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HIGHER DERIVATIONS AND DISTINGUISHED SUBFIELDS

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1. Introduction. Let L be a finitely generated extension of a field K having characteristic $p \neq 0$. A rank t higher derivation on L over K is a sequence

$$d = \{d_i | 0 \leqq i < t + 1\}$$

of additive maps of K into K such that

$$d_r(ab) = \sum \{d_i(a)d_j(b) \,|\, i + j = r\},$$

d_0 is the identity map and $d_i(x) = 0$, $i > 0$, $x \in K$. [**6**] contains the relevant background material on higher derivations. By Zorn's Lemma, there are maximal separable extensions of K in L. A maximal separable extension D of K in L is called distinguished if

$$L \subseteq K^{p^{-n}}(D) \quad \text{for some } n.$$

Dieudonné [**4**] established that any finitely generated extension always has distinguished subfields. L has the same dimension over any distinguished subfield [**5**], and this dimension is called the order of inseparability of L/K. The least n such that $K(L^{p^n})$ is separable over K is called the inseparable exponent of L/K, inex(L/K). This is the same as the least n such that

$$K(L^{p^n}) = K(D^{p^n})$$

for distinguished subfields D. This paper is generally concerned with extending higher derivations from distinguished subfields of L to L or a field containing L.

Extension properties of higher derivations can be used to characterize the distinguished subfields. A maximal separable subfield D of L/K is distinguished if and only if every higher derivation on D/K extends to some field H containing L (Proposition 1). It is then shown that there is a unique minimal field M containing L with the properties 1) every higher derivation on a distinguished subfield extends to M and 2) the field of constants of all higher derivations of M over K is the same as the field of constants of those of D over K. M is the modular closure, [**12**], [**13**], of L

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over K^s, the separable algebraic closure of K in L (Theorem 1). We also give a new construction of the modular closure L/K as a limit of the modular closures of $L/K(L^{p^n})$ (Theorem 2).

We then examine the question of which infinite rank higher derivations on a distinguished subfield D extend to L. There is a unique minimal intermediate field L_R of L/K over which L is regular and in the finitely generated case L_R is the field of constants of all infinite rank higher derivations on L/K [**1**, Theorem 1.1, p. 1304] and [**1**, Theorem 1.6, p. 1306]. Let $\mathscr{H}^\infty(D/K)$ be the group of all infinite higher derivations on D/K. The subgroup of $\mathscr{H}^\infty(D/K)$ which extends to L is $\mathscr{H}^\infty(D/D \cap L_R)$. Let H_D be the subgroup of $\mathscr{H}^\infty(L/K)$ which is generated by the extension to L of elements of $\mathscr{H}^\infty(D/D \cap L_R)$. The following are equivalent: 1) $H_D = \mathscr{H}^\infty(L/K)$; 2) $D \cap L_R$ is a distinguished subfield of L_R/K; 3) D is separable over $D \cap L_R$. Moreover, there is always a distinguished D such that $H_D = \mathscr{H}^\infty(L/k)$ (Theorem 3). We also present an example in which for different distinguished subfields D_1 and D_2, $H_{D_1} \neq H_{D_2}$.

In certain cases we are also able to determine which finite rank higher derivations on D over K extend to L. Thus, let D be a distinguished subfield of L/K such that D is separable over $D \cap L_R$, i.e., $H_D = H^\infty(L/K)$. Let L_M be the field of constants of all higher derivations of L/K. Assume

$$\{L:K(L^p)\} = p^{d+1}$$

where d is the transcendence degree of L over K or that L has inseparable exponent 1 over K. (This assumption is needed to force $D \supset L_M$.) Then

$$L = L_1 \otimes_{L_M} (D) = L_1 \otimes_{L_M} (D \cap L_R \otimes_{L_M} D_1).$$

These decompositions show that the higher derivations on D/K which extend to L are generated by those of $D \cap L_R/L_M$, which is a finite dimensional radical (purely inseparable) extension, and those of D_1/L_M which is a regular extension (Theorem 4).

2. Let $L \supset D \supset K$ be fields having characteristic $p \neq 0$ and let $\mathscr{H}(D/K)$ denote the set of all higher derivations $d = \{d_i\}$ mapping D into D such that $d_i(K) = 0$ for all $i > 0$; d may have finite or infinite rank. The higher derivations on D/K are said to extend through L if there is an overfield H of L such that each d in $\mathscr{H}(D/K)$ extends to a higher derivation on H, that is, extends to $\mathscr{H}(H/K)$. We begin with a characterization of distinguished subfields among maximal separable intermediate fields using this concept. Let D be a maximal separable intermediate field. Thus D/K is separable and there is no subfield of L properly containing D which is separable over K.

