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OPTIMIZATION UNDER THE $P^M_{\lambda,\tau}$ POLICY OF A FINITE DAM WITH BOTH CONTINUOUS AND JUMPWISE INPUTS

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Abstract

We consider a finite dam under the $P^M_{\lambda,\tau}$ policy, where the input of water is formed by a Wiener process subject to random jumps arriving according to a Poisson process. The long-run average cost per unit time is obtained after assigning costs to the changes of release rate, a reward to each unit of output, and a penalty that is a function of the level of water in the reservoir.

Keywords: Finite dam; $P^M_{\lambda,\tau}$ policy; Wiener process; compound Poisson process; long-run average cost

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1. Introduction

Since Faddy (1974) introduced a P^M_{λ} policy to a finite dam with input formed by a Wiener process, the model has been generalized in various ways by many authors. Lam (1985) and Lam and Lou (1987) introduced a more general policy, the $P^M_{\lambda,\tau}$ policy, to the finite dam with input formed by a Wiener process and obtained the long-run average cost per unit time after assigning costs to the changes of release rate, a reward to each unit of output, and a penalty depending on the level of water. Lee and Ahn (1998) applied the P^M_{λ} policy to an infinite dam with input formed by a compound Poisson process. Abdel-Hameed (2000) studied the $P^M_{\lambda,\tau}$ policy in the infinite dam in which the input process is a compound Poisson process with positive drift. Bae *et al.* (2003) generalized Abdel-Hameed's model to the case of a finite dam when the input is formed by a compound Poisson process and the level of water between inputs decreases linearly at a constant rate. Bae *et al.* (2003) obtained the long-run average cost per unit time after assigning the same costs to the dam as used by Lam (1985).

In this paper, we consider a finite dam under the $P^M_{\lambda,\tau}$ policy, where the input process is a Wiener process subject to compound Poisson jumps. The level of water is initially set at 0 and thereafter follows a Wiener process with drift μ ($-\infty < \mu < \infty$), variance $\sigma^2 > 0$, and reflecting barriers at both 0 and V, where V is the capacity of the reservoir. Meanwhile, the level of water also increases in jumps due to instantaneous inputs, such as rain, which occur according to a Poisson process with rate $\nu > 0$. The amounts of instantaneous inputs are independent and identically distributed with distribution function G and mean m. If the level of water exceeds V after an instantaneous input, then we assume that the amount of water

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exceeding V overflows immediately, so that the level of water becomes V. We also assume that $\nu m + \mu > 0$ so that the water level eventually increases. At the moment when the water level increases to cross a threshold λ $(0 < \lambda < V)$, we start to release water at a constant rate $M > 0$. Note that the water level now follows the Wiener process with drift $\mu - M$ and variance σ^2, which still has 0 and V as reflecting barriers and is subject to compound Poisson jumps. We continue to release water until the level reaches τ $(0 < \tau < \lambda)$, and at this moment we stop. We do not release any more water until the level of water exceeds the threshold λ again.

Our model reduces to the model of Lam (1985) when $\nu = 0$, and to that of Bae *et al.* (2003) when $\sigma^2 = 0$ with $\mu < 0$. After we assign the same costs to the dam as used by Lam (1985) and Bae *et al.* (2003), we determine the long-run average cost per unit time.

2. Long-run average cost per unit time

Let $Z(t)$ denote the level of water at time $t > 0$. We assign four costs to the dam. The cost of changing the release rate from 0 to M is given by $K_1 M$, and the cost of changing the release rate from M to 0 is given by $K_2 M$. A reward is given to each unit of output while the water is being released. A penalty function $f(z)$ is assigned to the dam per unit time when $Z(t) = z$ $(0 \le z \le V)$. Consider the points where $Z(t)$ falls to τ for the first time after we start to release water. These are the points at which we close the gate of the dam. Note that the sequence of these points forms an embedded delayed renewal process. From now on, we call the period between two successive renewal points a cycle. Let T_0 and T_M denote, in a cycle, the time periods of the release rate being equal to 0 and M, respectively. That is, T_0 is the time period from a renewal point to the point where $Z(t)$ increases to cross λ for the first time and T_M is the time period from the latter point to the next renewal point. Note that, in the model of Lam (1985), the level of water increases to cross λ always in a continuous path and in Bae *et al.* (2003) always by a jump. In our model, however, $Z(t)$ increases to cross λ in both ways. We denote the water level process by $Z_0(t)$ during T_0 and by $Z_M(t)$ during T_M.

