Can. J. Math., Vol. XXXII, No. 2, 1980, pp. 294–309

TOPOLOGICAL REFLECTION GROUPS

DRAGOMIR Ž. DJOKOVIĆ

0. Introduction. Let G be a closed subgroup of one of the classical compact groups $O(n)$, $U(n)$, $Sp(n)$. By a reflection we mean a matrix in one of these groups which is conjugate to the diagonal matrix diag $(-1, 1, \ldots, 1)$. We say that G is a *topological reflection group* (t.r.g.) if the subgroup of G generated by all reflections in G is dense in G.

It was shown recently by Eaton and Perlman [**5**] that, in case of $O(n)$, the whole group $O(n)$ is the unique infinite irreducible t.r.g. In this paper we solve the analogous problem for $U(n)$ and $Sp(n)$. Our method of proof is quite different from the one used in [**5**]. We treat simultaneously all the three cases.

The problem of classifying t.r.g.'s easily reduces to the case when the group is irreducible. In the irreducible case one has to distinguish the finite and infinite cases. Of course, the theory of finite reflection groups in $O(n)$ or $U(n)$ is well-known. More generally in case of $U(n)$ one knows all finite subgroups generated by pseudo-reflections [**9**], [**2**]. The list of all finite subgroups of $Sp(n)$ generated by pseudo-reflections does not seem to be known.

We shall consider only the case when G is an infinite irreducible t.r.g. Then the identity component G_0 of G has positive dimension. We again have to distinguish two cases: first when G_0 is reducible and the second when it is irreducible. The complete answer in the first case is provided by Theorem 11 and in the second case by Theorem 14. It is interesting that such a mild hypothesis allows us to completely classify these t.r.g.'s.

The proof of Theorem 14 relies heavily on the classification of totally geodesic submanifolds of the real, complex, and quaternionic projective spaces. In the real case this is well-known. The complex case is analysed in [**1**]. (I am indebted to Patrick Ryan for this reference.) We were not able to locate in the literature such a classification in the case of the quaternionic projective space. Therefore we have included the description of these submanifolds in Section 3. This is accomplished by classifying first some Lie triple systems in Section 2.

1. Preliminaries. We denote by $\mathbf{F}$ one of the following: the real field $\mathbf{R}$, the complex field $\mathbf{C}$, or the division algebra of real quaternions $\mathbf{H}$. By $U_n(\mathbf{F})$ we denote the corresponding unitary group. Thus $U_n(\mathbf{R}) = O(n)$, $U_n(\mathbf{C}) = U(n)$, and $U_n(\mathbf{H}) = Sp(n)$.

We denote by $1, i, j, k$ the standard basic units of $\mathbf{H}$ and we identify $\mathbf{C}$ with

Received June 22, 1978. This work was supported in part by NRC Grant A-5285.

$\mathbf{R} + \mathbf{R}i$, as usual. Hence $\mathbf{R} \subset \mathbf{C} \subset \mathbf{H}$, and consequently $O(n) \subset U(n) \subset Sp(n)$.

A *reflection* is a matrix $R \in U_n(\mathbf{F})$ which is conjugate to diag $(-1, 1, 1, \ldots, 1)$. We denote by $\mathscr{R}_n(\mathbf{F})$ the set of all reflections of $U_n(\mathbf{F})$. Clearly we have $\mathscr{R}_n(\mathbf{R}) \subset \mathscr{R}_n(\mathbf{C}) \subset \mathscr{R}_n(\mathbf{H})$.

By $\mathbf{F}^n$ we denote the right $\mathbf{F}$-vector space of dimension n consisting of column vectors. This vector space is equipped with the standard hermitian inner product $\langle x, y\rangle = x^*y$. The set of all unit vectors in $\mathbf{F}^n$ is a sphere which we denote by $S(\mathbf{F}^n)$. The dimension of this sphere (as a manifold) is $n - 1$ if $\mathbf{F} = \mathbf{R}$, $2n - 1$ if $\mathbf{F} = \mathbf{C}$, and $4n - 1$ if $\mathbf{F} = \mathbf{H}$.

The standard orthonormal basic vectors of $\mathbf{F}^n$ will be denoted by $e_1, \ldots, e_n$.

We shall denote by $P(\mathbf{F}^n)$ the projective space over $\mathbf{F}$ associated to $\mathbf{F}^n$, and by $p:\mathbf{F}^n\backslash\{0\} \to P(\mathbf{F}^n)$ the corresponding projection.

Any two reflections in $U_n(\mathbf{F})$ are conjugate. Hence $\mathscr{R}_n(\mathbf{F})$ is a homogeneous space for $U_n(\mathbf{F})$ and so a closed submanifold of $U_n(\mathbf{F})$. The map $\Phi:S(\mathbf{F}^n) \to \mathscr{R}_n(\mathbf{F})$, defined by $\Phi(a) = I_n - 2aa^*$, is smooth and surjective. It induces a diffeomorphism $\phi\colon P(\mathbf{F}^n) \to \mathscr{R}_n(\mathbf{F})$.

The centralizer of $R_1 = I_n - 2e_1e_1^* = \operatorname{diag}(-1, 1, 1, \ldots, 1)$ in $U_n(\mathbf{F})$ is the obvious subgroup $U_1(\mathbf{F}) \times U_{n-1}(\mathbf{F})$. The Lie algebra $\mathfrak{g}$ of $U_n(\mathbf{F})$ is the space of all skew-hermitian matrices, and the Lie algebra $\mathfrak{h}$ of $U_1(\mathbf{F}) \times U_{n-1}(\mathbf{F})$ consists of all skew-hermitian matrices of the form

$$\left(\begin{array}{c|c} \alpha & 0 \\ \hline 0 & X \end{array}\right)$$

where $\alpha \in \mathbf{F}$ and X is of size $(n - 1) \times (n - 1)$. Conjugation by R_1 is an involutive automorphism of $\mathfrak{g}$, $\mathfrak{h}$ is the $(+1)$-eigenspace and let $\mathfrak{m}$ be the (-1)-eigenspace. Clearly, $\mathfrak{m}$ consists of matrices of the form

$$\left(\begin{array}{c|c} 0 & -x^* \\ \hline x & 0 \end{array}\right), \quad x \in \mathbf{F}^{n-1}.$$

The projective space $P(\mathbf{F}^n)$ is the symmetric space associated with the symmetric pair described above (see [**7**]).

The differential of p at e_1 maps the space of tangent vectors $\begin{pmatrix} 0 \\ x \end{pmatrix}$, $x \in \mathbf{F}^{n-1}$ bijectively on the tangent space of $P(\mathbf{F}^n)$ at $p(e_1)$. Hence we may identify these spaces.

Let $f\colon (-\epsilon, +\epsilon) \to S(\mathbf{F}^n)$ be a smooth map and let $f(0) = a$, $f'(0) = h \in \mathbf{F}^n$. Then $f^*(t)f(t) = 1$ for all t and by differentiating and evaluating at $t = 0$ we obtain $h^*a + a^*h = 0$, or equivalently $\operatorname{Re}(a^*h) = 0$. Hence the tangent space of $S(\mathbf{F}^n)$ at a consists of all vectors $h \in \mathbf{F}^n$ satisfying the equation $\operatorname{Re}(a^*h) = 0$.