PROPOSITION 1. *The field D is a distinguished subfield of L/K if and only if the higher derivations on D/K extend through L.*

Proof. Assume that each d in $\mathscr{H}(D/K)$ extends to a field $H \supset L$, and let $\{x_1, \dots, x_n\}$ be a p-basis for D over K dual to the infinite higher derivations $\{d^{(1)}, \dots, d^{(n)}\}$ in $\mathscr{H}(D/K)$ [**6**, Definition 3.3, p. 266]. If $\{h^{(1)}, \dots, h^{(n)}\}$ are in $\mathscr{H}(H/K)$, where the restriction of $h^{(i)}$ to D is $d^{(i)}$, then $h^i_{p^s}$ is a derivation on $K(H^{p^s})$ which maps $x_i^{p^s}$ to 1 and $x_j^{p^s}$ to 0 for $j \neq i$. Thus $x_1^{p^s}, \dots, x_n^{p^s}$ are p-independent over $K(\mathscr{H}^{p^{s+1}})$ or, otherwise put, D and $K(\mathscr{H}^{p^{s+1}})$ are linearly disjoint over $K(D^{p^{s+1}})$. Hence, D and $K(L^{p^{s+1}})$ are linearly disjoint over $K(D^{p^{s+1}})$ for all $s \geqq 0$. By [**1**, Theorem 2.9, p. 1310], D is distinguished.

Conversely, assume D is distinguished. Then

$$L \subseteq K^{p^{-\infty}} D = K^{p^{-\infty}} \otimes_K D.$$

Higher derivations on D/K can be extended to $K^{p^{-\infty}} \otimes_K D$ by being trivial on $K^{p^{-\infty}}$. Thus all higher derivations on D/K extend through L.

A cannonical overfield of L/K associated with Proposition 1 is obtained by requiring H to be a minimal overfield of L such that 1) every d in $\mathscr{H}(D/K)$ extends to H for D a distinguished subfield and 2) the constant fields of $\mathscr{H}(D/K)$ and of $\mathscr{H}(H/K)$ are equal. There is a unique field fulfilling these requirements and it is the modular closure of L over K^s, the separable algebraic closure of K in L (and of K in D). We discuss related theory below.

Since D is a maximal separable extension of K in L, D must contain K^s, the separable algebraic closure of K in L. Using [**6**, Theorem 7.1 and Theorem 7.2, p. 273] it is clear that K^s is the field of constants of all higher derivations on D over K.

[**12**] and [**13**] describe the modular closure of a field extension L/K. Basically it is the intersection of all fields M containing L which are modular over K. If M is the modular closure of L/K^s, then

$$M = F \otimes_{K^s} D$$

where D is any distinguished subfield of L/K and F is radical modular over K^s [**12**, Theorem 5, p. 609]. Thus all higher derivations on D/K can be extended to M and the field of constants of all higher derivations of M/K is K^s. If M_1 is any field satisfying these two requirements, then M_1 must be modular over K^s, and hence $M_1 \supset M$. We have proven the following.

THEOREM 1. *There is a unique minimal overfield H of L/K such that* 1) *every d in $\mathscr{H}(D/K)$ extends to H for D a distinguished subfield of L/K and* 2) *the constant fields of $\mathscr{H}(D/K)$ and $\mathscr{H}(H/K)$ are equal. This field is the modular closure of L/K^s.*

Actually, the field above can be constructed as a type of limit. We describe this process, but first we need the following.

LEMMA 1. *Let L/K be a finite dimensional radical field extension of exponent $n + 1$. Then there is a radical extension M of K of exponent less than or equal to n such that $L(M)$ is the modular closure of L/K.*

