),$$

where $\beta(s) \to 0$, *as* $s \to +\infty$, *is a sequence of real numbers. Let* κ *be the largest integer such that* $\sum_s \beta^\kappa(s)$ *diverges. Define*

$$h(n) = \sum_{i=1}^{\kappa} \frac{(-1)^{i-1}}{i}\beta^i(n),$$

then

$$f(n) = F(n)\exp\left[\sum_{s=1}^{n-1} h(s)\right],$$

where $F(n) \to F > 0$ *as* $n \to +\infty$.

THEOREM 1. *Assume that* $1+\beta(n) \geq 0$ *for* $n > n_0$ *and that* $v(n)$ *and* $w(n)$ *satisfy the inequalities*

$$v(n) \leq \frac{(1+\beta(n))v(n-1)}{1+v(n-1)} - \beta(n), \qquad n \geq n_0+1 \tag{19}$$

$$w(n) \geq \frac{(1+\beta(n))w(n-1)}{1+w(n-1)} - \beta(n), \qquad n \geq n_0+1 \tag{20}$$

$$w(n_0) \geq v(n_0)$$

$$v(n) > -1, \qquad n \geq n_0.$$

If $X(n_0) \in [v(n_0), w(n_0)]$ *and* $X(n)$ *satisfies (17) for* $n \geq n_0+1$,
then $v(n) \leq X(n) \leq w(n)$, $n \geq n_0$.

THEOREM 2. *Assume that* $1+\beta(n) \geq 0$, $v(n)$ *satisfies* (19), $w(n)$ *satisfies* (20), $v(n) \geq w(n)$, $|v(n)| < 1$ *and* $|w(n)| < 1$ *for* $n \geq n_0$. *Then* (17) *has a solution* $X(n)$ *such that* $v(n) \geq X(n) \geq w(n)$ *for* $n \geq n_0$.

We will also need the following:

PROPOSITION 1. *For* λ_n *and* q_n *defined by* (2) *and* (6)–(8) *we have*

$$\prod_{i=1}^{n-1} \frac{\lambda - q_i}{2\lambda_i} = A(n)\exp\left[\sum_{i=1}^{n-1}\left(s(i) - r(i) + \frac{\lambda}{2i^\alpha} + \frac{1}{2}(r^2(i) - s^2(i)) - \frac{\lambda}{2i^\alpha}s(i) + \frac{1}{3}(s^3(i) - r^3(i))\right)\right],$$

$A(n)$ *is a sequence convergent to some positive constant.*

The proof of this proposition is based on the formula $\prod a_n = e^{\sum \ln a_n}$ and convergence of the series $\sum_i s^t(i)$, $\sum_i r^t(i)$ for $t \geq 4$.

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3. Asymptotics in the non-oscillatory case. In this section, we assume that condition (9) is fulfilled. It means that λ belongs to the set (denoted by Λ_-) of the real numbers for which (11) is valid. It may happen that $\Lambda_- = \emptyset$ or $\Lambda_- = \mathbb{R}$, it depends on the sign of ρ (see (12)) and on the numbers α_i.

In this section our aim is to find the formal solutions $X_\pm(n)$ of (17) modulo some terms of the order $\mathcal{O}(n^{-1-\widehat{\alpha}/2-\delta})$, here δ is a small positive real number. To find those solutions we assume that

$$X_\pm(n) = \pm\sqrt{-\beta(n+1)} + \gamma(n+1), \tag{21}$$

where $\gamma(n)$ is a sequence which tends to zero in infinity faster than $\sqrt{-\beta(n)}$. Our assumption on the form of the solutions $X_\pm(n)$ is based on the considerations of Janas in [**3**] (see also [**6**]). We do not want to repeat this reasoning because it is exactly the same. The difference between these works and ours starts in formal calculations which we present below, in shortest form of course.

Using the definitions of λ_n, q_n and $\beta(n)$ (see (2) and (15)) we may write

$$-\beta(n) = \eta(n) + \xi(n) - \zeta_1(n) - \zeta_2(n) + \frac{\alpha}{n} - \zeta_3(n) - \frac{3\lambda^2}{4n^{2\alpha}} - \zeta_4(n) + n^{-\widehat{\alpha}}\varepsilon^{(1)}(n), \tag{22}$$

here $\varepsilon^{(1)}(n)$ is some sequence which is in l^1. The sequences in (22) are given as follows by explicit formulas:

$$\eta(n) = 2(s'(n) - r'(n)) + \frac{\lambda}{n^\alpha}, \qquad \xi(n) = 2(s''(n) - r''(n)), \tag{23}$$

$$\zeta_1(n) = 3s^2(n) - 4r(n)s(n) + r^2(n),$$

$$\zeta_2(n) = 3\frac{\lambda}{n^\alpha}s(n) - 2\frac{\lambda}{n^\alpha}r(n) - 4s^3(n) + 6r(n)s^2(n) - 2r^2(n)s(n),$$

$$\zeta_3(n) = 6\frac{\lambda}{n^\alpha}r(n)s(n) - \frac{\lambda}{n^\alpha}r^2(n) - 6\frac{\lambda}{n^\alpha}s^2(n) + 5s^4(n) - 8r(n)s^3(n) + 3r^2(n)s^2(n),$$

$$\zeta_4(n) = \frac{2a_1(\alpha_1 - \alpha) + b_1(2\alpha - \alpha_1)}{n^{1+\alpha_1}} + \frac{\alpha\lambda}{n^{1+\alpha}}.$$