We have $\Phi(f(t)) = I_n - 2f(t)f^*(t)$, and $\Phi(f(0)) = \Phi(a) = R_a = I_n - 2aa^*$. Differentiating again at $t = 0$, we obtain

$$(d\Phi)_a(h) = -2(ha^* + ah^*).$$

It follows that the tangent space of $\mathscr{R}_n(\mathbf{F})$ at R_a consists of all matrices of the form $ah^* + ha^*$ where $h \in \mathbf{F}^n$ and Re $(a^*h) = 0$. If Re $(a^*h) = 0$ then

$$\begin{aligned} R_a(ah^* + ha^*)R_a &= (I_n - 2aa^*)(ah^* + ha^*)R_a \\ &= (ha^* - ah^* - 2aa^*ha^*)(I_n - 2aa^*) \\ &= -ha^* - ah^* + 2a(a^*h + h^*a)a^* = -(ah^* + ha^*). \end{aligned}$$

Thus conjugation by R_a induces the minus identity map in the tangent space of $\mathscr{R}_n(\mathbf{F})$ at R_a. Since this conjugation induces an automorphism of $\mathscr{R}_n(\mathbf{F})$, considered as a Riemannian globally symmetric space (see [**7**]), it follows that this conjugation is the geodesic symmetry of $\mathscr{R}_n(\mathbf{F})$ at the point R_a. We shall use this fact later in the proof of Theorem 14.

If P is a group we shall denote by $\mathrm{Mon}_n(P)$ the group of monomial $n \times n$ matrices whose nonzero entries belong to P. Clearly, this group is isomorphic to the wreath product of P with the symmetric group S_n.

2. Some Lie triple systems. We take here $\mathbf{F} = \mathbf{H}$ and identify $x \in \mathbf{F}^{n-1}$ with

$$\begin{pmatrix} 0 & -x^* \\ x & 0 \end{pmatrix} \in \mathfrak{m}.$$

An easy computation gives

$$\left[\left[\begin{pmatrix} 0 & -x^* \\ x & 0 \end{pmatrix}, \begin{pmatrix} 0 & -y^* \\ y & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & -z^* \\ z & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix}$$

where $u = yx^*z - xy^*z + z(x^*y - y^*x)$. Thus, using the above identification, we can write

$$(1) \quad [[x, y], z] = yx^*z - xy^*z + z(x^*y - y^*x).$$

A real vector subspace $V \subset \mathbf{F}^{n-1}$ is called a *Lie triple system* if $[[x, y], z] \in V$ for all $x, y, z \in V$.

$Sp(1) \times Sp(n-1)$ acts on $\mathfrak{g}$ by restricting the adjoint action of $Sp(n)$. The subspace $\mathfrak{m}$ is stable under $Sp(1) \times Sp(n-1)$. For

$$\begin{pmatrix} \lambda & 0 \\ 0 & A \end{pmatrix} \in Sp(1) \times Sp(n-1)$$

we have

$$\begin{pmatrix} \lambda & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} 0 & -x^* \\ x & 0 \end{pmatrix}\begin{pmatrix} \bar{\lambda} & 0 \\ 0 & A^* \end{pmatrix} = \begin{pmatrix} 0 & -\lambda x^* A^* \\ Ax\bar{\lambda} & 0 \end{pmatrix}.$$

Using dot to denote this module action and identifying $\mathfrak{m}$ with $\mathbf{F}^{n-1}$, we can write

$$\begin{pmatrix} \lambda & 0 \\ 0 & A \end{pmatrix} \cdot x = Ax\bar{\lambda} \quad (x \in \mathbf{F}^{n-1}).$$

If $a \in Sp(1) \times Sp(n-1)$ and if $V \subset \mathfrak{m}$ is a Lie triple system then $a \cdot V$ is also a Lie triple system; such pairs of systems are said to be *equivalent*.

It is easy to verify that the following are Lie triple systems in $\mathfrak{m}$:

Type 1.

$$\{(\xi_1, \ldots, \xi_m, 0, \ldots, 0)^t | \xi_1, \ldots, \xi_m \in \mathbf{R}\}\ 1 \leqq m \leqq n-1.$$

Type 2. Same as type 1 except that $\xi_1, \ldots, \xi_m \in \mathbf{C}$.

Type 3. Same as type 1 except that $\xi_1, \ldots, \xi_m \in \mathbf{H}$.

Type 4.

$$\{(\xi_1, 0, \ldots, 0)^t | \xi_1 \in \mathbf{H}, \xi_1 + \bar{\xi}_1 = 0\}.$$

THEOREM 1. *Every nonzero Lie triple system $V \subset \mathfrak{m}$ is equivalent to one of the systems of the four types given above.*

Proof. For nonzero $x \in V$ we have $x\mathbf{H} \cap V = x \cdot K_x$ for a unique real subspace K_x of $\mathbf{H}$ containing $\mathbf{R}$. For nonzero $\lambda \in K_x$ we have $K_{x\lambda} = \lambda^{-1}K_x = \bar{\lambda}K_x$. Let r_x be the real dimension of K_x.

We claim that $K_x = x^*V = \{x^*y | y \in V\}$. If $\lambda \in K_x$ then $y = x\lambda \in V$ and so $x^*y = x^*x\lambda \in x^*V$. This shows that $K_x \subset x^*V$. By specifying $z = x$ in (1), we obtain

$$yx^*x + x(x^*y + y^*x) - 3xy^*x \in V.$$

Since x^*x and $x^*y + y^*x$ are real numbers, it follows that $x, y \in V \Rightarrow xy^*x \in V$. Hence, then $y^*x \in K_x$ and since $\bar{K}_x = K_x$ we also have $x^*y \in K_x$. This shows that we also have $x^*V \subset K_x$, and so our claim is proved.

Let $r = \max(r_x)$ for nonzero $x \in V$. Then we have four cases according to whether $r = 1, 2, 3$ or 4.

Case $r = 1$. For all nonzero $x \in V$ we have $K_x = \mathbf{R}$. Since $K_x = x^*V$, it follows that $x^*y \in \mathbf{R}$ for all $x, y \in V$. This implies that the canonical map $V \otimes_{\mathbf{R}} \mathbf{H} \rightarrow \mathbf{H}^{n-1} = \mathfrak{m}$ is injective (see, for instance, [**4**, Lemma 3]). Hence there exists an orthonormal $\mathbf{R}$-basis $a_1, \ldots, a_m$ in V where $\dim_{\mathbf{R}}(V) = m$. Choose $A \in Sp(n-1)$ such that $Aa_s = e_s$, $1 \leqq s \leqq m$. Then $A \cdot V$ is a Lie triple system of type 1.

Case $r = 2$. Fix nonzero $a \in V$ such that $r_a = 2$ and set $\mathbf{K} = K_a$. Clearly $\mathbf{K}$ is a subfield of $\mathbf{H}$. We claim that for all nonzero $x \in V$ we have $K_x = \mathbf{K}$. It suffices to prove this in the case when $a^*x \neq 0$. Since $x^*a \in \bar{K}_a = \mathbf{K}$, we have $K_x = x^*V \supset x^*a\mathbf{K} = \mathbf{K}$. Now $r_x \leqq 2$ implies that $K_x = \mathbf{K}$, and our claim is proved.

