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ON EXTENSIONS OF NILPOTENT TORSION RINGS BY SEMISIMPLE RINGS

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A class of rings in which each member is the extension of a nilpotent torsion ring by a semisimple semiartinian ring is presented.

Throughout the following R will always designate an associative ring not necessarily containing an identity element.

Problem 75 cited in the second author's monograph seeks to determine the structure of rings R satisfying the following two conditions:

(A) if P is a prime ideal of R, R/P is a simple ring with non-zero socle; and

(B) the left annihilator of every homomorphic image of R is an MHR-ring (that is, a ring satisfying the minimum condition for principal right ideals).

We note that a left perfect ring is an MHR-ring with 1 (see [1] and [9]). Before giving a class of rings satisfying these two conditions, we pause to give certain definitions.

DEFINITION 1. An S-ring is a ring that satisfies conditions (A) and (B) simultaneously.

DEFINITION 2. A ring R is called a P-ring if for every homomorphic image B of R and for every right B-module M, $M = M_0 \oplus M_1$ where $M_0 B = \{0\}$ and $M_1 B = M_1$.

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DEFINITION 3. A *PS*-ring is a ring which is both a *P*-ring and an *S*-ring.

THEOREM 4. *A right Noetherian PS-ring* R *is the extension of a nilpotent torsion ring* B *by a semisimple ring* C . *Moreover, every non-zero right ideal of every homomorphic image of* C *contains a minimal right ideal.*

Proof. Since R is right Noetherian, every ideal of R contains a product of finitely many prime ideals of R (see [1] and [9]). In particular we can find prime ideals $P_1, \ldots, P_k$ of R whose product is the zero ideal. There will be no loss of generality if we suppose that

$$R = P_0 \supsetneq P_1 \supsetneq P_1P_2 \supsetneq \cdots \supsetneq P_1P_2 \cdots P_k = \{0\} .$$

Having this we build the following factor rings:

$$H_j = P_1 \cdots P_{j-1}/P_1 \cdots P_j , \quad j = 1, 2, \ldots, k+1 .$$

It is not hard to see that each H_j can be regarded as an R/P_j-module in the natural way. Our assumption shows that R/P_j is a simple MHR-ring that possesses minimal one-sided ideals. Thus, by a result of the second author [10, 11], the R/P_j-module H_j has a direct sum representation:

$$H_j = B_j \oplus C_j , \text{ where } B_jR = \{0\} \text{ and } C_jR = C_j .$$

This means that the perfect R/P_j-module C_j is a perfect R-module too. So C_j is a completely reducible R-module [11]. On the other hand,

$$B_j = \left(0 : R/P_1 \cdots P_j\right)_l .$$

But since the left annihilator of $R/P_1 \cdots P_j$ is an MHR-ring, the additive group of B_j is torsion [11, 12]. Moreover, building the union of the complete inverse images of the ring homomorphisms

$$\phi_j : B_j \to B_j/\left(B_j \cap \left(P_1 \cdots P_j\right)\right)$$

one obviously obtains a nil ideal B of R . But since R is right Noetherian, B should be nilpotent by the Levetzki theorem ([6], p. 199). Moreover, $(B, +)$ is torsion as we have seen.

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Finally the union C of the chain of the complete inverse images of the mappings

$$\psi : C_j \to C_j/\left(C_j \cap \left(P_1P_2 \ldots P_j\right)\right)$$

is a Jacobson semisimple ring with the required properties. This completes the proof. //

The importance of Theorem 4 may be best seen if we consider rings with 1 . A ring with 1 is obviously a P-ring. Moreover, a ring with 1 is an MHR-ring if and only if it is a left perfect ring. For characterizations of left perfect rings we refer to [1] and [9]. We recall that a ring R with 1 is right semiartinian if and only if every homomorphic image of right R-module R has non-zero socle. The study of such important rings can be found in [2], [3] and [9]. One can see that a right semiartinian ring is a PS-ring. On the other side, the class of PS-rings contains every right Artinian ring and every quasi-Frobenius ring. Also, a right Noetherian ring with 1 which is in the same time right seminoetherian is necessarily right Artinian. So, many of the known results concerning such rings can be drawn or reformulated in view of Theorem 4.

THEOREM 5. *A right Noetherian PS-ring R with* 1 *is right Artinian.*

Proof. Since R has 1 , Theorem 4 asserts that R is the extension of a nilpotent torsion ring by a right semiartinian ring. This shows that R itself is right Artinian. //

PROBLEM 6. R is a PS-ring and G is a finite (or soluble) group. Is the group ring $R[G]$ again a PS-ring?

PROBLEM 7. Give an example of a PS-ring which is not semiartinian.

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