We may write $-\beta(n) = \eta(n)\,(1 + Y_n)$, where

$$Y_n = \eta^{-1}(n)\left(\xi(n) - \zeta_1(n) - \zeta_2(n) - \frac{3\lambda^2}{4n^{2\alpha}} + \frac{\alpha}{n} - \zeta_3(n) - \zeta_4(n)\right) + \varepsilon^{(2)}(n).$$

In rest of the paper, all the remainders which are in l^1 will be denoted by $\varepsilon^{(i)}(n)$. Because Y_n is of the order $\mathcal{O}(n^{\widehat{\alpha}-\alpha_{i_0+1}}) + \mathcal{O}(n^{\widehat{\alpha}-2\alpha_1})$ for large n, we can see that (Y_n^4) is in l^1 (see (6), (7) and (8)). This observation implies

$$\sqrt{-\beta(n)} = \sqrt{\eta(n)}\left(1 + \frac{1}{2}Y_n - \frac{1}{8}Y_n^2 + \frac{1}{16}Y_n^3 + \varepsilon^{(3)}(n)\right).$$

Computing all the necessary powers of Y_n and multiplying $\sqrt{\eta(n)}$ by the expression in the brackets gives us

$$\sqrt{-\beta(n)} = \sqrt{\eta(n)} + \sum_{i=1}^{6}\omega_i(n) + n^{-\widehat{\alpha}/2}\varepsilon^{(4)}(n), \tag{24}$$

where

$$\omega_1(n) = \frac{\xi(n) - \zeta_1(n)}{2(\eta(n))^{1/2}},$$
$$\omega_2(n) = \frac{-\zeta_2(n) - \frac{3}{4}\lambda^2 n^{-2\alpha}}{2(\eta(n))^{1/2}} - \frac{(\xi(n) - \zeta_1(n))^2}{8(\eta(n))^{3/2}},$$
$$\omega_3(n) = \frac{\alpha n^{-1}}{2(\eta(n))^{1/2}},$$
$$\omega_4(n) = \frac{-\zeta_3(n)}{2(\eta(n))^{1/2}} + \frac{(\xi(n) - \zeta_1(n))\zeta_2(n)}{4(\eta(n))^{3/2}} + \frac{3\lambda^2 n^{-2\alpha}(\xi(n) - \zeta_1(n))}{16(\eta(n))^{3/2}} - \frac{\alpha n^{-1}(\xi(n) - \zeta_1(n))}{4(\eta(n))^{3/2}} + \frac{(\xi(n) - \zeta_1(n))^3}{16(\eta(n))^{5/2}},$$
$$\omega_5(n) = \frac{\alpha^2 n^{-2}}{8(\eta(n))^{3/2}}, \ \omega_6(n) = -\frac{\zeta_4(n)}{2(\eta(n))^{1/2}}.$$

Remark. By the definition of $\omega_i(n)$ one can easily check that $\omega_i(n)$ tends to zero when n goes to infinity (for $i = 1, \ldots, 6$). Because $\omega_1(n)$ is of the order $\mathcal{O}(n^{\widehat{\alpha}/2-\alpha_{i_0+1}}) + \mathcal{O}(n^{\widehat{\alpha}/2-2\alpha_1})$ and decreases the most slowly we have $\sqrt{-\beta(n)} = \sqrt{\eta(n)} + o(\sqrt{\eta(n)})$.

If we apply (21), (24) and the assumptions (6)–(8) to equation (18) then we obtain

$$X_\pm(n) - X_\pm(n-1) + \beta(n) + X_\pm^2(n-1) - X_\pm(n-1)(\beta(n) + X_\pm^2(n-1)) + n^{-\widehat{\alpha}/2}\varepsilon^{(5)}(n) = 0. \quad (25)$$

Now we must investigate all the terms in this equation to cancel out the left-hand side of it up to $n^{-\widehat{\alpha}/2}\varepsilon^{(i)}(n)$ terms. We will do this one by one. First we are going to examine the difference $X_\pm(n) - X_\pm(n-1)$.

Using (21) and (24) we see that

$$X_\pm(n) - X_\pm(n-1) = \pm(\sqrt{\eta(n+1)} - \sqrt{\eta(n)}) + \sum_{i=1}^{6}(\Delta\omega_i)(n) + (\Delta\gamma)(n) + n^{-\widehat{\alpha}/2}\varepsilon^{(6)}(n). \quad (26)$$

Here $(\Delta a)(n) := a(n+1) - a(n)$ for any sequence $(a(n))_{n\in\mathbb{N}}$. Applying the formulas for $\omega_i(n)$ we obtain that $(\Delta\omega_i)(n) = n^{-\widehat{\alpha}/2}\varepsilon^{(7)}(n)$ for $i = 1, \ldots, 6$ (see Remark mentioned earlier). One can easily check that

$$\sqrt{\eta(n+1)} = \sqrt{\eta(n)}\left(1 + \frac{\varphi(n)}{2\eta(n)} + \mathcal{O}(n^{-2})\right), \quad (27)$$

where

$$\varphi(n) = -\left[\sum_{i=1}^{i_0} \frac{2\alpha_i(b_i - a_i)}{n^{1+\alpha_i}} + \frac{\alpha\lambda}{n^{1+\alpha}}\right]. \quad (28)$$

Expressions (26)–(28) allow to write

$$X_\pm(n) - X_\pm(n-1) = \pm\frac{\varphi(n)}{2\sqrt{\eta(n)}} + n^{-\widehat{\alpha}/2}\varepsilon^{(8)}(n). \quad (29)$$