Thus for all $x, y \in V$ we have $x^*y \in \mathbf{K}$. Consequently, the canonical map $V \otimes_{\mathbf{K}} \mathbf{H} \rightarrow \mathfrak{m}$ is injective. Hence there exists an orthonormal $\mathbf{K}$-basis $a_1, \ldots, a_m$ of V where $m = \dim_{\mathbf{K}}(V)$. There exists $\lambda \in Sp(1)$ such that $\mathbf{K} = \lambda^{-1}\mathbf{C}\lambda$.

Choose $A \in Sp(n-1)$ such that $Aa_s = e_s\lambda$ for $1 \leqq s \leqq m$. Then

$$\left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & A \end{array}\right) \cdot V = AV\lambda^{-1}$$

is a Lie triple system of type 2.

Case $r = 4$. Fix nonzero $a \in V$ such that $K_a = \mathbf{H}$. We claim that $K_x = \mathbf{H}$ for all nonzero $x \in V$. It suffices to prove this when $a^*x \neq 0$. Then the claim follows from $K_x = x^*V \supset x^*a\mathbf{H} = \mathbf{H}$. Hence V is an $\mathbf{H}$-subspace of $\mathfrak{m}$ and it is clearly equivalent to a Lie triple system of type 3.

Case $r = 3$. Let $a \in V$ be a unit vector such that $r_a = 3$ and write $K_a = K$. We claim that $V = aK$. Otherwise let $b \in V \backslash aK$. Then $\alpha = a^*b \in a^*V = K_a = K$ and so $c = b - a\alpha \in V$. Clearly $c \neq 0$ and $a^*c = 0$. Let $\lambda \in K$ and put $x = a$, $y = c$, $z = a\lambda + c$. Then $x^*z = \lambda$, $y^*z = c^*c$, $x^*y = y^*x = 0$ and so it follows from (1) that $c\lambda - ac^*c \in V$. Hence $c\lambda \in V$, i.e., $\lambda \in K_c$. This shows that $K \subset K_c$ and since $r_c \leqq 3$ we must have $K_c = K$. Now let $\alpha, \beta \in K$ and specify x, y, z in (1) to be $x = a\alpha$, $y = c\beta$, $z = c$. Since now $x^*z = x^*y = 0$ and $y^*z = \bar{\beta}c^*c$, it follows that $-xy^*z = -a\alpha\bar{\beta}c^*c \in V$. i.e., $\alpha\bar{\beta} \in K$. Consequently K is closed under multiplication, which is impossible since $\dim_{\mathbf{R}}(K) = 3$. Thus $V = aK$.

We now claim that we can choose $\lambda \in K \cap Sp(1)$ such that if $b = a\lambda$ then $K_b = \mathbf{R} + \mathbf{R}i + \mathbf{R}j$. If $K = \mathbf{R} + \mathbf{R}i + \mathbf{R}j$ we take $\lambda = 1$. Otherwise $K \not\supset \mathbf{R}i + \mathbf{R}j$ and we can choose $\mu \in \mathbf{R}i + \mathbf{R}j$ such that $\mu \notin K$. If $W = K \cap (\mathbf{R} + \mathbf{R}i + \mathbf{R}j)$ then $\dim_{\mathbf{R}}(W) = 2$ and so $K \cap W\mu \neq 0$. Choose $\lambda \in W \cap Sp(1)$ such that $\lambda\mu \in K$. Since $W \supset \mathbf{R}$ and $\dim_{\mathbf{R}}(W) = 2$, W is a subfield of $\mathbf{H}$ and so $\lambda^{-1}K \supset \lambda^{-1}W = W$. Since $\mu \in \lambda^{-1}K$ and $\mu \notin W$ we must have $\lambda^{-1}K = W + \mathbf{R}\mu = \mathbf{R} + \mathbf{R}i + \mathbf{R}j$. Hence $K_b = K_{a\lambda} = \lambda^{-1}K_a = \lambda^{-1}K = \mathbf{R} + \mathbf{R}i + \mathbf{R}j$.

Now choose $A \in Sp(n-1)$ such that $Ab = e_1k$. Then

$$A \cdot V = AbK_b = e_1kK_b = e_1(\mathbf{R}i + \mathbf{R}j + \mathbf{R}k)$$

is precisely the Lie triple system of type 4.

This completes the proof of the theorem.

3. Totally geodesic submanifolds of the quaternionic projective space. As explained in Section 1, we identify $\mathbf{H}^{n-1}$ with the tangent space of $P(\mathbf{H}^n)$ at the point $p(e_1)$. We also identify $\mathfrak{m}$ with $\mathbf{H}^{n-1}$ via the map

$$x \mapsto \begin{pmatrix} 0 & -x^* \\ x & 0 \end{pmatrix}, \quad x \in \mathbf{H}^{n-1}.$$

There is a bijection between the connected totally geodesic submanifolds of $P(\mathbf{H}^n)$ containing the point $p(e_1)$ and the Lie triple systems in $\mathfrak{m}$ (see [**7**, p. 189]). If M is such a submanifold then the associated Lie triple system in $\mathfrak{m}$ is simply the tangent space of M at the point $p(e_1)$.

If W is a real subspace of $\mathbf{H}^n$ we shall write $p(W)$ instead of the more precise $p(W\backslash\{0\})$.

THEOREM 2. *Connected totally geodesic submanifolds M of $P(\mathbf{H}^n)$ of positive dimension are precisely the following:*

Type 1. *$M = p(W)$ where $W \subset \mathbf{H}^n$ is an $\mathbf{R}$-subspace such that $x^*y \in \mathbf{R}$ for all $x, y \in W$;*

Type 2. *$M = p(W)$ where $W \subset \mathbf{H}^n$ is a $\mathbf{C}$-subspace such that $x^*y \in \mathbf{C}$ for all $x, y \in W$ (here the embedding $\mathbf{C} \subset \mathbf{H}$ is not necessarily standard);*

Type 3. *$M = p(W)$ where $W \subset \mathbf{H}^n$ is an $\mathbf{H}$-subspace.*

Type 4. *$M = p(W)$ where $W = a\mathbf{R} + b(\mathbf{R}i + \mathbf{R}j + \mathbf{R}k)$ and a, b are non-zero vectors in $\mathbf{H}^n$ such that $a^*b = 0$.*

Proof. Let M be a totally geodesic submanifold of $P(\mathbf{H}^n)$ and $\dim M \geqq 1$. Since $Sp(n)$ acts transitively on $P(\mathbf{H}^n)$, we may assume that $p(e_1) \in M$. Let V be the tangent space of M at this point. Then V is a Lie triple system in $\mathfrak{m}$. The fixer of $p(e_1)$ in $Sp(n)$ is the subgroup $Sp(1) \times Sp(n-1)$. Hence, by Theorem 1, there exists A in this fixer such that $A \cdot V$ is one of the Lie triple systems listed just before Theorem 1. Using the correspondence between totally geodesic submanifolds containing $p(e_1)$ and Lie triple systems in $\mathfrak{m}$, the proof now reduces to the verification that $p(W)$ is totally geodesic in each of the following cases:

$$W = e_1\mathbf{R} + \ldots + e_m\mathbf{R},\ W = e_1\mathbf{C} + \ldots + e_m\mathbf{C},\ W = e_1\mathbf{H} + \ldots + e_m\mathbf{H},$$
$$\text{and } W = e_1\mathbf{R} + e_2(\mathbf{R}i + \mathbf{R}j + \mathbf{R}k), \quad 2 \leqq m \leqq n.$$

The geodesic $p(e_1 \cos t + e_2 \sin t)$ lies in $p(W)$ in the first three cases. In the first case the subgroup $\{1\} \times O(m-1) \times \{I_{n-m}\}$ of $Sp(n)$ leaves the subspace $V = e_2\mathbf{R} + \ldots + e_m\mathbf{R}$ invariant and acts transitively on the unit sphere $S(V)$. Since geodesic is mapped to geodesic by this action, it follows that all geodesics $g(t)$ such that $g(0) = p(e_1)$ and $g'(0) \in V$ lie in $p(W)$. Therefore $p(W)$ is totally geodesic. Similar arguments are applicable in the remaining three cases.