It can be shown that the expected total cost during a cycle is given by

$$\begin{aligned}(K_1 + K_2)M + \mathrm{E}\bigg[\int_0^{T_0} f(Z_0(t))\,\mathrm{d}t \;\bigg|\; Z_0(0) = \tau\bigg] - M\,\mathrm{E}[T_M] \\ + \mathrm{E}\bigg[\mathrm{E}\bigg[\int_0^{T_M} f(Z_M(t))\,\mathrm{d}t \;\bigg|\; Z_M(0) = Z_0(T_0),\ Z_0(0) = \tau\bigg]\bigg].\end{aligned}$$

By making use of the renewal reward theorem (Ross (1983, p. 78)), we can see that the long-run average cost per unit time is given by

$$\begin{aligned}C(M, \lambda, \tau) &= \frac{\mathrm{E}[\text{cost during a cycle}]}{\mathrm{E}[\text{length of a cycle}]} \\ &= \frac{KM + w(\tau) - M\,\mathrm{E}[T_M] + \mathrm{E}[u(Z_0(T_0)) \mid Z_0(0) = \tau]}{\mathrm{E}[T_0] + \mathrm{E}[T_M]},\end{aligned}$$

where $K = K_1 + K_2$,

$$w(x) = \mathrm{E}\bigg[\int_0^{T_0} f(Z_0(t))\,\mathrm{d}t \mid Z_0(0) = x\bigg], \quad \text{for } 0 \le x \le \lambda,$$

$$u(x) = \mathrm{E}\bigg[\int_0^{T_M} f(Z_M(t))\,\mathrm{d}t \mid Z_M(0) = x\bigg], \quad \text{for } \tau \le x \le V.$$

3. Evaluations

In this section, we evaluate the functions $w(x)$ for all $0 \leq x \leq \lambda$, $u(x)$ for all $\tau \leq x \leq V$, and the distribution of $Z_0(T_0)$, given that $Z_0(0) = x$ for all $0 \leq x \leq \lambda$, by establishing backward differential equations and converting the equations into renewal-type equations. Then $w(\tau)$ is nothing but $w(x)|_{x=\tau}$ and $\mathrm{E}[u(Z_0(T_0)) \mid Z_0(0) = \tau]$ is easily obtained from $u(x)$ by conditioning on $Z_0(T_0)$ while setting $x = \tau$. Note also that $\mathrm{E}[T_0]$ and $\mathrm{E}[T_M]$ can easily be derived from $w(\tau)$ and $\mathrm{E}[u(Z_0(T_0)) \mid Z_0(0) = \tau]$ by setting $f \equiv 1$. Throughout this paper, we denote the Wiener process with drift μ, variance σ^2, and reflecting barrier 0 by $B_0(t)$, and the Wiener process with drift $M^* = \mu - M$, variance σ^2, and reflecting barrier V by $B_M(t)$. We also denote the increment of $B_0(t)$ in an interval of length h by Δ_0, and the increment of $B_M(t)$ in an interval of length h by Δ_M.

3.1. The function $w(x)$ for $0 \leq x \leq \lambda$

To evaluate $w(x)$, we first need to show that $w(x)$ satisfies the boundary conditions given in the following lemma.