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Again by (21) we see that

$$\beta(n) + X_{\pm}^2(n-1) = \pm 2\sqrt{-\beta(n)}\gamma(n) + \gamma^2(n), \tag{30}$$

which implies

$$X_{\pm}(n-1)(\beta(n) + X_{\pm}^2(n-1)) = -2\beta(n)\gamma(n) \pm 3\sqrt{-\beta(n)}\gamma^2(n) + \gamma^3(n). \tag{31}$$

Combining the expressions (29)–(31) we can rewrite (25) as

$$\pm\left[\frac{\varphi(n)}{2\sqrt{\eta(n)}} + 2\sqrt{-\beta(n)}\gamma(n) + 3\sqrt{-\beta(n)}\gamma^2(n)\right]$$
$$+ \gamma(n+1) - \gamma(n) + \gamma^2(n) - 2\beta(n)\gamma(n) + \gamma^3(n) + n^{-\widehat{\alpha}/2}\varepsilon^{(9)}(n) = 0. \tag{32}$$

Now we will try to *guess* the form of the sequence $(\gamma(n))$ so that we could reduce the left-hand side of (32) modulo $n^{-\widehat{\alpha}/2}\varepsilon^{(9)}(n)$. If we put

$$\gamma(n) = \frac{-\varphi(n)}{4\eta(n)} + \delta(n),$$

with $\delta(n) = o(\frac{\varphi(n)}{4\eta(n)})$. Then (32) reads as

$$\pm 2\sqrt{\eta(n)}\delta(n) - \frac{1}{2}\varphi(n) - 2\beta(n)\delta(n) + n^{-\widehat{\alpha}/2}\varepsilon^{(10)}(n) = 0. \tag{33}$$

Looking at the expression (33) it is obvious that if we define

$$\delta(n) = \pm\frac{\varphi(n)}{4\sqrt{\eta(n)}},$$

then we will achieve our goal which is to cancel out the left-hand side of (25) up to $n^{-\widehat{\alpha}/2}\varepsilon^{(i)}(n)$, here again $(\varepsilon^{(i)}(n))$ is a sequence from l^1.

Now we present the asymptotic formulas of the solutions $u_{\pm}(n)$.

THEOREM 3. *Let λ_n and q_n be defined by* (2) *and* (6)–(8). *Assume that condition* (11) *is fulfilled. Then equation* (1) *has two linearly independent solutions $u_-(n)$ and $u_+(n)$ with the asymptotics given by*

$$u_{\pm}(n) = F_{\pm}(n)n^{\frac{\widehat{\alpha}-2\alpha}{4}}\exp\left[\pm\sum_{k=1}^{n-1} f(k)\right](1 + o(1)),$$

where $F_{\pm}(n) \longrightarrow F_{\pm} > 0$, as $n \longrightarrow +\infty$, and $f(k)$ is given in (41).

Proof. Define

$$v(n) := \sqrt{-\beta(n+1)} - \frac{\varphi(n+1)}{4\eta(n+1)} + \frac{A_1}{4(n+1)^{1+\widehat{\alpha}/2}}$$

and put it into equation (17) instead of $X(n)$. Repeating all the previous calculations we get that

$$v(n) - (1+\beta(n))\frac{v(n-1)}{1+v(n-1)} + \beta(n) \tag{34}$$

reads as

$$(A_1 - \widehat{\alpha}\sqrt{-\rho})\frac{\sqrt{-\rho}}{2n^{1+\widehat{\alpha}}} + n^{-\widehat{\alpha}}\varepsilon^{(11)}(n). \tag{35}$$

We used here $\sqrt{\eta(n)} = \sqrt{-\rho}n^{-\widehat{\alpha}/2}(1+o(1))$ and $\varphi(n) = \widehat{\alpha}\rho n^{-1-\widehat{\alpha}}(1+o(1))$, ρ is given by (12). Choosing A_1 appropriately we are able to obtain, for large n,

$$(A_1 - \widehat{\alpha}\sqrt{-\rho})\frac{\sqrt{-\rho}}{2n^{1+\widehat{\alpha}}} + n^{-\widehat{\alpha}}\varepsilon^{(11)}(n) < 0,$$

it is possible because $\varepsilon^{(11)}(n)$ is of the order $o(n^{-1})$.

Now let

$$w(n) := \sqrt{-\beta(n+1)} - \frac{\varphi(n+1)}{4\eta(n+1)} + \frac{A_2}{4(n+1)^{1+\widehat{\alpha}/2}}.$$

Proceeding exactly like before, choosing appropriate A_2, we are able to obtain

$$(A_2 - \widehat{\alpha}\sqrt{-\rho})\frac{\sqrt{-\rho}}{2n^{1+\widehat{\alpha}}} + n^{-\widehat{\alpha}}\varepsilon^{(12)}(n) > 0,$$

for n sufficiently large.

