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THE DISTRIBUTION OF FINITE VALUES OF MEROMORPHIC FUNCTIONS WITH DEFICIENT POLES

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Abstract A result is presented giving conditions on a set of open discs in the complex plane that ensure that a transcendental meromorphic function with Nevanlinna deficient poles omits at most one finite value outside the set of discs. This improves a previous result of Langley, and goes some way towards closing a gap between Langley's result and a theorem of Toppila in which the omitted values considered may include ∞.

Keywords: exceptional set; value distribution; Picard set; Nevanlinna theory

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1. Introduction

Picard's theorem states that a non-constant function f that is meromorphic in the complex plane $\mathbb{C}$ omits at most two values of the extended complex plane $\mathbb{C} \cup \{\infty\}$. The example $f(z) = \mathrm{e}^z$, which omits the values 0 and ∞, shows that this is best possible.

Lehto [**5**] introduced the concept of a *Picard set*. A subset E of the plane is a Picard set for a family $\mathcal{F}$ of functions meromorphic in $\mathbb{C}$ if every transcendental f in $\mathcal{F}$ takes every value in the extended complex plane, with at most two exceptions, infinitely often in $\mathbb{C} \setminus E$.

Thus, for example, the set $E = \{2\mathrm{i}n\pi : n \in \mathbb{Z}\}$ is *not* a Picard set for the family of entire functions, because the function $f(z) = \mathrm{e}^z$ fails to take any of the three values 0, 1, ∞ on $\mathbb{C} \setminus E$.

For the family of entire functions, Toppila proved [**8**] that a countable set of points $E = \{a_m\}_{m=1}^{\infty}$, where the a_m converge to ∞, is a Picard set if there exists $\varepsilon > 0$ such that the a_m satisfy

$$|a_m - a_{m'}| > \frac{\varepsilon |a_m|}{\log |a_m|}, \quad m \neq m'. \tag{1.1}$$

In the same paper he gives an example showing that this condition is sharp.

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Baker and Liverpool [**1**] proved further that if the set $E = \{a_m\}$ above satisfies (1.1), then there exists a sequence of small radii $d_m \to 0$ such that the countable union of open discs

$$\bigcup_{m=1}^{\infty} \{z : |z - a_m| < d_m\}$$

forms a Picard set for the family of entire functions.

For the family of meromorphic functions, Toppila proved in [**6**] that if $E = \{a_m\}$ is a countable set of points converging to infinity, which satisfy

$$|a_m|^2 = O(|a_{m+1}|),$$

then E is a Picard set. He also gave an example showing that this is essentially best possible.

In clear contrast to the case for entire functions, Toppila showed in [**7**] that no countable union of open discs tending to infinity can be a Picard set for the family of meromorphic functions.

We recall some of the standard concepts and definitions of Nevanlinna theory, a standard reference for which is Hayman's book [**2**]. For a non-constant meromorphic function f we define $n(r, f)$ to be the number of poles of f in $|z| \leqslant r$, where a pole of multiplicity p is counted p times. We then define

$$N(r, f) = \int_0^r \frac{n(t, f)}{t} \, \mathrm{d}t.$$

We also define

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \max(\log |f(re^{\mathrm{i}\theta})|, 0) \, \mathrm{d}\theta$$

and set

$$T(r, f) = m(r, f) + N(r, f).$$

We write $N(r, a, f) = N(r, 1/(f - a))$ and define $m(r, a, f)$ similarly. We also sometimes write $N(r, \infty, f)$ for $N(r, f)$.

Nevanlinna's first fundamental theorem states that, for any fixed complex number a, we have

$$T(r, f) = N(r, a, f) + m(r, a, f) + O(1).$$

We define the (Nevanlinna) *deficiency* of a value a as

$$\delta(a, f) = \liminf_{r \to \infty} \frac{m(r, a, f)}{T(r, f)} = 1 - \limsup_{r \to \infty} \frac{N(r, a, f)}{T(r, f)}.$$

Nevanlinna's second fundamental theorem states that the number of values a for which $\delta(a, f) > 0$ is countable and that

$$\sum_{a \in \mathbb{C}} \delta(a, f) \leqslant 2.$$

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(The theorem is actually stronger than this (see [**2**] for details) but the above result is all that we require for our present purposes.)