Proof. Let L_1 be a subfield of L maximal with respect to its number of generators r such that

$$L_1 = K(b_1, \dots, b_r) \quad \text{and} \quad [L_1:K] = p^{(n+1)r}.$$

That is, L_1 is modular over K and has all its generators of exponent $n + 1$. Then L is a subfield of $L_1K^{p^{-n}}$. Since L_1 is modular over K, there is a p-basis for K of the form

$$\{b_1^{p^{n+1}}, \dots, b_r^{p^{n+1}}\} \cup \{c_\alpha\}.$$

Now

$$L \subseteq L_1^{p^{-n}} = K(b_1^{p^{-n}}, \dots, b_r^{p^{-n}}, \{c_\alpha^{p^{-n}}\})$$

and

$$L \subseteq K^{p^{-n-1}} = K(b_1, \dots, b_r, \{c_\alpha^{p^{-n-1}}\}).$$

Thus L is a subfield of the intersection which is

$$K(b_1, \dots, b_r, \{c_\alpha^{p^{-n}}\}).$$

i.e., $L \subseteq L_1K^{p^{-n}}$. Since $L_1K^{p^{-n}}$ is modular over K and has r elements of exponents $n + 1$ over K, the modular closure has at most r elements of exponent $n + 1$ over K. Thus the field generated over K by the elements of a subbasis of the modular closure having exponent n or less, will satisfy the requirements of the lemma.

Now let L/K be a finitely generated field extension. To simplify notation, let M_n denote the modular closure of L over $K(L^{p^n})$.

THEOREM 2. *For $n \geqq 0$, $M_n \subseteq M_{n+1}$ and for some t,*

$$M_t = M_{t+n} = M(L/K^s),$$

the modular closure of L over K^s.

Proof. We note first that the distinguished subfields of L/K are distinguished subfields of L/K^s and conversely. Thus

$$M(L/K^s) = D \otimes_{K^s} M_1$$

where D is a distinguished subfield of L/K and M_1/K^s is modular radical. Since D is modular over $K^s(D^{p^r})$ for all r and

$$K^s(D^{p^r}) = K^s(L^{p^r}) = K(L^{p^r}) \quad \text{for } r \geqq \text{inex}\ (L/K),$$

it follows that $M(L/K^s)$ is modular over $K(L^{p^r})$ for $r \geqq$ inex (L/K). Thus

$$M(L/K^s) \supseteq M_n$$

for large n. We now consider the inclusion $M_n \subseteq M_{n+1}$. It suffices to show M_{n+1} is modular over $K(L^{p^n})$. By Lemma 1, $M_{n+1} = L(N)$ where N is of exponent at most n over $K(L^{p^{n+1}})$. But M_{n+1} is modular over

$$K(L^{p^{n+1}})(M_{n+1}^{p^n}) = K(\,(L^{p^{n+1}})(\,(L(N)\,)^{p^n})\,) = K(L^{p^n}).$$

Thus

$$M_{n+1} \supseteq M_n \quad \text{and} \quad M(L/K^s) \supseteq M_n \quad \text{for all } n.$$

Since $M(L/K^s)$ is finite dimensional over L, there is a t such that

$$M_t = M_{t+n} \quad \text{for all } n \geqq 0.$$

Thus M_t being modular over $K(L^{p^{t+n}})$ for all n, is modular over

$$\bigcap_n K(L^{p^{t+n}}) = K^s$$

[**14**, Theorem 1.1, p. 39]. Thus $M_t = M(L/K^s)$.

The observation that K^s is separably algebraically closed in $M(L/K^s)$ demonstrates that K^s is the constant field of $\mathscr{H}(M(L/K^s)/K)$. Since clearly d in $\mathscr{H}(D/K)$ extends to $M(L/K)^s$, the latter field fulfills the requirements listed in the paragraph preceding Theorem 1.

Let P be a perfect field and let $\{u, v, x\}$ be algebraically independent over P. Set

$$K = P(u^p, v^p) \quad \text{and} \quad L = K(x, ux^{p^n} + v).$$

Then L/K is of exponent 1, $M(L/K^s) = P(u, x, v)$. Since L is modular over $K(L^{p^n})$, the least t such that $M_t = M(L/K^s)$ is $n + 1$.

Since $M(L/K^s)$ is modular over K^s, we have

$$M(L/K) \supset M(L/K^s).$$

The following example illustrates that in general

$$M(L/K) \neq M(L/K^s).$$

A non-trivial algebraic extension K' of K is exceptional if it is inseparable and has the property $K^{p^{-1}} \cap K' = K$. If $[K:K^p] > p$, then K has finite dimensional exceptional extensions and hence a finite exceptional extension K' with inseparability exponent 1 [**8**, Proposition 5, p. 1179]. If $N = K'(x)$, x an indeterminate, then N/K^s is modular, being a modular radical extension followed by a separable extension [**7**, Lemma 5, p. 303]. However, N/K is not modular [**7**, Theorem 6, p. 303].