Lemma 1. *We have $w(\lambda) = 0$ and $w'(0) = 0$, where the prime denotes differentiation.*

Proof. It is clear that $w(\lambda) = 0$, since the water level is already at λ. To show that $w'(0) = 0$, we adopt a similar argument to that given by Cox and Miller (1965, pp. 231–232). Suppose that $Z_0(0) = 0$. After time h, let p be the probability that $B_0(t)$ in $Z_0(t)$ makes an increment $\Delta_0 = O((h)^{1/2})$ and let q be the probability of $B_0(t)$ still being at 0. Then

$$w(0) = \begin{cases} \mathrm{E}[w(\Delta_0)] + O(h) & \text{with probability } (1-\nu h)p, \\ w(0) + O(h) & \text{with probability } (1-\nu h)q, \\ \mathrm{E}[w(Y)] + O(h) & \text{with probability } \nu h G(\lambda), \\ O(h) & \text{with probability } \nu h (1 - G(\lambda)), \end{cases}$$

where Y is the random variable with distribution G. Hence, we obtain

$$w(0) = (1-\nu h)(p\,\mathrm{E}[w(\Delta_0)] + q w(0)) + \nu h \int_0^\lambda w(y)\,\mathrm{d}G(y) + O(h).$$

Making a Taylor series expansion of $w(\Delta_0)$ about 0 gives

$$w(0) = (1-\nu h)(p[w(0) + \mathrm{E}[\Delta_0]w'(0)] + q w(0)) + \nu h \int_0^\lambda w(y)\,\mathrm{d}G(y) + O(h).$$

Solving this equation for $w'(0)$ and letting $h \to 0$, we can see that $w'(0) = 0$.

We now derive the backward differential equation for $w(x)$. Suppose that $Z_0(0) = x$, $0 \leq x \leq \lambda$. Conditioning on whether or not a jump occurs during $[0, h]$ gives

$$w(x) = \begin{cases} \mathrm{E}\bigg[\displaystyle\int_0^h f(B_0(t))\,\mathrm{d}t + w(x + \Delta_0)\bigg] & \text{if no jump occurs,} \\ \mathrm{E}\bigg[\displaystyle\int_0^h f(B_0(t))\,\mathrm{d}t\bigg] & \text{if a jump occurs and } Y \geq \lambda - x - \Delta_0, \\ \mathrm{E}\bigg[\displaystyle\int_0^h f(B_0(t))\,\mathrm{d}t + w(x + \Delta_0 + Y)\bigg] & \text{if a jump occurs and } Y < \lambda - x - \Delta_0. \end{cases}$$

Hence, we obtain

$$\begin{aligned}w(x) = {} & (1-\nu h)\,\mathrm{E}\bigg[\int_0^h f(B_0(t))\,\mathrm{d}t + w(x+\Delta_0)\bigg] + o(h)\\ & + \nu h\,\mathrm{E}\bigg[\int_0^h f(B_0(t))\,\mathrm{d}t \;\bigg|\; x+\Delta_0+Y \ge \lambda\bigg]\mathrm{P}(x+\Delta_0+Y\ge\lambda)\\ & + \nu h\,\mathrm{E}\bigg[\int_0^h f(B_0(t))\,\mathrm{d}t + w(x+\Delta_0+Y) \;\bigg|\; x+\Delta_0+Y<\lambda\bigg]\\ & \times \mathrm{P}(x+\Delta_0+Y<\lambda).\end{aligned}$$

Making a Taylor series expansion of $w(x+\Delta_0)$ gives

$$\begin{aligned}w(x) = {} & (1-\nu h)\bigg(\mathrm{E}\bigg[\int_0^h f(B_0(t))\,\mathrm{d}t\bigg] + w(x) + \mathrm{E}[\Delta_0]w'(x) + \frac{\mathrm{E}[\Delta_0^2]}{2}w''(x)\bigg)\\ & + \nu h\,\mathrm{E}\bigg[\int_0^{\lambda-x-\Delta_0} w(x+\Delta_0+y)\,\mathrm{d}G(y)\bigg] + o(h).\end{aligned}$$

Re-arranging this equation, dividing by h, and letting $h\to 0$ yields

$$0 = f(x) + \mu w'(x) + \frac{\sigma^2}{2}w''(x) - \nu w(x) + \nu\int_0^{\lambda-x} w(x+y)\,\mathrm{d}G(y). \tag{1}$$

For the convenience of analysis, we define $\bar{w}(x) = w(\lambda - x)$.