We observe that the second fundamental theorem includes Picard's theorem. If f omits a value a, then $N(r, a, f) = 0$ for all r and therefore $\delta(a, f) = 1$; the second fundamental theorem states that this cannot happen for more than two such values. In some sense, therefore, the deficiency of a value measures the extent to which that value is taken less often than other values.

For the family $\mathcal{F}$ of meromorphic functions f that have deficient poles, i.e.

$$\delta(\infty, f) > 0,$$

Toppila [**9**] proved the following theorem.

Theorem A. *Let $\{a_m\}_{m=1}^{\infty}$ be a sequence of complex numbers with*

$$\lim_{m\to\infty} a_m = \infty$$

and $|a_1| > \mathrm{e}$ and such that, for some

$$0 < \alpha < 1, \quad \beta > 2\alpha,$$

we have

$$|a_m - a_{m'}| > \frac{|a_m|}{(\log |a_m|)^{\alpha}} \tag{1.2}$$

for all $m \neq m'$. If radii d_m are given by

$$\log 1/d_m = (\log |a_m|)^{2+\beta}, \tag{1.3}$$

then the set

$$E = \bigcup_{m=1}^{\infty} B(a_m, d_m) = \bigcup_{m=1}^{\infty} \{z : |z - a_m| < d_m\}$$

is a Picard set for $\mathcal{F}$.

He also gives an example showing that this result is essentially best possible.

This result shows that, for any transcendental $f \in \mathcal{F}$, the preimage of at most two values in the extended complex plane $\mathbb{C} \cup \{\infty\}$ may be contained in the set E.

The fact that any such f has deficient poles means that at most one finite value may be omitted in the whole plane, by Nevanlinna's second fundamental theorem.

This suggests the following question. If two exceptional values do exist for a given $f \in \mathcal{F}$, may they both be finite or must one be the deficient value ∞?

In this direction, Langley [**4**] has proved the following theorem.

Theorem B. *Let $\{a_m\}_{m=1}^{\infty}$ be a sequence of complex numbers with*

$$\lim_{m\to\infty} a_m = \infty$$

and

$$|a_m - a_{m'}| > \varepsilon |a_m|, \quad m \neq m',$$

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for some $0 < \varepsilon < \frac{1}{2}$. *Then there exists* $K = K(\varepsilon) > 0$ *such that, if radii* d_m *are given by*

$$\log 1/d_m > K(\log |a_m|)^2$$

and

$$E = \bigcup_{m=1}^{\infty} B(a_m, d_m),$$

then every transcendental $f \in \mathcal{F}$ *takes every finite value, with at most one exception, infinitely often in* $\mathbb{C} \setminus E$.

The points a_m are further apart in Langley's result than in Toppila's. In this paper we go some way towards closing the gap between these two results. We prove the following theorem.

Theorem 1.1. *Let* $\{a_m\}_{m=1}^{\infty}$ *be a sequence of complex numbers with*

$$\lim_{m\to\infty} a_m = \infty$$

and, for some

$$0 < \alpha < \tfrac{1}{4}, \quad \beta > 4\alpha, \tag{1.4}$$

let the a_m *satisfy the spacing condition (1.2) and let radii* d_m *satisfy condition (1.3). If*

$$E = \bigcup_{m=1}^{\infty} B(a_m, d_m),$$

then every transcendental $f \in \mathcal{F}$ *takes every finite value, with at most one exception, infinitely often in* $\mathbb{C} \setminus E$.

The tighter constraints on α and β mean that we have not closed the gap completely between Langley's result and Toppila's. The question of whether both the omitted values permitted by Toppila's result may be finite, when $\frac{1}{4} \leqslant \alpha < 1$ or when $2\alpha < \beta \leqslant 4\alpha$, remains open.

The restrictions on α and β may be relaxed when the a_m lie on a ray. We have the following theorem.