We now examine which infinite rank higher derivations on a distinguished subfield D of L/K will extend to higher derivations on L/K. $\mathscr{H}^\infty(D/K)$ will denote the group of all infinite rank higher derivations on D/K. The fields of constants of groups of infinite rank higher derivations on L/K are the intermediate fields over which L is regular, i.e., separable and relatively algebraically closed. For any field extension L/K there is a unique minimal intermediate field L_R over which L is regular [**1**, Theorem 1.1, p. 1304]. If L/K is finitely generated, L_R will be the field of constants of all infinite higher derivations of L/K [**1**, Theorem 1.6, p. 1306]. If D_1 is distinguished for L_R over K, then L_R is radical over D_1. Thus

$$L = L_R \otimes_{D_1} D_2$$

where D_2 is separable over D_1 [**7**, Proposition 1, p. 302]. Since

$$L_R \subseteq K^{p^{-\infty}} D_1,$$

$L \subseteq K^{p^{-\infty}} D_2$, i.e., D_2 is distinguished for L/K. All infinite higher derivations on D_2 over D_1 can be extended to L by being trivial on L_R. By counting transcendence degree, the field of constants of the extensions will be L_R. Thus for this distinguished subfield a largest possible subgroup of infinite rank higer derivations can be extended to L. The general situation is summarized in the following result.

THEOREM 3. *Let D be a distinguished subfield of L/K. The subgroup of $\mathscr{H}^\infty(D/K)$ which extends to L is $\mathscr{H}^\infty(D/D \cap L_R)$. Let H_D be the subgroup of $\mathscr{H}^\infty(L/K)$ which is generated by the extension to L of elements of $\mathscr{H}^\infty(D/D \cap L_R)$. The following are equivalent*:

1) $H_D = \mathscr{H}^\infty(L/K)$;

2) *$D \cap L_R$ is a distinguished subfield of L_R over K*;

3) *D is separable over $D \cap L_R$.*

Moreover, there is always a distinguished subfield D such that $H_D = \mathscr{H}^\infty(L/K)$.

Proof. Let F be the field of constants of all infinite rank higher derivations on D/K which extend to L. Necessarily, $F \supseteq L_R \cap D$ and hence the subgroup which extends is at most $\mathscr{H}^\infty(D/D \cap L_R)$.

Conversely, let F_1 be the field of constants of $\mathscr{H}^\infty(D/D \cap L_R)$. L is radical over D, hence L_R is radical over $D \cap L_R$, and hence $L_R F_1$ is radical over F_1. Since D is separable over F_1,

$$L_R F_1 \otimes_{F_1} D = L_R D$$

is a subfield of L/K. Since $L_R D$ contains a distinguished subfield of L/K and has the same order of inseparability over K as does L, $L_R D = L$ [**3**, Theorem 2.2, p. 659]. Thus every infinite higher derivation on D/F_1 extends to L by being trivial on $L_R F_1$. Thus $\mathscr{H}^\infty(D/D \cap L_R)$ extends to L.

We now establish the equivalence of 1, 2, and 3 of Theorem 3. Assume D is a distinguished subfield of L/K and $H_D = H^\infty(L/K)$. Let D_F be the field of constants of $\mathscr{H}^\infty(D/D \cap L_R)$, the group of those infinite higher derivations which extend to L. By assumption, the extended ones have

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field of constants L_R and hence $D_F \subseteq L_R$. By a transcendence degree argument, L_R is algebraic over D_F. Thus L_R and D are linearly disjoint over D_F since D is regular over D_F. Thus $L_R \otimes_{D_F} D$ is a subfield of L which has the same order of inseparability over K as does L because L is separable over L_R. Since $L_R \otimes_{D_F} D$ contains a distinguished subfield of L/K,

$$L_R \bigotimes_{D_F} D = L$$

[**3**, Theorem 2.2, p. 659]. Thus $[L_R:D_F] = [L:D]$ and D_F is a distinguished subfield of L_R over K. Since $D \cap L_R \supseteq D_F$ and $D \cap L_R$ is separable over K, $D \cap L_R = D_F$.

Assume $D \cap L_R$ is a distinguished subfield of L_R over K. Then

$$[L_R:D \cap L_R] = [L:D]$$

since L and L_R have the same order of inseparability over K. Since L is separable over L_R, the order of inseparability of L over $D \cap L_R$ is the same as the order of inseparability of L over K. Since

$$L \subseteq (D \cap L_R)^{p^{-\infty}} D,$$

any distinguished subfield of D over $D \cap L_R$ will also be one for L over $D \cap L_R$. But if D_1 is distinguished for L over $D \cap L_R$, $[L:D_1] = [L:D]$. Thus $D_1 = D$ and D is separable over $D \cap L_R$.