The proof is now completed.

Let $M = p(W)$ as in Theorem 2. Let $W\mathbf{H}$ be the $\mathbf{H}$-subspace of $\mathbf{H}^n$ spanned by W. Then $p(W\mathbf{H})$ is the smallest quaternionic projective subspace containing M. The $\mathbf{H}$-dimension of $W\mathbf{H}$ will be called the *width* of M and denoted by $w(M)$. Note that if M is of type 4 then $w(M) = 2$. Later we shall be in particular interested in totally geodesic submanifolds M of $P(\mathbf{H}^n)$ of maximal width i.e., $w(M) = n$. Thus type 4 will be relevant only if $n = 2$.

Results similar to Theorem 2 are valid for $P(\mathbf{C}^n)$ and $P(\mathbf{R}^n)$. We did not include the case of $P(\mathbf{C}^n)$ because it occurs in the literature (see [**1**, Lemma 4]). The complex totally geodesic subspaces of $P(\mathbf{C}^n)$ are also described in [**8**, pp. 273-278]. Of course, totally geodesic submanifolds of $P(\mathbf{R}^n)$ are just the projective subspaces.

https://doi.org/10.4153/CJM-1980-022-1 Published online by Cambridge University Press

4. Topological reflection groups. We shall say that a subgroup G of $U_n(\mathbf{F})$ ($\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$) is a *topological reflection group* (t.r.g.) if it coincides with the closure of the subgroup generated by $G \cap \mathscr{R}_n(\mathbf{F})$, i.e., the set of reflections in G.

In the sequel G denotes a t.r.g. We consider $\mathbf{F}^n$ as a $(G, \mathbf{F})$-bimodule with G operating on the left and $\mathbf{F}$ on the right. We say that G is *irreducible* if this bimodule is irreducible.

The following proposition is well-known (at least in the cases $\mathbf{F} = \mathbf{R}$, or $\mathbf{F} = \mathbf{C}$).

PROPOSITION 3. *Let $G \subset U_n(\mathbf{F})$ be a* t.r.g. *There exists an orthogonal decomposition $\mathbf{F}^n = V_1 \oplus \ldots \oplus V_k$ into simple $(G, \mathbf{F})$-bimodules. If G_i $(1 \leqq i \leqq k)$ is the subgroup of $U(V_i)$ induced by G then G_i is a* t.r.g. *and $G = G_1 \times \ldots \times G_k$.*

Since the proof is the same as in [**6**] for $\mathbf{F} = \mathbf{R}$, we shall omit it.

This Proposition reduces the study of t.r.g.'s to the irreducible case. Hence from now on we shall assume that G is an irreducible t.r.g.

The case when G is finite has different flavor from the infinite case. Finite reflection groups have been extensively studied by Coxeter [**3**] and Shephard and Todd [**9**] in the real and complex case. A more recent study of complex finite reflection groups has been carried out by A. M. Cohen [**2**]. All these authors use the word "reflection" in a more general sense, and so their results are more general.

We shall be concerned exclusively with the case when G is infinite. It turns out that a complete classification is possible in all three cases real, complex, and quaternionic. As mentioned in the introduction, the real case has been dealt with by Eaton and Perlman [**5**]. Our method is completely different and if we were to consider just the real case, our proof would be much simpler. In fact the complex case is also easy; only the quaternionic case presents a few surprises.

We repeat once more that from now on G will be an infinite irreducible t.r.g. in $U_n(\mathbf{F})$. By G_0 we denote the identity component of G.

5. Several lemmas. In order to keep the proofs of the main theorems reasonably short, we shall prove in this section seven lemmas.

LEMMA 4. *Let $U(1)$ be embedded in $Sp(n)$ by $\lambda \mapsto \lambda I_n$. Then the normalizer of $U(1)$ in $Sp(n)$ is $U(n) \cup jU(n)$.*

Proof. Let $A = (\alpha_{rs})$ be in this normalizer. Since $i \in U(1)$ there exists $\lambda \in U(1)$ such that $AiA^{-1} = \lambda$, i.e., $Ai = \lambda A$. Thus $\alpha_{rs}i = \lambda\alpha_{rs}$ for all r, s. It follows that $\lambda = \pm i$. If $\lambda = i$ then $\alpha_{rs}i = i\alpha_{rs}$ implies that $\alpha_{rs} \in \mathbf{C}$ and so $A \in U(n)$. Similarly, if $\lambda = -i$ then $\alpha_{rs}i = -i\alpha_{rs}$ implies $\alpha_{rs} \in j\mathbf{C}$ and so $A \in jU(n)$.

Conversely, it is clear that $U(n) \cup jU(n)$ normalizes $U(1)$.

LEMMA 5. *Let $Sp(1)$ be embedded in $Sp(n)$ by $\lambda \mapsto \lambda I_n$. Then the normalizer of $Sp(1)$ in $Sp(n)$ is $Sp(1) \cdot O(n)$.*

Proof. Let $A = (\alpha_{rs}) \in Sp(n)$ be in this normalizer. Then for $\lambda \in Sp(1)$ there exists $\mu \in Sp(1)$ such that $A\lambda A^{-1} = \mu$. Hence $\alpha_{rs}\lambda = \mu\alpha_{rs}$ for all r, s. If $\alpha_{rs} \neq 0$ then $\alpha_{rs}\lambda\alpha_{rs}^{-1} = \mu$, and so $\alpha_{rs}\lambda\alpha_{rs}^{-1}$ is independent of r, s. Consequently, if also $\alpha_{pq} \neq 0$ then $\alpha_{rs}^{-1}\alpha_{pq}$ commutes with λ. Since $\lambda \in Sp(1)$ is arbitrary this implies that $\alpha_{rs}^{-1}\alpha_{pq} \in \mathbf{R}$. Hence $A \in Sp(1) \cdot O(n)$. The converse is clear.