Theorem 1.2. *Let* $\{a_m\}_{m=1}^{\infty}$ *be a sequence of positive real numbers with*

$$\lim_{m\to\infty} a_m = \infty.$$

Then Theorem 1.1 holds with condition (1.4) replaced by

$$0 < \alpha < \tfrac{1}{2}, \quad \beta > 2\alpha.$$

The proof of Theorem 1.2 is omitted as the method is the same as for Theorem 1.1.

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2. Preliminaries

The following is a modification by Langley [**4**] of an argument of Toppila.

Lemma C. *Let* $0 < t < s < r$ *and assume that*

$$s_j > 0, \quad t < |b_j| - s_j < |b_j| + s_j < s$$

for $j = 1, \dots, M$. *Set*

$$\Omega = \{z : t < |z| < r\} - \bigcup_{j=1}^{M} E_j,$$

where E_j *is the closed disc* $\{z : |z - b_j| \leqslant s_j\}$. *Let* u *be subharmonic and non-positive on* Ω *and continuous on the closure of* Ω, *and let* $v(z)$ *be the Poisson integral*

$$v(z) = \frac{1}{2\pi} \int_0^{2\pi} -u(re^{i\theta}) \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \, d\theta$$

of $-u$ *in* $B(0, r)$. *Then for* z *in* Ω *we have*

$$u(z) \leqslant -v(z) + C(z) m_0(r, -u) \leqslant \left(\frac{|z| - r}{|z| + r} + C(z) \right) m_0(r, -u), \tag{2.1}$$

in which

$$m_0(r, -u) = \frac{1}{2\pi} \int_0^{2\pi} -u(re^{i\theta}) \, d\theta$$

and

$$C(z) = \frac{1 + t/r}{1 - t/r} \frac{\log(r/|z|)}{\log(r/t)} + \frac{1 + s/r}{1 - s/r} \sum_{j=1}^{M} \frac{\log(2r/|z - b_j|)}{\log(2r/s_j)}. \tag{2.2}$$

3. Proof of Theorem 1.1

We follow Langley's method [**4**]. The improvement obtained in the result stems primarily from a refinement of the conditions applied at (3.4), (3.5) and (3.12).

Let the a_m, d_m and α, β be as in the statement of the theorem. Suppose that there exists an f that is transcendental and meromorphic with

$$\delta = \delta(\infty, f) > 0 \tag{3.1}$$

and which has (without loss of generality) all but finitely many of its zeros and 1-points in $E = \bigcup_{m=1}^{\infty} B(a_m, d_m)$.

We set

$$g = \frac{f - 1}{f} = 1 - \frac{1}{f},$$

so that

$$\delta(1, g) = \delta > 0 \tag{3.2}$$

and all but finitely many of g's poles and zeros lie in E. Throughout the proof, C_j will denote positive constants.

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Lemma 3.1. *Choose constants k, l with*

$$\mathrm{e} < k < l < \mathrm{e}^{9/8}.$$

There exists a constant $c > 0$ and a sequence $r_n \to \infty$ with

$$kr_n < r_{n+1} < lr_n \tag{3.3}$$

such that, for each $m, n \in \mathbb{N}$,

$$B\biggl(a_m, \frac{c|a_m|}{(\log|a_m|)^{2\alpha}}\biggr) \cap B_n = \emptyset, \tag{3.4}$$

where

$$B_n = \biggl\{z : r_n - \frac{cr_n}{(\log r_n)^{2\alpha}} < |z| < r_n + \frac{cr_n}{(\log r_n)^{2\alpha}}\biggr\}. \tag{3.5}$$