Assume D is separable over $D \cap L_R$. Since L is radical over D, L_R is radical over $D \cap L_R$. Thus $L_R \otimes_{D \cap L_R} D$ is a subfield of L and as before

$$L_R \bigotimes_{D \cap L_R} D = L.$$

Since L is regular over L_R, D is regular over $D \cap L_R$. Thus the field of constants of $\mathscr{H}_D$ is L_R and hence

$$\mathscr{H}_D = H^\infty(L/K).$$

The following example shows that in general for distinct distinguished subfields D_1 and D_2, $\mathscr{H}_{D_1} \neq \mathscr{H}_{D_2}$.

Example 1.

$$\begin{tikzcd}[arrows=dash]
 & L = K(x, \mu x + v, z) & \\
 & & D = K(z, zx + (\mu x + v)) \\
L_R = K(x, \mu x + v) \arrow[uur] \arrow[urr] & & \\
K(x^p) \arrow[u] & & \\
K = P(\mu^p, v^p) \arrow[u] & &
\end{tikzcd}$$

Consider the above diagram of fields where P is a field of characteristic $p \neq 0$ and $\{\mu, x, v, z\}$ is algebraically independent over P. Since L is separable over $K(x, \mu x + v)$ and $K(x, \mu x + v)$ is not separable over any field containing K[**1**, Example 1.3, p. 1305], it follows that

$$L_R = K(x, \mu x + v).$$

D is distinguished because D is pure transcendental over K and $L \subset D(\mu, v)$. We claim

$$D \cap L_R = K(x^p).$$

This follows once we show $K(x^p)$ is algebraically closed in D. But

$$K(x^p) = P(x^p, \mu^p, x^p + v^p, v^p) \quad \text{and}$$

$$D = K(x^p)(z, zx + (\mu x + v)).$$

Thus the same argument which shows $P(\mu^p, v^p)$ is algebraically closed in

$$P(\mu^p, v^p)(x, \mu x + v),$$

shows $K(x^p)$ is algebraically closed in D (see [**1**, Example 1.3, p. 1305]). Thus, for this D, only the trivial infinite higher derivations extends to L since the transcendence degree of L over L_R is one. But Theorem 3 guarantees there is some D_1 such that

$$\mathscr{H}(D_1) = \mathscr{H}(L/K) = \mathscr{H}^{\infty}(L/L_R) \neq \{e\}.$$

The question of which finite rank higher derivations on an arbitrary distinguished subfield extend to L seems to be quite complicated. However, if we restrict our attention to distinguished subfields D such that $D \cap L_R$ is distinguished for L_R over K, some interesting results can be obtained. At this point, we consider groups of higher derivations of different rank, thus, for example $\bar{\mathscr{H}}(L/K)$ represents the group of all higher derivations on L/K [**6**, p. 172].

THEOREM 4. *Assume that either* $[L:K(L^p)] = p^{d+1}$ *where* d *is the transcendence degree of* L/K *or* $K(L^p)$ *is separable over* K. *Let* L_M *be the field of constants of all higher derivations of* L *over* K *and let* D *be a distinguished subfield of* L/K *such that* $D \cap L_R$ *is distinguished for* L_R/K. *Then*

$$L = L_1 \otimes_{L_M} D, \quad D = (D \cap L_R) \otimes_{L_M} D_1 \quad \textit{and}$$
$$L = L_1 \otimes_{L_M} (D \cap L_R \otimes_{L_M} D_1)$$

where L_1/L_M *and* $D \cap L_R/L_M$ *are radical and* D_1/L_M *is regular. The subgroup of* $\mathscr{H}(D/K)$ *which extends to* L *is generated over* D *by the extensions to* D *of the groups*

$$\bar{\mathscr{H}}(D \cap L_R/L_M) \quad \textit{and} \quad \bar{\mathscr{H}}(D_1/L_M).$$

Proof. Consider the following diagram of intermediate fields of L/K.

$$\begin{tikzcd}[arrows=dash]
 & L & \\
L_R \arrow[ur] & & D \\
D \cap L_R \arrow[u] \arrow[urr] & & D_1 \\
L_M \arrow[u] \arrow[urr] & & \\
K \arrow[u] & &
\end{tikzcd}$$