LEMMA 6. *Let R be a reflection of the form $R = jA$ where $A \in U(n)$. Then $n = 2$ and $A = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$ where $\alpha \in U(1)$.*

Proof. Clearly $n \geqq 2$. Since $Aj = j\bar{A}$ we have $I_n = R^2 = jAjA = -\bar{A}A$. Since $A^* = A^{-1}$ we also have $\bar{A} = -A^*$, and so $A' = -A$, A' being the transpose of A. Thus A is a unitary skew-symmetric matrix. So let $A = (\alpha_{rs})$, $\alpha_{rs} \in \mathbf{C}$, $\alpha_{sr} = -\alpha_{rs}$. Since $R = jA$ is a reflection, the matrix $R - I = jA - I = j(A + j)$ must have rank one. Thus

$$A + j = \begin{pmatrix} j & \alpha_{12} & \alpha_{13} \dots \\ -\alpha_{12} & j & \alpha_{23} \\ -\alpha_{13} & -\alpha_{23} & j \\ \vdots & & \end{pmatrix}$$

Multiplying the first row on the left by $-\alpha_{12}j$ and adding it to the second row we conclude that $\alpha_{12}j\alpha_{12} = j$. Consequently α_{12} has unit norm and it follows that $\alpha_{1s} = \alpha_{2s} = 0$ for $s \geqq 3$ because A is unitary. Since rank $(A + j) = 1$, we can conclude now that $n = 2$. This proves the lemma.

LEMMA 7. *If $n \geqq 3$ then every reflection in $Sp(1) \cdot O(n)$ lies in $O(n)$. If $n = 2$ and R is a reflection in $Sp(1) \cdot O(2)$ not lying in $O(2)$, then it is conjugate to $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ in $Sp(1) \cdot O(2)$.*

Proof. Let $R = \alpha A$ be a reflection where $\alpha \in Sp(1)$ and $A \in O(n)$. Since $Sp(1)$ and $O(n)$ centralize each other, we have $\alpha^2A^2 = I_n$ which forces $\alpha^2 = \pm 1$. If $R \notin O(n)$ then we must have $\alpha^2 = -1$ and replacing R by a suitable conjugate $\lambda R\lambda^{-1}$, $\lambda \in Sp(1)$, we may assume that $\alpha = i$. Since $A^2 = -I_n$, it follows that n is even, say $n = 2m$, and that A is conjugate in $O(n)$ to the block-diagonal matrix with m blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal. We may replace A by this conjugate. Then R is block-diagonal with m blocks $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ on the diagonal. Since R is a reflection we must have $m = 1$, $n = 2$.

LEMMA 8. *The normalizer in $Sp(n)$ of the diagonal subgroup $D = Sp(1) \times \dots \times Sp(1)$ is the group $\mathrm{Mon}_n(Sp(1))$ of all monomial matrices in*

$Sp(n)$. *(Recall that a matrix is monomial if in each row and each column it has precisely one nonzero entry.)*

Proof. The center of D has order 2^n and it consists of all diagonal matrices with diagonal entries ± 1. If $A \in Sp(n)$ normalizes D then it also normalizes its center. This implies that A permutes the subspaces e_sF, $1 \leqq s \leqq n$, i.e., A is a monomial matrix.

Let $D = Sp(1) \times \ldots \times Sp(1)$ be the diagonal subgroup of $Sp(n)$.

LEMMA 9. *The normalizer of $D \cap U(n)$ in $Sp(n)$ is the group $\mathrm{Mon}_n(U(1) \cup jU(1))$. If $n \geqq 2$ then the normalizer of $D \cap SU(n)$ in $Sp(n)$ is $\mathrm{Mon}_n(U(1)) \cup j\mathrm{Mon}_n(U(1))$.*

Proof. If A normalizes $D \cap U(n)$ then it also normalizes the subgroup of $D \cap U(n)$ consisting of all matrices having ± 1 as diagonal entries. This implies that A is a monomial matrix. Since $i = iI_n$ has order 4 and no eigenvalues ± 1, it follows that AiA^{-1} has $\pm i$ as diagonal entries. Hence if $A = (\alpha_{rs})$ then $i^{-1}\alpha_{rs}i = \pm \alpha_{rs}$ for all r, s. This implies that $\alpha_{rs} \in \mathbf{C}$ or $\alpha_{rs} \in j\mathbf{C}$. Since $A \in Sp(n)$ and is monomial, it follows that $\alpha_{rs} \in U(1) \cup jU(1)$ if $\alpha_{rs} \neq 0$. The first assertion is proved.

Now let $n \geqq 2$. The centralizer of $D \cap SU(n)$ in $Sp(n)$ is $D \cap U(n)$. Hence the normalizer of $D \cap SU(n)$ in $Sp(n)$ is contained in $\mathrm{Mon}_n(U(1) \cup jU(1))$. On the other hand this normalizer contains $\mathrm{Mon}_n(U(1))$. The index of $\mathrm{Mon}_n(U(1))$ in $\mathrm{Mon}_n(U(1) \cup jU(1))$ is 2^n and we can take as coset representatives the diagonal matrices with diagonal entries 1 or j. It is clear that only two of these representatives, namely I_n and jI_n, normalize $D \cap SU(n)$. This proves the second assertion.

LEMMA 10. *Let Γ_n be the complete graph on vertices $1, \ldots, n$. Assume that the edges of Γ_n are colored black or white so that each triangle has precisely one or all three of its edges white. Then either all edges are white or the white edges induce two disjoint complete subgraphs.*

Proof. Let Ω be a connected component of the subgraph W of Γ_n induced by all the white edges. Using the property that if two edges of a triangle are white so is the third, we see at once that Ω is a complete subgraph. It remains to show that W has at most two components. Otherwise let Ω_1, Ω_2, Ω_3 be 3 different components of W and assume that $1 \in \Omega_1$, $2 \in \Omega_2$, $3 \in \Omega_3$. Then the triangle $\{1, 2, 3\}$ must have all 3 edges black, contrary to our hypothesis.

The lemma is proved.

6. Case when G_0 is reducible. In this section we classify infinite irreducible t.r.g.'s in $U_n(\mathbf{F})$ when $\mathbf{F}^n$ is not irreducible as a $(G_0, \mathbf{F})$-bimodule. Recall that G_0 is the identity component of G. Since G_0 has positive dimension and G is a t.r.g., it follows that $n \geqq 2$.

https://doi.org/10.4153/CJM-1980-022-1 Published online by Cambridge University Press

We often need to consider $U(1)$ and $Sp(1)$ as embedded in $Sp(n)$. Unless otherwise stated we shall always use the embedding $\lambda \mapsto \lambda I_n$.

Let $\tilde{U}(1)$ be the subgroup of $Sp(2)$ consisting of the matrices

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \text{ and } \begin{pmatrix} 0 & j\lambda \\ -j\lambda & 0 \end{pmatrix}, \lambda \in U(1).$$

Our first main result is the following theorem.