Proof. The proof is by induction on n. Given r_n, using the spacing condition (1.2) we see that the annulus $A_n = \{z : kr_n < |z| < lr_n\}$ contains at most

$$C_1(\log r_n)^{2\alpha}$$

of the a_m. We can then find r_{n+1} and an annulus B_{n+1} of width at least

$$\frac{C_2 r_n}{(\log r_n)^{2\alpha}}$$

that satisfy (3.3), (3.4) and (3.5). This concludes the proof of the lemma. □

Lemma 3.2. *Let γ, ε be positive constants with ε/c and γ/ε small, where c is as in Lemma 3.1. Then for each large n there exist S_n, S_n' with*

$$\biggl(1 + \frac{\gamma}{(\log r_n)^{2\alpha}}\biggr)r_n < S_n < \biggl(1 + \frac{\varepsilon}{(\log r_n)^{2\alpha}}\biggr)r_n \tag{3.6}$$

and

$$S_n' = S_n + 1/T(S_n, g) < r_n + \frac{cr_n}{(\log r_n)^{2\alpha}}$$

such that

$$\left.\begin{aligned} T(S_n', g) &< 2T(S_n, g),\\ m(S_n, g'/g(g-1)) &< C_3 \log(S_n T(S_n, g)). \end{aligned}\right\} \tag{3.7}$$

Proof. Such S_n and S_n' exist, by [**2**, p. 38] applied to the function $\phi(s) = T(\mathrm{e}^s, g)$, and the lemma is proved. □

Lemma 3.3. *There exist positive constants C_4, C_5 such that, for large n, we have*

$$|g'(z)/g(z)| \leqslant C_4 S_n T(S_n, g)^3 \tag{3.8}$$

for all z satisfying

$$C_5 \leqslant |z| \leqslant S_n, \quad z \notin \bigcup_{m=1}^{\infty} B(a_m, 1).$$

Proof. We fix a large integer L and assume that n is large compared with L. We have, using (3.6),

$$\begin{aligned} T(S_n, g) &\geqslant n(r_n, g)\log(S_n/r_n) \\ &\geqslant \frac{\gamma}{2(\log r_n)^{2\alpha}} n(r_n, g) \end{aligned} \tag{3.9}$$

and similarly for $n(r_n, 1/g)$ so that (since by construction there are no zeros or poles of g in $r_n < |z| < S'_n$)

$$n(S'_n, g) + n(S'_n, 1/g) = n(r_n, g) + n(r_n, 1/g) < C_6(\log r_n)^{2\alpha} T(S_n, g).$$

Now an application of the differentiated Poisson–Jensen formula (see, for example, [**2**, p. 22]) in $B(0, S'_n)$ gives

$$|g'(z)/g(z)| \leqslant C_4 S_n T(S_n, g)^3$$

as long as $C_5 \leqslant |z| \leqslant S_n$ and $|z - a_m| \geqslant 1$ for all m. The proof of the lemma is therefore complete. □

Lemma 3.4. *For large enough n we have*

$$m(S_n, g/g') > (\delta/2)T(S_n, g). \tag{3.10}$$

Proof. We have

$$\frac{1}{g-1} = \frac{g'}{g(g-1)}\frac{g}{g'}.$$

Now (3.2) and (3.7) give (3.10) for large enough n. Lemma 3.4 is proved. □

Lemma 3.5. *Let ε_1 be small and positive, in particular with $\varepsilon_1 < \frac{1}{2}c$, where c is as in Lemma 3.1. There then exists $C_7 > 0$ such that, for all large n, we have*

$$\log|g(z) - 1| < -C_7 \delta T(r_n, g)$$

for all z satisfying

$$r_{n-1} \leqslant |z| \leqslant r_n, \quad z \notin \bigcup_{m=1}^{\infty} B\left(a_m, \frac{\varepsilon_1 |a_m|}{(\log|a_m|)^{2\alpha}}\right).$$

Proof. We apply Lemma C to the function

$$u(z) = \log|g'(z)/g(z)| - \log[C_4 S_{n+1} T(S_{n+1}, g)^3] \tag{3.11}$$

with $r = S_{n+1}$, $t = r_{n-L}$ and $s = r_{n+1}$ and with the $B(b_j, s_j)$ those discs $B(a_m, 1)$ for which $t < |a_m| < r$.