From these considerations we obtain two inequalities

$$v(n) - \frac{(1+\beta(n))v(n-1)}{1+v(n-1)} + \beta(n) \le 0 \tag{36}$$

and

$$w(n) - \frac{(1+\beta(n))w(n-1)}{1+w(n-1)} + \beta(n) \ge 0 \tag{37}$$

which are valid for n large enough. Now applying Theorem 1 we get a solution $X_+(n)$ of equation (17) such that

$$\left| X_+(n) - \sqrt{-\beta(n+1)} + \frac{\varphi(n+1)}{4\eta(n+1)} \right| \le \frac{C_1}{n^{1+\widehat{\alpha}/2}}.$$

Similarly, applying Theorem 2 with

$$v(n) := -\sqrt{-\beta(n+1)} - \frac{\varphi(n+1)}{4\eta(n+1)} + \frac{A_3}{4(n+1)^{1+\widehat{\alpha}/2}}$$

and

$$w(n) := -\sqrt{-\beta(n+1)} - \frac{\varphi(n+1)}{4\eta(n+1)} + \frac{A_4}{4(n+1)^{1+\widehat{\alpha}/2}}$$

we can find a solution $X_{-}(n)$ such that

$$\left| X_{-}(n) + \sqrt{-\beta(n+1)} + \frac{\varphi(n+1)}{4\eta(n+1)} \right| \leq \frac{C_2}{n^{1+\widehat{\alpha}/2}}.$$

Using the facts that

$$\sqrt{-\beta(n+1)} = \sqrt{-\beta(n)} + \varepsilon^{(13)}(n)$$

and

$$\frac{\varphi(n+1)}{4\eta(n+1)} = -\frac{\widehat{\alpha}}{4n} + \varepsilon^{(14)}(n)$$

we have that

$$X_{\pm}(n) = \pm\sqrt{-\beta(n)} + \frac{\widehat{\alpha}}{4n} + \varepsilon^{(15)}(n) \tag{38}$$

forms a pair of solutions of equation (17).

This is only a half way. We want to find asymptotic formulas of base vectors of a space of solutions of (1). In order to do that we need to recall the definition of $X(n)$ (see (16)),

$$\frac{w(n+1)}{w(n)} = 1 + X(n).$$

Dividing both sides of this equality by $w(n)$ gives us

$$w(n+1) = (1 + X(n))w(n),$$

by induction we have

$$w(n) = \prod_{i=1}^{n-1}(1 + X(i)).$$

Applying Lemma 1 and replacing $X(n)$ by $X_{\pm}(n)$ (given by (38)) we obtain two linearly independent solutions $(w_{\pm}(n))_{n\in\mathbb{N}}$ of equation (14),

$$w_{\pm}(n) = F_{1,\pm}(n) \exp\left[\sum_{k=1}^{n-1}\sum_{l=1}^{7} \frac{(-1)^{l-1}}{l} X_{\pm}^{l}(k)\right],$$

here $F_{1,\pm}(n)$ are some real sequences tending to some positive constants. In the above formula there are powers of $X_{\pm}(k)$ up to order 7 because $\left(X_{\pm}^{r}(k)\right)_{k\in\mathbb{N}} \in l^1$ for $r > 7$.

Now using (13) we are able to find $\left\{(u_{-}(n))_{n=1}^{+\infty}, (u_{+}(n))_{n=1}^{+\infty}\right\}$ – a base of solutions of the recurrences (1), namely,

$$u_{\pm}(n) = F_{1,\pm}(n) \prod_{i=1}^{n-1} \frac{\lambda - q_i}{2\lambda_i} \exp\left[\sum_{k=1}^{n-1}\sum_{l=1}^{7} \frac{(-1)^{l-1}}{l} X_{\pm}^{l}(k)\right]. \tag{39}$$

To write more explicit form of $u_\pm(n)$ we need to do some more calculations. Let us first calculate the even powers of $X_\pm(n)$ in (39). From (38) and (24) we get that

$$\begin{aligned}
X_\pm^2(k) &= \eta(k) + \omega_1^2(k) + 2\sqrt{\eta(k)}\omega_1(k) \\
&\quad + 2\sqrt{\eta(k)}\omega_2(k) + 2\sqrt{\eta(k)}\omega_3(k) + 2\omega_1(k)\omega_3(k) + \varepsilon^{(16)}(k), \\
X_\pm^4(k) &= \eta^2(k) + 4\eta^{3/2}(k)\omega_1(k) + 4\eta^{3/2}(k)\omega_3(k) + \varepsilon^{(17)}(k), \\
X_\pm^6(k) &= \eta^3(k) + \varepsilon^{(18)}(k).
\end{aligned}$$

Using the formulas for $\eta(k)$ and $\omega_i(k)$ $(i = 1, 2, 3)$ we have

$$\begin{aligned}
&-\frac{1}{2}X_\pm^2(k) - \frac{1}{4}X_\pm^4(k) - \frac{1}{6}X_\pm^6(k) \\
&\quad = r(k) - s(k) - \frac{\lambda}{k^\alpha} + \frac{1}{2}((s'(k))^2 - (r'(k))^2) + s'(k)s''(k) - r'(k)r''(k) \\
&\qquad + \frac{\lambda}{2k^\alpha}s'(k) - \frac{\alpha}{2k} - \frac{1}{3}((s'(k))^3 - (r'(k))^3) + \varepsilon^{(19)}(k),
\end{aligned}$$

which along with Proposition 1 and the facts that

$$\begin{aligned}
(r(k))^2 &= (r'(k))^2 + 2r'(k)r''(k) + \varepsilon^{(20)}(k), \\
(s(k))^2 &= (s'(k))^2 + 2s'(k)s''(k) + \varepsilon^{(21)}(k),
\end{aligned}$$

and

$$\begin{aligned}
(r(k))^3 &= (r'(k))^3 + \varepsilon^{(22)}(k), \\
(s(k))^3 &= (s'(k))^3 + \varepsilon^{(23)}(k),
\end{aligned}$$

allow us to rewrite the formula (39) as

$$u_\pm(n) = F_{2,\pm}(n)n^{-\alpha/2}\exp\left[\sum_{k=1}^{n-1}\left(X_\pm(k) + \frac{1}{3}X_\pm^3(k) + \frac{1}{5}X_\pm^5(k) + \frac{1}{7}X_\pm^7(k)\right)\right], \tag{40}$$