THEOREM 11. *Let G be an infinite irreducible* t.r.g. *in $U_n(\mathbf{F})$, $\mathbf{F} = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$. Assume that $\mathbf{F}^n$ is not irreducible as a $(G_0, \mathbf{F})$-bimodule. Then $\mathbf{F} \neq \mathbf{R}$ and $n \geqq 2$. If $\mathbf{F} = \mathbf{C}$ then G is conjugate in $U(n)$ to the subgroup*

1) *monomial matrices in $U(n)$ with determinant ± 1.*

If $\mathbf{F} = \mathbf{H}$ then G is conjugate in $Sp(n)$ to one of the following groups:

2) $\mathrm{Mon}_n(Sp(1))$;

3) *the group* 1) *above;*

4) $\mathrm{Mon}_n(U(1) \cup jU(1))$;

5) *(if $n = 2$) $Sp(1) \cdot Q$ where $Q \subset O(2)$ is $O(2)$, cyclic of order 4, or dihedral of order a multiple of 8;*

6) *(if $n = 2$) $\tilde{U}(1) \cdot T$ where T is a finite subgroup of $SU(2)$ containing* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and T is cyclic of order 4, binary dihedral, or binary polyhedral.*

Proof. Let $\mathbf{F}_n = V_1 \oplus \ldots \oplus V_r$ be the canonical decomposition of $\mathbf{F}^n$ as a $(G_0, \mathbf{F})$-bimodule into its homogeneous components. Assume that the $\mathbf{F}$-dimension of one of these components, say V_1, is larger than one. If $R \in G$ is a reflection, it follows that R fixes a nonzero vector in V_1. Since $V_1, \ldots, V_r$ is a system of imprimitivity of G, it follows that $RV_1 = V_1$. Since this holds for all reflections $R \in G$ and G is a t.r.g. we deduce that $G \cdot V_1 = V_1$. The irreducibility of G now implies that $V_1 = V$. The other possibility is that each of $V_1, \ldots, V_r$ has $\mathbf{F}$-dimension one (and so $r = n$).

We shall treat these two cases separately.

Case 1. $\mathbf{F}^n$ is homogeneous as a $(G_0, \mathbf{F})$-bimodule. Let W be a simple $(G_0, \mathbf{F})$-submodule of $\mathbf{F}^n$. Assume that the $\mathbf{F}$-dimension of W is bigger than one. If $R \in G$ is a reflection then R fixes a nonzero vector $a \in W$. Hence $RW \cap W \neq 0$. Since W and also RW are simple $(G_0, \mathbf{F})$-submodules, this implies that $RW = W$. This is valid for each reflection $R \in G$, and since G is a t.r.g. it follows that W is G-stable. The irreducibility of G then implies that $W = \mathbf{F}^n$, i.e., $\mathbf{F}^n$ is a simple $(G_0, \mathbf{F})$-bimodule. This contradicts the hypothesis of the theorem.

Thus, we must have $\dim_{\mathbf{F}}(W) = 1$. Without loss of generality we may assume that $e_s\mathbf{F}$, $1 \leqq s \leqq n$ are G_0-submodules. Since all these submodules are isomorphic we may assume that G_0 is a subgroup of $U_1(\mathbf{F})$, the latter being embedded in $U_n(\mathbf{F})$ by $\lambda \mapsto \lambda I_n$.

We cannot have $\mathbf{F} = \mathbf{R}$ because $U_1(\mathbf{R}) = O(1)$ is a group of order two and G_0 has positive dimension. Assume that $\mathbf{F} = \mathbf{C}$. Then the same dimension

https://doi.org/10.4153/CJM-1980-022-1 Published online by Cambridge University Press

argument implies that $G_0 = U_1(\mathbf{C}) = U(1)$. On the other hand since G is a t.r.g. in $U(n)$, we must have $\det(A) = \pm 1$ for all $A \in G$ and in particular for $A \in G_0$. This gives a contradiction.

Hence, we must have $\mathbf{F} = \mathbf{H}$. Since G_0 is now a closed connected subgroup of $Sp(1)$ of positive dimension, we have either $G_0 = Sp(1)$ or otherwise we may assume that $G_0 = U(1)$.

Let first $G_0 = Sp(1)$. By Lemma 5, $G \subset Sp(1) \cdot O(n)$. If $n \geqq 3$ then, by Lemma 7, every reflection in $Sp(1) \cdot O(n)$ is in fact in $O(n)$. It follows that $G \subset O(n)$, contradicting the fact that $G_0 = Sp(1)$. Therefore we must have $n = 2$. Since $G \not\subset O(2)$ and G is a t.r.g., it follows from Lemma 7 that G contains all reflections of the form $\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$ where α is a pure quaternion and $\alpha\bar{\alpha} = 1$. These reflections generate the group consisting of all matrices

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}, \quad \lambda \in Sp(1).$$

We have $G = Sp(1) \cdot (G \cap O(2))$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G \cap O(2)$. It follows that $G \cap O(2)$ is generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and some reflections. Therefore $G \cap O(2)$ is one of the following: the cyclic group of order 4 generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, a dihedral group containing $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, or $O(2)$. This gives case 5) of our theorem.

Now let $G_0 = U(1)$. Then, by Lemma 4, $G \subset U(n) \cup jU(n)$. We cannot have $G \subset U(n)$ because G is a t.r.g. and this would imply $\det(A) = \pm 1$ for all $A \in G$, contrary to $G_0 = U(1)$. Hence G must contain a reflection belonging to $jU(n)$. By Lemma 6, it follows that $n = 2$ and that $G \supset \tilde{U}(1)$. Therefore $G = \tilde{U}(1) \cdot T$ where $T = G \cap SU(2)$. Since $G_0 = U(1)$, T must be finite. If a reflection $R \in G$ is not in $\tilde{U}(1)$, then $R \in U(2)$ and $R = iA$ where $A \in SU(2)$ is conjugate to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Therefore T must be generated by elements conjugate to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since T contains $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, it follows by consulting a list of finite subgroups of $SU(2) \cong Sp(1)$ [**10**, Theorem 2.6.7] that T is one of the following: cyclic of order 4, binary dihedral, or binary polyhedral.

This gives case 6) of the theorem.

Case 2. Each of the homogeneous components $V_1, \ldots, V_r$ has $\mathbf{F}$-dimension one, and so $r = n$.

Without any loss of generality we may assume that these homogeneous components are the subspaces $e_1\mathbf{F}, \ldots, e_n\mathbf{F}$. Since G permutes these subspaces we have

$$G \subset \mathrm{Mon}_n(U_1(\mathbf{F})) \text{ and } G_0 \subset D_n(\mathbf{F}) = U_1(\mathbf{F}) \times \ldots \times U_1(\mathbf{F}).$$

We cannot have $\mathbf{F} = \mathbf{R}$ because $D_n(\mathbf{R})$ is a finite group and G_0 has positive dimension.

Let $\theta: G \to S_n$ be the restriction of the canonical homomorphism $\mathrm{Mon}_n(U_1(\mathbf{F})) \to S_n$. If $R \in G$ is a non-diagonal reflection then $\theta(R)$ is a transposition. Since G is irreducible, $\theta(G)$ is a transitive subgroup of S_n. Using that G is a t.r.g., it follows now that $\theta(G) = S_n$.

For $g \in G_0$ let $\lambda_1(g), \ldots, \lambda_n(g)$ be the diagonal entries of g. Then each $\lambda_s: G_0 \to U_1(\mathbf{F})$ is a group homomorphism.