We take z to satisfy

$$r_{n-1} \leqslant |z| \leqslant r_n, \qquad |z - a_m| \geqslant \frac{\varepsilon_1 |a_m|}{(\log|a_m|)^{2\alpha}} \tag{3.12}$$

for all m.

We then have

$$\frac{|z|-r}{|z|+r} \leqslant \frac{-(k-1)}{k+1} \equiv -\tau < 0. \tag{3.13}$$

Furthermore, using (3.6),

$$\frac{1+s/r}{1-s/r} < \frac{3}{\gamma}(\log r_{n+1})^{2\alpha}. \tag{3.14}$$

The number of a_m between $|z| = r_{n-L}$ and $|z| = r_{n+1}$ is at most

$$C_8 l^{2L+2}(\log r_{n-L})^{2\alpha}. \tag{3.15}$$

For each a_m with $r_{n-1} < |a_m| < r_n$, we have, using (3.12),

$$\frac{\log(2S_{n+1}/|z-a_m|)}{\log(2S_{n+1})} \leqslant \frac{C_9 + 2\alpha\log\log r_{n-1}}{\log 2S_{n+1}}. \tag{3.16}$$

Also,

$$\frac{1+r_{n-L}/S_{n+1}}{1-r_{n-L}/S_{n+1}} \frac{\log(S_{n+1}/|z|)}{\log(S_{n+1}/r_{n-L})} < \frac{C_{10}}{L}. \tag{3.17}$$

Therefore, in (2.2) we have, using (3.14)–(3.17),

$$\begin{aligned} 0 < C(z) &\leqslant \frac{C_{10}}{L} + \frac{C_{11}\log\log r_{n-1}}{(\log S_{n+1})^{1-4\alpha}} \\ &< \tfrac{1}{2}\tau \end{aligned} \tag{3.18}$$

for large enough L, n since $\alpha < \frac{1}{4}$.

Now (3.13) and (3.18) give that

$$\frac{|z|-r}{|z|+r} + C(z) < -\tfrac{1}{2}\tau. \tag{3.19}$$

Also, we obtain, from (3.8) and (3.11), that

$$m_0(r, -u) \geqslant m(r, g/g')$$

and this, together with (2.1) and (3.19), gives

$$\log|g'(z)/g(z)| - \log(C_4 S_{n+1} T(S_{n+1}, g)^3) \leqslant -(\tfrac{1}{2}\tau) m(S_{n+1}, g/g').$$

Therefore, using (3.10), we obtain

$$\log|g'(z)/g(z)| \leqslant -C_{12}\delta T(r_n, g) \tag{3.20}$$

for n large enough and z as in (3.12). Using (3.2) to find a point where g is close to 1, and integrating (3.20) along a path that begins at that point and lies away from the a_m, we obtain

$$|\log g(z)| \leqslant \exp[-C_{13}\delta T(r_n, g)],$$

and therefore

$$\log|g(z) - 1| < -C_7\delta T(r_n, g). \tag{3.21}$$

The proof of Lemma 3.5 is therefore complete. □

Lemma 3.6. *g has the same number of zeros as poles inside each $B(a_m, d_m)$, for large enough m.*

Proof. This follows from (3.21) and the 'argument principle', since $g(z)$ stays close to 1 as z traverses the circle

$$\left\{z : |z - a_m| = \frac{\varepsilon_1 |a_m|}{(\log a_m)^{2\alpha}}\right\}$$

and all of g's large zeros and poles are inside the smaller discs $B(a_m, d_m)$. Lemma 3.6 is proved. □

Lemma 3.7. *There exists $\rho > 0$ such that*

$$T(r, g) < r^{\rho} \tag{3.22}$$

for all large enough r.