Lemma 2. *The function $\bar{w}(x)$ satisfies the renewal-type equation*

$$\bar{w}(x) = \bar{w}'(0)x - \frac{2}{\sigma^2}\int_0^x F_\lambda(t)\,\mathrm{d}t + \int_0^x \bar{w}(x-t)\,\mathrm{d}W(t),$$

with boundary conditions $\bar{w}(0)=0$ and $\bar{w}'(\lambda)=0$, where $\rho=\nu m$, $F_\lambda(x)=\int_0^x f(\lambda-t)\,\mathrm{d}t$, and $W(x)=\int_0^x(2\mu/\sigma^2+(2\rho/\sigma^2)G_\mathrm{e}(t))\,\mathrm{d}t$ (where $G_\mathrm{e}(t)=(1/m)\int_0^t(1-G(y))\,\mathrm{d}y$ is the equilibrium distribution of G).

Proof. From (1), we can see that $\bar{w}(x)$ satisfies

$$0 = f(\lambda-x) - \mu\bar{w}'(x) + \frac{\sigma^2}{2}\bar{w}''(x) - \nu\bar{w}(x) + \nu\int_0^x \bar{w}(x-y)\,\mathrm{d}G(y). \tag{2}$$

Integrating both sides of (2) with respect to x, with the boundary condition $\bar{w}(0)=0$, we obtain

$$\frac{\sigma^2}{2}\bar{w}'(x) = \frac{\sigma^2}{2}\bar{w}'(0) + \mu\bar{w}(x) - F_\lambda(x) + \nu\int_0^x (1-G(x-y))\bar{w}(y)\,\mathrm{d}y. \tag{3}$$

If we integrate (3) with respect to x, then we obtain

$$\bar{w}(x) = \bar{w}'(0)x - \frac{2}{\sigma^2}\int_0^x F_\lambda(t)\,\mathrm{d}t + \int_0^x \bar{w}(x-t)\bigg(\frac{2\mu}{\sigma^2} + \frac{2\rho}{\sigma^2}G_\mathrm{e}(t)\bigg)\,\mathrm{d}t.$$

Hence, we obtain the given renewal-type equation, as required.

It is well known (see, for example, Asmussen (1987, p. 113)) that the unique solution of the renewal-type equation in Lemma 2 is

$$\bar{w}(x) = \bar{w}'(0) \int_{0-}^{x} (x-t)\,\mathrm{d}M(t) - \frac{2}{\sigma^2} \int_0^x M(x-t) F_\lambda(t)\,\mathrm{d}t, \tag{4}$$

where $M(x) = \sum_{n=0}^{\infty} W^{(n)}(x)$. Here, $W^{(n)}$ denotes the n-fold Stieltjes convolution of W, with $W^{(0)}$ being the Heaviside function. To find $\bar{w}'(0)$, we differentiate (4) with respect to x, and set $x = \lambda$ with boundary condition $\bar{w}'(\lambda) = 0$; then we obtain

$$\bar{w}'(0) = \frac{(2/\sigma^2)(\int_0^\lambda M'(\lambda - t) F_\lambda(t)\,\mathrm{d}t + F_\lambda(\lambda))}{M(\lambda)}.$$

Finally, $w(x) = \bar{w}(\lambda - x),\ 0 \le x \le \lambda$.

3.2. The function $u(x)$ for $\tau \le x \le V$

Note again that, in our model, the level of water can increase to cross λ either along a continuous path or by a jump. Hence, we first assume that V is infinite and obtain the distribution of $L(x) = Z_0(T_0) - \lambda$, the excess amount over λ, given that $Z_0(0) = x,\ 0 \le x \le \lambda$, which is needed later to obtain $\mathrm{E}[u(Z_0(T_0)) \mid Z_0(0) = \tau]$.