If $R \in G$ is a non-diagonal reflection then R fixes $n - 2$ of the standard basic vectors of $\mathbf{F}^n$, say the vectors $e_3, \ldots, e_n$. Furthermore $Re_1 = e_2\alpha$ and $Re_2 = e_1\alpha^{-1}$ for some $\alpha \in U_1(\mathbf{F})$. We have

$$RgRe_1 = Rge_2\alpha = Re_2\lambda_2(g)\alpha = e_1\alpha^{-1}\lambda_2(g)\alpha$$

and so $\lambda_1(RgR) = \alpha^{-1}\lambda_2(g)\alpha$. In particular $\lambda_1(G_0)$ and $\lambda_2(G_0)$ are conjugate in $U_1(F)$. Replacing g by RgR in $\lambda_1(RgR) = \alpha^{-1}\lambda_2(g)\alpha$, we obtain $\lambda_2(RgR) = \alpha\lambda_1(g)\alpha^{-1}$. If $s \geqq 2$ it is easy to check that $\lambda_s(RgR) = \lambda_s(g)$. Hence

$$RgRg^{-1} = \mathrm{diag}\,(\mu, \nu, 1, \ldots, 1)$$

where

$$\mu = \lambda_1(RgRg^{-1}) = \alpha^{-1}\lambda_2(g)\alpha\lambda_1(g)^{-1},$$
$$\nu = \lambda_2(RgRg^{-1}) = \alpha\lambda_1(g)\alpha^{-1}\lambda_2(g)^{-1}.$$

If $\mathbf{F} = \mathbf{C}$ then $\nu = \mu^{-1}$ and since λ_1 and λ_2 are different characters of G_0 all matrices of the form $\mathrm{diag}\,(\mu, \mu^{-1}, 1, \ldots, 1)$ are in G_0. By using other reflections in G, we may now conclude that $G_0 \supset D_n(\mathbf{C}) \cap SU(n)$. Since $G_0 \subset D_n(\mathbf{C})$ and $A \in G$ implies $\det(A) = \pm 1$ we must have $G_0 = D_n(\mathbf{C}) \cap SU(n)$. It is now easy to see that G consists of all monomial matrices in $U(n)$ having determinant ± 1. This is case 1) of the theorem.

Next assume that $\mathbf{F} = \mathbf{H}$ and that $\lambda_s(G_0)$ is a circle subgroup of $Sp(1)$ for all s. Clearly we may assume that $\lambda_s(G_0) = U(1)$ for all s. Since $\lambda_1(RgR) = \alpha^{-1}\lambda_2(g)\alpha$, we have $\alpha^{-1}U(1)\alpha = U(1)$ and so $\alpha \in U(1)$ or $\alpha \in jU(1)$. Since a similar conclusion is valid for each non-diagonal reflection in G, we deduce that G is contained in the group

$$M = \mathrm{Mon}_n(U(1) \cup jU(1)).$$

If $\alpha \in U(1)$ we have again $\nu = \mu^{-1}$ while if $\alpha \in jU(1)$ we obtain $\nu = \mu$. Now assume that there exist reflections R_1, R_2 in G such that $\theta(R_1) = \theta(R_2) = (1, 2)$, $R_1e_1 = e_2\alpha_1$ and $R_2e_1 = e_2\alpha_2$ with $\alpha_1 \in U(1)$, $\alpha_2 \in jU(1)$. In that case all matrices of the form

$$\mathrm{diag}(\mu, \mu^{-1}, 1, \ldots, 1),\ \mu \in U(1),$$

or

$$\mathrm{diag}(\mu, \mu, 1, \ldots, 1),\ \mu \in U(1),$$

are in G_0. Consequently $G_0 = D_n(\mathbf{C})$ and it follows easily that $G = M$. Of course the same conclusion is obtained if the indices 1,2 are replaced by r, s $(r \neq s)$. This is case 4) of the theorem.

https://doi.org/10.4153/CJM-1980-022-1 Published online by Cambridge University Press

Now assuming that such a pair of indices does not exist, i.e., for each pair of indices r, s $(r \neq s)$ all reflections $R \in G$ satisfying $\theta(R) = (r, s)$ are of the same type: either all have corresponding α in $U(1)$ or all have α in $jU(1)$. Using Lemma 10, we may assume that there is an m $(1 \leqq m \leqq n)$ such that $\alpha \in U(1)$ precisely for pairs r, s satisfying $r, s \leqq m$ or $r, s \geqq m + 1$. Hence each of the reflections $R \in G$ belongs to the group

$$M_m = J_m^{-1}\,\mathrm{Mon}_n(U(1))J_m$$

where

$$J_m = \left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & jI_{n-m} \end{array}\right).$$

Therefore $G \subset M_m$ and replacing G by a conjugate we may assume that $m = n$, i.e., $G \subset \mathrm{Mon}_n(U(1))$. It is clear that $D_n(\mathbf{C}) \cap SU(n) \subset G_0 \subset D_n(\mathbf{C})$ and since $A \in G$ now implies $\det(A) = \pm 1$, we conclude that G consists of monomial matrices in $U(n)$ having determinant ± 1. This is case 3) of the theorem.

Finally, let $\mathbf{F} = \mathbf{H}$ and $\lambda_s(G_0) = Sp(1)$. The kernel N_s of λ_s is connected because $G_0/N_s \cong Sp(1)$ and $Sp(1)$ is simply connected. Therefore there exists a unique normal subgroup P_s of G_0 such that $G_0 = N_s \times P_s$ and, of course, $P_s \cong Sp(1)$. Since λ_1 and λ_2 are non-equivalent representations of G_0, we have $N_1 \neq N_2$. This implies that $P_1 \subset N_2$. In general we have $P_r \subset N_s$ if $r \neq s$. Hence the elements of P_1 are precisely the matrices

$$\mathrm{diag}(\mu, 1, \ldots, 1), \ \mu \in Sp(1).$$

Hence we have $G_0 = D_n(\mathbf{H})$ and since $\theta(G) = S_n$, we have $G = \mathrm{Mon}_n(Sp(1))$. Thus we have case 2) of our theorem.

This completes the proof.

7. Case when G_0 is irreducible. In this section we consider the case when $\mathbf{F}^n$ is irreducible as a $(G_0, \mathbf{F})$-bimodule.

We shall need the following lemmas.

LEMMA 12. *The normalizer of $SO(n)$ in $Sp(n)$ is $Sp(1) \cdot O(n)$.*

Proof. We use induction on n. The claim is trivial for $n = 1$. Let $n = 2$. Then every automorphism of $SO(2)$ is induced by conjugation by some element of $O(2)$. Hence the normalizer of $SO(2)$ is the product of $O(2)$ and the centralizer of $SO(2)$. This centralizer is $Sp(1) \cdot SO(2)$ and the claim follows.

Now let $n \geqq 3$ and let

$$P = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right).$$

If A normalizes $SO(n)$ then APA^{-1} has $n - 2$ eigenvalues equal $+1$ and has order 4. Hence APA^{-1} is conjugate to P in $O(n)$. Thus there exists $B \in O(n)$

such that BA commutes with P. Therefore we may assume that A is of the form

$$A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$$

where $A_1 \in Sp(2)$ and $A_2 \in Sp(n-2)$. By induction, we have $A_1 \in Sp(1) \cdot O(2)$ and $A_2 \in Sp(1) \cdot O(n-2)$. This allows us to assume further that $A_1 = \lambda I_2$, $A_2 = \mu I_{n-2}$ where $\lambda, \mu \in Sp(1)$. It is now easy to see that $\lambda\mu^{-1}$ must be real and so $A = \lambda C$, $C \in O(n)$.

This completes the proof.

LEMMA 13. *If $n \geqq 2$ then $U(n) \cup jU(n)$ is the normalizer in $Sp(n)$ of both $SU(n)$ and $U(n)$.*

Proof. It is clear that the centralizer of $SU(n)$ in $Sp(n)$ is $U(1)$. On the other hand, the centralizer of $U(1)$ in $Sp(n)$ is $U(n)$. Consequently $U(1)$, $U(n)$ and $SU(n)$ have the same normalizer in $Sp(n)$. Now it suffices to apply Lemma 4.