Proof. For large enough n,

$$T(r_n, g) \leqslant \frac{1}{C_7 \delta} m\left(r_{n-1}, \frac{1}{g-1}\right) \leqslant C_{14} \delta^{-1} T(r_{n-1}, g)$$

from (3.21), and so we can find ρ such that (3.22) holds for all sufficiently large r. The proof of the lemma 3.7 is complete. □

Lemma 3.8. *We have $g(z) = 1 + o(1)$ as $z \to \infty$ outside the union of the discs $B(a_m, \sqrt{d_m})$.*

Proof. Lemma 3.5 shows that this result holds outside the discs

$$B\left(a_m, \frac{\varepsilon_1 |a_m|}{(\log |a_m|)^{2\alpha}}\right),$$

so it suffices to prove that $g(z) = 1 + o(1)$ for

$$\sqrt{d_m} \leqslant |z - a_m| \leqslant \frac{\varepsilon_1 |a_m|}{(\log a_m)^{2\alpha}}. \tag{3.23}$$

This is proved as in [**4**]. We factor out the zeros and poles of g in

$$B\left(a_m, \frac{\varepsilon_1 |a_m|}{(\log a_m)^{2\alpha}}\right).$$

Since these are equal in number, from Lemma 3.6, and lie in $B(a_m, d_m)$, the result follows from the maximum principle (using Lemma 3.7). Lemma 3.8 is therefore proved. □

This shows in particular that $|f(z)|$ is large for large z outside the $B(a_m, \sqrt{d_m})$.

Lemma 3.9. *$T(r,f) = o((\log r)^2)$ as $r \to \infty$.*

Proof. We take n large and apply Lemma C with

$$r = r_n, \quad t = r_{n'}, \quad r^{1/100} < t < r^{1/70}, \quad s = r - \frac{cr}{2(\log r)^{2\alpha}}, \quad u(z) = -\log|f(z)|,$$

where c is the constant in Lemma 3.1, and with the $B(b_j, s_j)$ those $B(a_m, \sqrt{d_m})$ for which $t < |a_m| < r$.

Then

$$\frac{1+s/r}{1-s/r} < C_{14}(\log r)^{2\alpha}. \tag{3.24}$$

We have that $u(z) \leqslant 0$ for z outside the $B(b_j, s_j)$ since f is large there, by Lemma 3.8.

For $|z| = r_{n-1}$, we have

$$\frac{1+t/r}{1-t/r}\,\frac{\log r/|z|}{\log r/t} < \frac{8}{7\log r} \tag{3.25}$$

using the fact that $r_n/r_{n-1} < \mathrm{e}^{9/8}$, and

$$\log\frac{2r}{|z-b_j|} \leqslant C_{15}\alpha\log\log r \tag{3.26}$$

using (3.5) for r_{n-1}. Also,

$$\log\frac{2r}{s_j} \geqslant \log\frac{2r}{\sqrt{d_{n'}}} \tag{3.27}$$

for every j, since $|b_j| \geqslant t$.

The maximum number of a_m in the annulus $t < |z| < r$ is no more than

$$C_{16}(\log r)^{1+2\alpha}. \tag{3.28}$$

So, using (1.3) and (3.24)–(3.28), Lemma C gives, for $|z| = r_{n-1}$,

$$\begin{aligned} -\log|f(z)| = u(z) &\leqslant -v(z) + m(r,f)\left[\frac{8}{7\log r} + C_{17}(\log r)^{1+4\alpha}\frac{\alpha\log\log r}{(\log t)^{2+\beta}}\right] \\ &\leqslant -v(z) + m(r,f)\frac{7}{6\log r} \end{aligned}$$

for large enough n, using the fact that $\beta > 4\alpha$.

But v is harmonic in $B(0,r)$ with $v(0) = m_0(r_n, -u) = m(r_n, f)$, and so integrating round $|z| = r_{n-1}$ we obtain

$$-m(r_{n-1}, f) \leqslant m(r_n, f)\left[-1 + \frac{7}{6\log r_n}\right]$$

and therefore

$$\frac{m(r_n,f)}{m(r_{n-1},f)} \leqslant 1 + \frac{6}{5\log r_n} \leqslant 1 + \frac{5}{4n}$$