with $F_{2,\pm}(n) \longrightarrow F_{2,\pm} > 0$, as $n \longrightarrow +\infty$. $F_{2,\pm}(n)$ converge because they are of the form $e^{\sum_1^n \varepsilon(k)}$, where $\varepsilon(k)$ is a sequence from l^1. The odd powers (1, 3, 5, 7) of $X_\pm(k)$ are as follows:

$$\begin{aligned}
X_\pm(k) &= \pm(\eta^{1/2}(k) + \omega_1(k) + \omega_2(k) + \omega_3(k) + \omega_4(k)) + \frac{\widehat{\alpha}}{4k} + \varepsilon^{(24)}(k), \\
X_\pm^3(k) &= \pm(\eta^{3/2}(k) + 3\eta(k)\omega_1(k) + 3\eta(k)\omega_2(k) \\
&\quad + 3\eta(k)\omega_3(k) + 3\eta^{1/2}(k)\omega_1^2(k)) + \varepsilon^{(25)}(k), \\
X_\pm^5(k) &= \pm(\eta^{5/2}(k)5\eta^2(k)\omega_1(k)) + \varepsilon^{(26)}(k), \\
X_\pm^7(k) &= \pm\eta^{7/2}(k) + \varepsilon^{(27)}(k).
\end{aligned}$$

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All the remainders $\varepsilon^{(i)}(n)$ are in l^1 of course. When we put these expressions into (40) we obtain the final form of $u_\pm(n)$. If we define

$$\begin{aligned} f(k) := A(k) &+ \frac{1}{2}A^{-1}(k)B(k) - \frac{1}{2}A^{-1}(k)C(k) + \frac{\alpha}{2k}A^{-1}(k) \\ &- \frac{3\lambda^2}{8k^{2\alpha}}A^{-1}(k) - \frac{1}{8}A^{-3}(k)B^2(k) - \frac{1}{2}A^{-1}(k)D(k) \\ &+ \frac{1}{4}A^{-3}(k)B(k)C(k) + \frac{1}{16}A^{-5}(k)B^3(k) + \frac{1}{3}A^3(k) \\ &+ \frac{1}{2}A(k)B(k) - \frac{1}{2}A(k)C(k) + \frac{1}{8}A^{-1}(k)B^2(k) \\ &+ \frac{1}{5}A^5(k) + \frac{1}{2}A^3(k)B(k) + \frac{1}{7}A^7(k), \end{aligned} \tag{41}$$

where

$$A(k) := \sqrt{2(s'(k) - r'(k)) + \frac{\lambda}{k^\alpha}}, \tag{42}$$

$$B(k) := 2(s''(k) - r''(k)) - 3s^2(k) + 4r(k)s(k) - r^2(k), \tag{43}$$

$$C(k) := \frac{3\lambda}{k^\alpha}s(k) - \frac{2\lambda}{k^\alpha}r(k) - 4s^3(k) + 6r(k)s^2(k) - 2r^2(k)s(k), \tag{44}$$

$$D(k) := \frac{6\lambda}{k^\alpha}r(k)s(k) - \frac{\lambda}{k^\alpha}r^2(k) - \frac{6\lambda}{k^\alpha}s^2(k) + 5s^4(k) - 8r(k)s^3(k) + 3r^2(k)s^2(k). \tag{45}$$

The above formulas complete the proof. □

One can easily check that when (11) is valid, then for k large enough $A(k)$, $B(k)$, $C(k)$ and $D(k)$ (see formulas (42)–(45)) are real and so is $f(k)$. Moreover, if k is large enough, then $f(k)$ is positive. It can be seen from the fact that (11) implies $A(k) > 0$, for large k, and the other terms in formula (41) tend to zero faster than $A(k)$. For instance, $\frac{1}{2}A^{-1}(k)B(k)$ is of the order $\mathcal{O}(k^{\widehat{\alpha}/2-\alpha_{i_0+1}}) + \mathcal{O}(k^{-3\widehat{\alpha}/2})$ which is smaller (from (8)) than the order of $A(k)$.

Formula (41) looks complicated but in special cases it reduces to a much simpler one. For example, let us consider the case when $K = 2$ (see (2)). Assume condition (6) and $\alpha_1 = 1$, $\alpha_2 = 2\alpha$ which implies $\alpha_{i_0+1} = 1$ and conditions (7) and (8) are trivially fulfilled. In this case formula (41) reduces to

$$f(k) = A(k) + \frac{1}{2}A^{-1}(k)B(k) + \frac{\alpha}{2k}A^{-1}(k) - \frac{3\lambda^2}{8k^{2\alpha}}A^{-1}(k) + \frac{1}{3}A^3(k), \tag{46}$$

modulo some l^1 remainder. Moreover, if we use the definitions of $A(k)$ and $B(k)$, then we can rewrite (46) as

$$f(k) = \sqrt{\lambda}k^{-\alpha/2} + \frac{\alpha + 2(b_1 - a_1)}{2\sqrt{\lambda}}k^{\alpha/2-1} - \left(\frac{\lambda^{3/2}}{24} + \frac{\sqrt{\lambda}}{b_2 - a_2}\right)k^{-3\alpha/2}$$

plus some l^1 remainder. The example described above was considered by Janas in **[3]**.