The assumption that $[L:K(L^p)] = p^{d+1}$ or $K(L^p)$ is separable over K forces L to be modular over D. Since L is also radical over D, $D \supseteq L_M$. Since $D \cap L_R$ is distinguished for L_R/K,

$$L = L_R \otimes_{D \cap L_R} D.$$

L_R is radical over L_M [**1**, Theorem 1.1, p. 1304] and hence $D \cap L_R$ is radical over L_M. Thus

$$D = D \cap L_R \otimes_{L_M} D_1$$

where D_1/L_M is separable and in fact regular since L is regular over L_R. We now want to see that

$$L_R = D \cap L_R \otimes_{L_M} L_1 \quad \text{for some } L_1.$$

Since $D \cap L_R$ is distinguished for L_R/K, $D \cap L_R$ and $K(L_R^{p^n})$ are linearly disjoint over $K((D \cap L_R)^{p^n})$ for all n [**1**, Theorem 2.9, p. 1310]. Using the standard lemma on linear disjointness [**9**, Lemma, p. 162], it follows that $D \cap L_R$ and $L_M(L_R^{p^n})$ are linearly disjoint over $L_M((D \cap L_R)^{p^n})$ for all n. Thus $D \cap L_R$ is a pure intermediate field of the radical modular extension L_R/L_M, and hence it is a tensor factor, i.e.,

$$L_R = D \cap L_R \otimes_{L_M} L_1$$

where L_1 can be chosen modular over L_M [**2**, Theorem 4, p. 1163]. Thus

$$\begin{aligned} L &= L_R \otimes_{D \cap L_R} D \\ &= (L_1 \otimes_{L_M} D \cap L_R) \otimes_{D \cap L_R} (D \cap L_R \otimes_{L_M} D_1) \\ &= L_1 \otimes_{L_M} (D \cap L_R \otimes_{L_M} D_1). \quad (*) \end{aligned}$$

The group $\mathscr{H}(D/L_M)$ is generated over D by a set of higher derivations dual to a tensor basis of D/L_M [**6**, p. 171 and 174]. Thus, with the observation that every higher derivation on D/K which extends to L must have L_M in its constant field, the last sentence of Theorem 4 follows by observing that a tensor basis for D/L_M is the union of a tensor basis for $D \cap L_R/L_M$ with a tensor basis for D_1/L_M. This follows from the fact that the minimal subfield of $D \cap L_R/L_M$ over which $D \cap L_R$ is modular is also the minimal subfield of D/L_M over which D is modular.

The expression (*) is valid for any distinguished subfield D such that $D \cap L_R$ is distinguished for L_R/K. However, we should mention that two different distinguished subfields D_1 and D_2 of L/K could have different "size" groups which extend to L even though both $D_1 \cap L_R$ and $D_2 \cap L_R$ are distinguished for L_R/K. The difference would be in the finite rank higher derivations of $D_1 \cap L_R/L_M$ and those of $D_2 \cap L_R/L_M$. The best possible situation would be when $D \cap L_R/L_M$ is modular over L_M.

COROLLARY 1. *Assume that either* $[L:K(L^p)] = p^{d+1}$ *where* d *is the transcendence degree of* L/K *or* $K(L^p)$ *is separable over* K. *Then*

$$L = L_1 \otimes_{L_M} (D \cap L_R \otimes_{L_M} D_1)$$

for some D *a distinguished subfield of* L/K. *All higher derivations of* L/K *are generated by the extension to* L *of finite rank higher derivations of* L_1/L_M *and* $D \cap L_R/L_M$ *and infinite rank higher derivations of* D_1/L_M.

Proof. We need only to observe that L_1/L_M is modular and that D can be chosen so that $D \cap L_R/L_M$ is modular. The former follows from the hypothesis. The latter is established as follows. Let e be the inseparable exponent of L_R/K. The assumptions guarantee that L_R is modular over each of its distinguished subfields. Thus

$$L_M \subseteq K(L_R^{p^e})$$

[**1**, Theorem 2.2, p. 1308]. Any subbasis of L over L_M contains a separating transcendence basis $\{x_1, \dots, x_d\}$ of a distinguished subfield D_1 of L_R/K [**5**]. Since $D_1 \supset L_M$,

$$D_1 = L_M(x_1, \dots, x_r).$$

Since L/L_R is separable and L_R/D_1 is purely inseparable

$$L = L_R \otimes_{D_1} D \quad \text{for some } D \text{ of } L/K.$$

This D satisfies the conditions of the corollary.

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