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for large enough n, and so

$$\log m(r_n, f) \leqslant O(1) + \tfrac{3}{2}\log n \leqslant O(1) + \tfrac{3}{2}\log\log r_n$$

so that, by (3.1),

$$T(r_n, f) \leqslant \frac{2}{\delta} m(r_n, f) = o((\log r_n)^2).$$

Now, for any large r we have $r_{n-1} < r \leqslant r_n$ for some n, and therefore

$$T(r, f) = o((\log r)^2).$$

And the proof of Lemma 3.9 is complete. □

In particular, we note that Lemma 3.9 implies that

$$n(r, f) + n(r, 1/f) = o(\log r). \tag{3.29}$$

Lemma 3.10. *Let $0 < \sigma < \varepsilon_1$ and let m be large. Then f has at least as many poles as zeros, counting multiplicity, in $B(a_m, (\sigma|a_m|/(\log|a_m|)^{2\alpha}))$.*

Proof. Set

$$h(z) = f(z)\prod_{\mu=1}^{p}(z - z_\mu)^{-1}\prod_{\nu=1}^{q}(z - w_\nu), \tag{3.30}$$

where the z_μ, $1 \leqslant \mu \leqslant p$, are the zeros and the w_ν, $1 \leqslant \nu \leqslant q$, are the poles of f in

$$B\biggl(a_m, \frac{\sigma|a_m|}{(\log|a_m|)^{2\alpha}}\biggr).$$

Then h is analytic and non-zero in

$$B\biggl(a_m, \frac{\sigma|a_m|}{(\log|a_m|)^{2\alpha}}\biggr).$$

We have

$$\begin{aligned} T(4|a_m|, h) &\leqslant T(4|a_m|, f) + O(n(2|a_m|, f) + n(2|a_m|, 1/f))\log|a_m| \\ &= o((\log|a_m|)^2), \end{aligned} \tag{3.31}$$

by Lemma 3.9 and (3.29).

We apply the Poisson–Jensen formula to h in $|z| < 2|a_m|$. This gives, using (3.31),

$$\log|h(z)| = o((\log|a_m|)^2) \tag{3.32}$$

for $z \in B(a_m, 1)$.

We choose z with

$$\sqrt{d_m} \leqslant |z - a_m| \leqslant 4\sqrt{d_m} \quad \text{and} \quad |z - w_\nu| \geqslant \sqrt{d_m}/q$$

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for each ν. Then Lemma 3.8 and Lemma 3.9 and (3.30) and (3.32) give

$$\begin{aligned}0 &\leqslant \log|f(z)| \\ &\leqslant \log|h(z)| + p\log|4\sqrt{d_m} + d_m| - q\log\sqrt{d_m}/q \\ &\leqslant o((\log|a_m|)^2) + (p-q)\log\sqrt{d_m}.\end{aligned}$$

Now

$$\log\sqrt{d_m} = \tfrac{1}{2}(\log|a_m|)^{2+\beta}$$

and in particular

$$-\log\sqrt{d_m} \neq o((\log|a_m|)^2)$$

and we therefore conclude that $p \leqslant q$. Lemma 3.10 is proved. □

Lemma 3.11. *For large n we have*

$$N(r_n, 1/f) \leqslant (1+o(1))N(r_n, f).$$

Proof. By Lemma 3.8, f has infinitely many zeros. If m is large and $|a_m| < r_n$, then Lemma 3.10 shows that to each zero z_μ of f in $B(a_m, d_m)$ there corresponds a pole w_ν of f with

$$w_\nu = z_\mu(1+o(1)), \qquad \log\frac{r_n}{|z_\mu|} \leqslant \log\frac{r_n}{|w_\nu|} + o(1).$$

This gives

$$N(r_n, 1/f) \leqslant N(r_n, f) + O(\log r_n) + o(n(r_n, 1/f)) = N(r_n, f) + O(\log r_n)$$

and Lemma 3.11 follows. □

But now we can complete the proof of the theorem. Since f is large on $|z| = r_n$, by Lemma 3.8, we have, for large n,

$$T(r_n, f) = N(r_n, 1/f) + O(1) \leqslant (1+o(1))N(r_n, f) \leqslant (1-\delta/2)T(r_n, f),$$

which is a contradiction. Theorem 1.1 is proved.

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