Let $P_l(x) = \mathrm{P}(L(x) > l),\ l \ge 0$. Then, by an argument similar to that used in Lemma 1, we have $P_l(\lambda) = 0$ and $P_l'(0) = 0$ as boundary conditions.

Lemma 3. *The probability $\bar{P}_l(x) = P_l(\lambda - x)$ satisfies the renewal-type equation*

$$\bar{P}_l(x) = \bar{P}_l'(0)x - \frac{2}{\sigma^2} \int_0^x G_l(t)\,\mathrm{d}t + \int_0^x \bar{P}_l(x-t)\,\mathrm{d}W(t),$$

with boundary conditions $\bar{P}_l(0) = 0$ and $\bar{P}_l'(\lambda) = 0$, where $G_l(x) = \rho[G_{\mathrm{e}}(x+l) - G_{\mathrm{e}}(l)]$.

Proof. Conditioning on whether or not a jump in the water level occurs during the time interval $[0, h]$ gives

$$P_l(x) = \begin{cases} \mathrm{E}[P_l(x+\Delta_0)] & \text{if no jump occurs,} \\ \mathrm{E}[P_l(x+\Delta_0+Y)] & \text{if a jump occurs and } x+Y+\Delta_0 \le \lambda, \\ \mathrm{P}(x+\Delta_0+Y > \lambda + l) & \text{if a jump occurs and } x+Y+\Delta_0 > \lambda. \end{cases}$$

Hence, we obtain, for $0 \le x \le \lambda$,

$$\begin{aligned} P_l(x) &= (1-\nu h)\,\mathrm{E}[P_l(x+\Delta_0)] + \nu h\,\mathrm{P}(x+\Delta_0+Y > \lambda + l) \\ &\quad + \nu h\,\mathrm{E}[P_l(x+\Delta_0+Y) \mid x+Y+\Delta_0 \le \lambda]\,\mathrm{P}(x+Y+\Delta_0 \le \lambda) + o(h). \end{aligned}$$

Making a Taylor series expansion of $P_l(x+\Delta_0)$ gives

$$\begin{aligned} P_l(x) &= (1-\nu h)\bigg[P_l(x) + \mathrm{E}[\Delta_0]P_l'(x) + \frac{\mathrm{E}[\Delta_0^2]}{2}P_l''(x)\bigg] + \nu h\,\mathrm{P}(Y > \lambda + l - x - \Delta_0) \\ &\quad + \nu h\,\mathrm{E}\bigg[\int_0^{\lambda - x - \Delta_0} P_l(x+\Delta_0+y)\,\mathrm{d}G(y)\bigg] + o(h). \end{aligned}$$

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Dividing by h and letting $h \to 0$, we obtain

$$0 = \mu P_l'(x) + \frac{\sigma^2}{2} P_l''(x) - \nu P_l(x) + \nu \, \mathrm{P}(Y > \lambda + l - x) + \nu \int_0^{\lambda - x} P_l(x + y) \, \mathrm{d}G(y).$$

Replacing x with $\lambda - x$ gives

$$0 = -\mu \bar{P}_l'(x) + \frac{\sigma^2}{2} \bar{P}_l''(x) - \nu \bar{P}_l(x) + \nu \, \mathrm{P}(Y > x + l) + \nu \int_0^{x} \bar{P}_l(x - y) \, \mathrm{d}G(y).$$

Integrating the above equation twice with respect to x, with boundary condition $\bar{P}_l(0) = 0$, and using the identity

$$\begin{aligned} \nu \int_0^x (1 - G(t + l)) \, \mathrm{d}t &= \nu \int_l^{x+l} (1 - G(y)) \, \mathrm{d}y \\ &= \rho [G_{\mathrm{e}}(x + l) - G_{\mathrm{e}}(l)] \\ &= G_l(x), \end{aligned}$$

we obtain

$$\bar{P}_l(x) = \bar{P}_l'(0) x - \frac{2}{\sigma^2} \int_0^x G_l(t) \, \mathrm{d}t + \int_0^x \bar{P}_l(x - t) \bigg(\frac{2\mu}{\sigma^2} + \frac{2\rho}{\sigma^2} G_{\mathrm{e}}(t) \bigg) \, \mathrm{d}t.$$

This equation simplifies to the renewal-type equation for $\bar{P}_l(x)$.