From the construction of the solutions $u_\pm$ we have that $f(k)$ is of the order $\mathcal{O}(k^{-\widehat{\alpha}/2})$, this implies that $\sum_1^n f(k)$ increases to infinity. Because of Lemma 1 and Proposition 1, we have that $F_\pm > 0$. It becomes obvious that $u_-(n)$ decays exponentially, contrary to $u_+(n)$ which grows to infinity, when $n \longrightarrow +\infty$.

4. Asymptotics in the oscillatory case. Throughout this section we assume that λ belongs to a subset of reals such as for some large integer $N_0 = N_0(\lambda)$ the following inequality holds

$$n^{\alpha}\left(\frac{2(a_1-b_1)}{n^{\alpha_1}}+\cdots+\frac{2(a_{i_0}-b_{i_0})}{n^{\alpha_{i_0}}}\right)-\lambda>0, \qquad n\geq N_0. \tag{47}$$

We denote this set by Λ_+. This expression is similar to (11), the difference is the direction of the inequality. Condition (47) implies that for large n the expression $f(n)$ defined by (41) which belongs to $i\mathbb{R}$, because if $\lambda\in\Lambda_+$, then $A(n)$ (defined by (42)) is a square root of a negative real number and $B(n)$, $C(n)$, $D(n)$ (see (43)–(45)) are real numbers. Following the proof of Theorem 4.1 in [**3**] we make an *ansatz* to find asymptotics of solutions of (1). The form of the ansatz comes from Theorem 3 of course.

Let us make some notations.

$$z_n := n^{\gamma}\exp\left[\sum_{k=1}^{n} f(k)\right], \quad \text{and} \quad S_n := \begin{pmatrix} \bar{z}_{n-1} & z_{n-1}\\ \bar{z}_n & z_n\end{pmatrix}, \quad n\in\mathbb{N}, \tag{48}$$

where $f(k)$ is given by (41) and $\gamma := \frac{\widehat{\alpha}-2\alpha}{4}$, the bar over z_n denotes the complex conjugate. It is obvious that $\bar{z}_n = n^{\gamma}\exp(-\sum_{k=1}^{n} f(k))$ because $f(k)\in i\mathbb{R}$ for large k. By B_n we denote the transfer matrix (see (3) and (4))

$$B_n := \begin{pmatrix} 0 & 1\\ -\dfrac{\lambda_{n-1}}{\lambda_n} & \dfrac{\lambda-q_n}{\lambda_n}\end{pmatrix}, \qquad n\in\mathbb{N},$$

here λ_n and q_n are the sequences introduced in previous sections.

We want to prove that

$$S_{n+1}^{-1}B_nS_n = I + R_n, \tag{49}$$

where $(R_n)_{n\in\mathbb{N}}$ is a matrix sequence such that $(||R_n||)_{n\in\mathbb{N}}$ is in l^1. Equation (49) implies that we can rewrite (3) as

$$\begin{aligned}\vec{u}(n+1) &= B_n\cdot\dots\cdot B_2\vec{u}(2) = S_{n+1}S_{n+1}^{-1}B_nS_n\cdot\dots\cdot B_2S_2S_2^{-1}\vec{u}(2)\\ &= S_{n+1}(I+R_n)(I+R_{n-1})\cdot\dots\cdot(I+R_2)\vec{w}(2),\end{aligned}$$

where $\vec{w}(2) := S_2^{-1}\vec{u}(2)$. From the assumption that (R_n) is in l^1 we know that $\vec{w}(n) := \prod_{i=2}^{n}(I+R_i)\vec{w}(2)$ is convergent which gives us that $\vec{u}(n+1) = S_{n+1}\vec{w}(n)$. This reasoning leads us to the fact that any solution of (1), for λ's fulfilling (47), behaves in infinity like a linear combination of z_n and $\bar{z}_n$.

Now we turn to the proof of (49). First we need to calculate the inverse of $\det S_{n+1}$,

$$\begin{aligned}\det S_{n+1} &= \bar{z}_n z_{n+1} - z_n\bar{z}_{n+1}\\ &= n^{\gamma}(n+1)^{\gamma}\exp[f(n+1)] - n^{\gamma}(n+1)^{\gamma}\exp[-f(n+1)].\end{aligned}$$

Using (41), the Taylor expansion of e^t and the above equality we have

$$(\det S_{n+1})^{-1} = n^{-2\gamma}(2\sqrt{\eta(n)})^{-1}(1+o(1)). \tag{50}$$

After some simple calculations we get

$$S_{n+1}^{-1} B_n S_n = (\det S_{n+1})^{-1} \left[\begin{pmatrix} p_n & g_n \\ -\overline{g}_n & -\overline{p}_n \end{pmatrix} + \begin{pmatrix} v_n & t_n \\ -\overline{t}_n & -\overline{v}_n \end{pmatrix} \right], \tag{51}$$

where

$$p_n = |z_n|^2 \left(\frac{\overline{z}_{n-1}}{\overline{z}_n} + \frac{z_{n+1}}{z_n} - 2 \right), \qquad g_n = z_n^2 \left(\frac{z_{n-1}}{z_n} + \frac{z_{n+1}}{z_n} - 2 \right),$$
$$v_n = |z_n|^2 \left(-\Psi(n) - \Phi(n) \frac{\overline{z}_{n-1}}{\overline{z}_n} \right), \; t_n = z_n^2 \left(-\Psi(n) - \Phi(n) \frac{z_{n-1}}{z_n} \right),$$

and

$$\Psi(n) = \frac{\lambda - q_n}{\lambda_n} - 2, \qquad \Phi(n) = 1 - \frac{\lambda_{n-1}}{\lambda_n}.$$

Recalling the forms of λ_n and q_n we may write

$$\begin{aligned} \Psi(n) = {} & \eta(n) + \xi(n) + 2r^2(n) - 2r(n)s(n) - \frac{\lambda}{n^\alpha} r(n) - 2r^3(n) \\ & + 2s(n)r^2(n) + \frac{\lambda}{n^\alpha} r^2(n) + 2r^4(n) - 2r^3(n)s(n) + n^{-\widehat{\alpha}/2} \varepsilon^{(28)}(n), \end{aligned} \tag{52}$$

and

$$\Phi(n) = \frac{\alpha}{n} + n^{-\widehat{\alpha}/2} \varepsilon^{(29)}(n). \tag{53}$$

The sequences $r(n)$, $s(n)$ and $\eta(n)$, $\xi(n)$ are given by (2) and (23). As in previous sections $\varepsilon^{(i)}(n)$ are l^1 remainders.

Looking at formula (51) it is obvious that if we want to prove (49) we need to show

$$(\det S_{n+1})^{-1} (p_n + v_n) = 1 + \varepsilon^{(30)}(n), \quad (\det S_{n+1})^{-1} (-\overline{p}_n - \overline{v}_n) = 1 + \varepsilon^{(31)}(n),$$

and

$$(\det S_{n+1})^{-1} (g_n + t_n) = \varepsilon^{(32)}(n), \qquad (\det S_{n+1})^{-1} (-\overline{g}_n - \overline{t}_n) = \varepsilon^{(33)}(n).$$

Repeating the calculations (formula 4.10) from [**3**] we see that

$$|p_n + v_n - \det S_{n+1}| = |g_n + t_n|.$$

This observation leaves us only one equality to prove, namely

$$(\det S_{n+1})^{-1} (g_n + t_n) = \varepsilon^{(34)}(n). \tag{54}$$

First let us calculate the quotients $\frac{z_{n-1}}{z_n}$ and $\frac{z_{n+1}}{z_n}$. From the definition of z_n we have

$$\frac{z_{n-1}}{z_n} = 1 + \sum_{j=1}^{8} (-1)^j \frac{1}{j!} f^j(n) - \frac{\gamma}{n} + \frac{\gamma}{n} f(n) + n^{-\widehat{\alpha}/2} \varepsilon^{(35)}(n) \tag{55}$$

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and

$$\frac{z_{n+1}}{z_n} = 1 + \frac{\varphi(n)}{2A(n)} + \sum_{j=1}^{8} \frac{1}{j!} f^j(n) + \frac{\gamma}{n} + \frac{\gamma}{n} f(n) + n^{-\widehat{\alpha}/2} \varepsilon^{(36)}(n), \tag{56}$$

where $\varphi(n)$ and $A(n)$ are given by (28) and (42). Formulas (53) and (55) imply

$$\phi(n) \frac{z_{n-1}}{z_n} = \frac{\alpha}{n} - \frac{\alpha}{n} A(n) + n^{-\widehat{\alpha}/2} \varepsilon^{(37)}(n). \tag{57}$$

Here we omit some lengthy calculations which are quite simple but very tedious. Combining (52), (55)–(57) and we obtain

$$(g_n + t_n) = n^{2\gamma - \widehat{\alpha}/2} \varepsilon^{(38)}(n)$$

which along with the fact that $(\det S_{n+1})^{-1} = \mathcal{O}(n^{-2\gamma + \widehat{\alpha}/2})$ (see (50)) proves (54). With this sentence we complete the proof of the following theorem.

THEOREM 4. *Let* λ_n *and* q_n *be defined by* (2) *and* (6)–(8). *Assume that condition* (47) *is fulfilled. Then equation* (1) *has two linearly independent solutions* $u_-(n)$ *and* $u_+(n)$ *with the asymptotics given by*

$$u_\pm(n) = n^{\frac{\widehat{\alpha} - 2\alpha}{4}} \exp\left[\pm \sum_{k=1}^{n-1} f(k) \right] (1 + o(1)),$$

where $f(k)$ *is as in* (41).

In this theorem, the main part of the asymptotic formulas of the solutions $u_\pm(n)$ looks exactly the same as in Theorem 3. The difference is hidden in the sequence $f(k)$, particularly in its ‘main’ part $A(k) = \sqrt{\eta(k)}$. In the case (Theorem 3) when condition (11) is valid, $f(k)$ is real for k large enough. Assumptions (47) imply that all of $f(k)$, except of a finite number, are complex and have zero real parts, so for large n we have $u_-(n) = \overline{u_+(n)}$.

5. Applications to the spectral theory of Jacobi operators. Let $l^2(\mathbb{N}; \mathbb{C})$ be the Hilbert space of all complex sequences $x = (x(n))_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{+\infty} |x(n)|^2 < +\infty$. Let $\mathcal{J}$ be a Jacobi operator defined in $l^2(\mathbb{N}; \mathbb{C})$ by

$$\begin{aligned} (\mathcal{J}u)(n) &= \lambda_{n-1} u(n-1) + q_n u(n) + \lambda_n u(n+1), \qquad n > 2, \\ (\mathcal{J}u)(1) &= q_1 u(1) + \lambda_1 u(2). \end{aligned}$$

$\mathcal{J}$ acts on its maximal domain

$$D(\mathcal{J}) = \left\{ u \in l^2(\mathbb{N}; \mathbb{C}) : \mathcal{J}u \in l^2(\mathbb{N}; \mathbb{C}) \right\}.$$

We call equation (1) the *generalized eigenequation* of $\mathcal{J}$. If λ_n and q_n are defined by (2), then by the Carleman condition ($\sum \frac{1}{\lambda_n} = +\infty$) we have $\mathcal{J} = \mathcal{J}^*$, so we can apply the Gilbert–Khan–Pearson *subordination theory* [7]. For the reader convenience we recall the notion of subordination.