The renewal-type equation in Lemma 3 has the following unique solution:

$$\bar{P}_l(x) = \bar{P}_l'(0) \int_{0-}^x (x - t) \, \mathrm{d}M(t) - \frac{2}{\sigma^2} \int_0^x M(x - t) G_l(t) \, \mathrm{d}t.$$

Differentiating the above equation with respect to x and using the boundary condition $\bar{P}_l'(\lambda) = 0$, we obtain

$$\bar{P}_l'(0) = \frac{(2/\sigma^2)(\int_0^\lambda M'(\lambda - t) G_l(t) \, \mathrm{d}t + G_l(\lambda))}{M(\lambda)}.$$

Now, when $V < \infty$, note that the survival function of $L(x)$ is still $P_l(x)$, for $0 \le l < V - \lambda$, but with a discrete probability $P_{V-\lambda}(x)$ at $l = V - \lambda$.

Remark 1. Note that $P_0(x)$ is the probability that $Z_0(t)$ increases to cross λ by a jump and $1 - P_0(x)$ is the probability that $Z_0(t)$ increases to cross λ along a continuous path.

Now, we evaluate $u(x)$ for $\tau \le x \le V$. Let $\bar{u}(x) = u(V - x)$.

Lemma 4. *Using arguments similar to those used to derive $w(x)$, we have the renewal-type equation*

$$\bar{u}(x) = \bigg(1 - \frac{2M^*}{\sigma^2} x - \frac{2\rho}{\sigma^2} \int_0^x G_{\mathrm{e}}(t) \, \mathrm{d}t \bigg) \bar{u}(0) - \frac{2}{\sigma^2} \int_0^x F_V(t) \, \mathrm{d}t + \int_0^x \bar{u}(x - t) \, \mathrm{d}U(t),$$

with boundary conditions $\bar{u}(V-\tau)=0$ *and* $\bar{u}'(0)=0$*, where* $F_V(x)=\int_0^x f(V-t)\,\mathrm{d}t$ *and* $U(x)=\int_0^x (2M^*/\sigma^2+(2\rho/\sigma^2)G_{\mathrm{e}}(t))\,\mathrm{d}t$.

Proof. Conditioning on whether or not a jump occurs during the time interval $[0,h]$, we obtain

$$u(x)=\begin{cases}\mathrm{E}\Big[\int_0^h f(B_M(t))\,\mathrm{d}t+u(x+\Delta_M)\Big] & \text{if no jump occurs,}\\ \mathrm{E}\Big[\int_0^h f(B_M(t))\,\mathrm{d}t+u(x+\Delta_M+Y)\Big] & \text{if a jump occurs and } Y\le V-x-\Delta_M,\\ \mathrm{E}\Big[\int_0^h f(B_M(t))\,\mathrm{d}t+u(V+\Delta_M)\Big] & \text{if a jump occurs and } Y> V-x-\Delta_M.\end{cases}$$

Therefore, for $\tau\le x\le V$,

$$\begin{aligned}u(x)&=(1-\nu h)\,\mathrm{E}\Big[\int_0^h f(B_M(t))\,\mathrm{d}t+u(x+\Delta_M)\Big]+o(h)\\&\quad+\nu h\,\mathrm{E}\Big[\int_0^h f(B_M(t))\,\mathrm{d}t+u(x+\Delta_M+Y)\;\Big|\;Y\le V-x-\Delta_M\Big]\\&\quad\times \mathrm{P}(Y\le V-x-\Delta_M)\\&\quad+\nu h\,\mathrm{E}\Big[\int_0^h f(B_M(t))\,\mathrm{d}t+u(V+\Delta_M)\;\Big|\;Y> V-x-\Delta_M\Big]\\&\quad\times \mathrm{P}(Y> V-x-\Delta_M).\end{aligned}$$

Making a Taylor series expansion of $u(x+\Delta_M)$ gives

$$\begin{aligned}u(x)&=(1-\nu h)\bigg(\mathrm{E}\Big[\int_0^h f(B_M(t))\,\mathrm{d}t\Big]+u(x)+\mathrm{E}[\Delta_M]u'(x)+\frac{\mathrm{E}[\Delta_M^2]}{2}u''(x)\bigg)\\&\quad+\nu h\,\mathrm{E}\Big[\int_0^{V-x-\Delta_M} u(x+\Delta_M+y)\,\mathrm{d}G(y)\Big]\\&\quad+\nu h\,\mathrm{E}[u(V+\Delta_M)\mid Y>V-x-\Delta_M]\,\mathrm{P}(Y>V-x-\Delta_M)+o(h).\end{aligned}$$

Dividing by h, letting $h\to 0$, and replacing x with $V-x$, we obtain

$$\begin{aligned}0&=f(V-x)-M^*\bar{u}'(x)+\frac{\sigma^2}{2}\bar{u}''(x)-\nu\bar{u}(x)\\&\quad+\nu\bar{u}(0)(1-G(x))+\nu\int_0^x \bar{u}(x-y)\,\mathrm{d}G(y).\end{aligned}$$

Integrating the above equation with respect to x, and using the boundary condition $\bar{u}'(0)=0$, we obtain

$$\begin{aligned}\frac{\sigma^2}{2}\bar{u}'(x)&=M^*\bar{u}(x)-M^*\bar{u}(0)-F_V(x)-\nu\bar{u}(0)\int_0^x (1-G(t))\,\mathrm{d}t\\&\quad+\nu\int_0^x (1-G(x-y))\bar{u}(y)\,\mathrm{d}y.\end{aligned}\tag{5}$$

If we integrate (5) with respect to x, we then find that

$$\bar{u}(x) = \left(1 - \frac{2M^*}{\sigma^2}x - \frac{2\rho}{\sigma^2}\int_0^x G_e(t)\,dt\right)\bar{u}(0) - \frac{2}{\sigma^2}\int_0^x F_V(t)\,dt + \int_0^x \bar{u}(x-y)\left(\frac{2M^*}{\sigma^2} + \frac{2\rho}{\sigma^2}G_e(y)\right)dy.$$

Hence, we obtain the given renewal-type equation for $\bar{u}(x)$.

The unique solution of the renewal-type equation in Lemma 4 is given by

$$\bar{u}(x) = \left(N(x) - \frac{2M^*}{\sigma^2}\int_{0-}^x (x-t)\,dN(t) - \frac{2\rho}{\sigma^2}\int_0^x N(x-t)G_e(t)\,dt\right)\bar{u}(0) - \frac{2}{\sigma^2}\int_0^x N(x-t)F_V(t)\,dt,$$

where $N(x) = \sum_{n=0}^{\infty} U^{(n)}(x)$. To get $\bar{u}(0)$, we put $x = V - \tau$ in the above equation and use the boundary condition $\bar{u}(V-\tau) = 0$. Then we obtain

$$\bar{u}(0) = \frac{(2/\sigma^2)\int_0^{V-\tau} N(V-\tau-t)F_V(t)\,dt}{(1 - 2M^*(V-\tau)/\sigma^2)N(V-\tau) - (2\rho/\sigma^2)\int_0^{V-\tau} N(V-\tau-t)G_e(t)\,dt}.$$

Finally, using $u(x) = \bar{u}(V-x)$, for $\tau \le x \le V$, and conditioning on $L(\tau)$ gives

$$E[u(Z_0(T_0)) \mid Z_0(0) = \tau] = E[u(\lambda + L(\tau))] = \int_0^{V-\lambda} u(\lambda + l)\,dP_l(\tau).$$

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