DEFINITION 1. A (non-trivial) solution $(u(n))_{n\in\mathbb{N}}$ of the recurrence relations (1) is said to be subordinated if and only if

$$\lim_{N\to+\infty}\frac{\sqrt{\sum_{n=1}^{N}|u(n)|^2}}{\sqrt{\sum_{n=1}^{N}|v(n)|^2}}=0,$$

for any solution $(v(n))_{n\in\mathbb{N}}$ of (1) not a constant multiple of $(u(n))_{n\in\mathbb{N}}$.

For example, every non-trivial l^2 solution of (1) is subordinated. From constancy of the Wronskian $W(u, v)(n) := \lambda_n(u(n)v(n+1) - v(n)u(n+1))$, we know that there can be at most only one linearly independent l^2 solution. By Theorem 3 in **[7]** we have that a real number λ belongs to the absolutely continuous spectrum of the operator $\mathcal{J}$, if there are no subordinated solutions of its generalized eigenequation. According to this theory we know that from the behaviour of solutions of (1), called *generalized eigenvectors*, we may obtain some spectral properties of $\mathcal{J}$. Theorems 3 and 4 tell us what happen in infinity with the generalized eigenvectors $(u_\pm(n;\lambda))_{n\in\mathbb{N}}$. Here, we see that the sequence $(\sqrt{\eta(n;\lambda)})$ defined by (23) plays the major role (see also formulas (11) and (47)). If for $\lambda \in (a, b)$ its values (for large n) are strictly complex with zero real part (such solutions cannot be subordinated), then (a, b) is in the absolutely continuous part of the spectrum of $\mathcal{J}$, contrary if $\sqrt{\eta(n;\lambda)} \in \mathbb{R}$, then we might have some eigenvalues in (a, b). We sketch this briefly below in few cases.

In all further cases we assume that λ_n and q_n are like in (2), ρ is defined by (12) and conditions (6)–(8) are fulfilled.

Case 1. Let $\widehat{\alpha} = \alpha_1 < \alpha$ and $\rho = 2(a_1 - b_1) < 0$ then condition (11) is valid for all real λ. Applying Theorem 3 we see that $u_-(n;\lambda)$ behaves like $n^{(\alpha_1-2\alpha)/4}\exp(\frac{-\sqrt{-\rho}}{1-\frac{\alpha_1}{2}}n^{1-\frac{\alpha_1}{2}})$, because $\rho < 0$ and $1 - \frac{\alpha_1}{2} > 0$ we obtain $(u_-(n;\lambda)) \in l^2$, hence it is subordinated. We see that the spectrum of $\mathcal{J}$ is pure point (from the subordination theory we know that the continuous part of the spectrum is empty).

Case 2. Now, let $\widehat{\alpha} = \alpha_1 < \alpha$ and $\rho = 2(a_1 - b_1) > 0$. In this case we have to apply Theorem 4 because for any real number λ condition (47) is valid. This time for n large enough $\sqrt{\eta(n;\lambda)} \in i\mathbb{R}$, which makes $u_-(n;\lambda)$ as the conjugate of $u_+(n;\lambda)$. The latter implies that the spectrum of $\mathcal{J}$ is purely absolutely continuous.

From these two examples we can see that $\rho = 0$ is, in some sense, a critical point. If ρ changes from positive to negative, then the spectrum of $\mathcal{J}$ goes from absolutely continuous to pure point. This is a new situation which is impossible for Jacobi operators considered by Janas in **[3]**. If $\lambda_n = n^\alpha(1 + \frac{a}{n} + \frac{D}{n^{2\alpha}} + \frac{V(n)}{n})$ and $q_n = -2n^\alpha(1 + \frac{b}{n} + \frac{E}{n^{2\alpha}} + \frac{W(n)}{n})$, like in **[3]**, then we have $\rho = -\lambda$ and $\widehat{\alpha} = \alpha$. In this case for $\lambda \in (0, +\infty)$ condition (11) is fulfilled, if $\lambda \in (-\infty, 0)$ then (47) is true. Applying Theorem 3 or Theorem 4 and subordination theory we obtain that $(-\infty, 0)$ is in the absolutely continuous part of the spectrum of the operator $\mathcal{J}$ and in $(0, +\infty)$ we may have some eigenvalues.

Do we always have this kind of spectrum when $\rho = 0$? If $\rho = 2(a_1 - b_1) = 0$ but $a_2 \neq b_2$ and α_2 is still less than α, then again we are in Case 1 or Case 2. A new situation appears when $a_i = b_i$ for all $i = 1, \ldots, i_0$, or equivalently when $\alpha \leq \alpha_1$, this case is discussed as follows:

Case 3. When $\widehat{\alpha} = \alpha = \alpha_1$ and, of course $a_1 \neq b_1$ then we can rewrite the conditions (11) and (47) as $2(a_1 - b_1) - \lambda < 0$ and $2(a_1 - b_1) - \lambda > 0$. Depending on the sign of the expression $2(a_1 - b_1) - \lambda$ we can apply Theorem 3 or Theorem 4 and conclude that if λ lies on the right from $2(a_1 - b_1)$, then it might be an eigenvalue but all the λ's smaller than $2(a_1 - b_1)$ are in the absolutely continuous part of the spectrum of the considered operator. More details will be given in another work.

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