THEOREM 14. *Let G be an infinite irreducible* t.r.g. *in $U_n(\mathbf{F})$, $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. Assume that $\mathbf{F}^n$ is irreducible as a $(G_0, \mathbf{F})$-bimodule. We have $n \geqq 2$.*

If $\mathbf{F} = \mathbf{R}$ then $G = O(n)$.

If $\mathbf{F} = \mathbf{C}$ then either $G = \{A \in U(n) | \det(A) = \pm 1\}$ or $n \geqq 3$ and G is a conjugate of $O(n)$.

If $\mathbf{F} = \mathbf{H}$ then G is one of the following: $Sp(n)$, a conjugate of $\{A \in U(n) | \det(A) = \pm 1\}$ $(n \geqq 3)$, a conjugate of $O(n)$ $(n \geqq 3)$, or a conjugate of $Sp(1) \cdot O(2)$ $(n = 2)$.

Proof. Let R_0 be a fixed reflection in G. Recall that $\mathscr{R}_n(\mathbf{F})$ is diffeomorphic to $P(\mathbf{F}^n)$ and is a Riemannian globally symmetric space. The group $U^n(\mathbf{F})$ acts on $\mathscr{R}_n(\mathbf{F})$ by conjugation.

The orbit of R_0 for the action of G_0 on $\mathscr{R}_n(\mathbf{F})$ is a closed submanifold M. Note that M has positive dimension because G_0 is connected and $\mathbf{F}^n$ is an irreducible $(G_0, \mathbf{F})$-bimodule.

We claim that M is a totally geodesic submanifold of $\mathscr{R}_n(\mathbf{F})$. If $R \in M$ then RMR is again an orbit of G_0 and since $R \in M \cap RMR$ we must have $RMR = M$.

Let X be a tangent vector of M at R_0. Choose a smooth regular curve $f: (-\epsilon, +\epsilon) \rightarrow M$ such that $f(0) = R_0$ and $f'(0) = X$. Moreover we assume that f is injective and that its image lies in a sufficiently small neighborhood of R_0. For $\alpha \in (-\epsilon, +\epsilon)$, $\alpha \neq 0$ let g_α be the unique geodesic in $\mathscr{R}_n(\mathbf{F})$ joining the points R_0 and $f(\alpha)$. Moreover we assume that α is chosen so that the ratio of the distance from $f(0)$ to $f(\alpha)$ to the total length of the geodesic is irrational. This condition is easy to satisfy because all geodesics in $\mathscr{R}_n(\mathbf{F})$ have the same length (see [**7**, p. 356]). We have $g_\alpha(0) = f(0) = R_0$ and $g_\alpha(\beta) = f(\alpha)$. Since $g_\alpha(0)$ and $g_\alpha(\beta)$ are in M and for $R \in M$ we have $RMR = M$, it follows that

$g_\alpha(k\beta) \in M$ for all integers k. From our choice of α it follows that the points $g_\alpha(k\beta)$ are dense on this geodesic and since M is closed we conclude that the whole geodesic g_α lies in M. Now we let $\alpha \to 0$, still subject to the same conditions as before. Then the limiting position of g_α is the geodesic g with $g(0) = R_0$, and $g'(0)$ a scalar multiple of $f'(0) = X$. It follows that the whole geodesic g lies in M, and so M is totally geodesic.

By Theorem 2 and corresponding results for $P(\mathbf{C}^n)$ and $P(\mathbf{R}^n)$, we have $M = (\phi \circ p)(W)$ where W is a suitable subspace of $\mathbf{F}^n$, and $p:\mathbf{F}^n\backslash\{0\} \to P(\mathbf{F}^n)$ and $\phi:P(\mathbf{F}^n) \to \mathscr{R}_n(\mathbf{F})$ are the canonical maps defined in Section 1. Clearly, W is G_0-invariant and so $W\mathbf{F}$ must coincide with $\mathbf{F}^n$, i.e., the width of M must be n.

If $\mathbf{F} = \mathbf{R}$ then $W = \mathbf{R}^n$, $M = \mathscr{R}_n(\mathbf{R})$ and so $G = O(n)$.

If $\mathbf{F} = \mathbf{C}$ then either $W = \mathbf{C}^n$ or we may assume that $W = \mathbf{R}^n$. In the first case $M = \mathscr{R}_n(\mathbf{C})$ and so $G = \{A \in U(n)|\det(A) = \pm 1\}$. In the second case $M = \mathscr{R}_n(\mathbf{R})$ and so $G \supset O(n)$. Since M is stable under G_0, it follows that G_0 normalizes $O(n)$. By Lemma 12, $G_0 \subset U(1) \cdot O(n)$, and since G_0 is irreducible we have $n \geqq 3$. We have $G_0 = SO(n)$ or $G_0 = U(1) \cdot SO(n)$, so G normalizes $SO(n)$ and $G \subset U(1) \cdot O(n)$, by Lemma 12. Since every reflection $R \in G$ lies in $O(n)$ (Lemma 7), we have $G = O(n)$.

Now let $\mathbf{F} = \mathbf{H}$. Consider first the case when M is of type 1 (see Theorem 2). In that case we can assume that $W = \mathbf{R}^n$ and so $M = \mathscr{R}_n(\mathbf{R})$, $G \supset O(n)$. Clearly G_0 stabilizes M and so it normalizes $O(n)$. By Lemma 12 $G_0 \subset Sp(1) \cdot SO(n)$. Since G_0 is irreducible, it is non-abelian. It follows that in all cases G normalizes $SO(n)$ and so $G \subset Sp(1) \cdot O(n)$.

If $n \geqq 3$ then, by Lemma 7, every reflection $R \in G$ lies in $O(n)$ forcing $G = O(n)$. If $n = 2$ the irreducibility of G_0 implies that $G_0 = Sp(1) \cdot SO(2)$, and so $G = Sp(1) \cdot O(2)$.

Now consider the case when M is of type 2 (see Theorem 2). We can now assume that $W = \mathbf{C}^n$, and so $M = \mathscr{R}_n(\mathbf{C})$, $G \supset \{A \in U(n)|\det(A) = \pm 1\}$. Clearly, G_0 normalizes $SU(n)$ and so, by Lemma 13, $G_0 \subset U(n)$. Thus G_0 is either $SU(n)$ or $U(n)$. Lemma 13 implies that $G \subset U(n) \cup jU(n)$. If $n \geqq 3$, using Lemma 6, we can conclude that $G \subset U(n)$ and so $G = \{A \in U(n)| \det(A) = \pm 1\}$. If $n = 2$ we cannot have $G_0 = SU(2)$ because the subspace $\begin{pmatrix} 1 \\ j \end{pmatrix}\mathbf{H}$ is $SU(2)$-invariant. Then $G_0 = U(2)$ and so $G = U(2) \cup jU(2) = A^{-1}Sp(1) \cdot O(2)A$, where

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -j \\ i & k \end{pmatrix}.$$

If M is of type 3 (see Theorem 2) then $W = \mathbf{H}^n$, $M = \mathscr{R}_n(\mathbf{H})$, and so $G = Sp(n)$.

Finally, let M be of type 4. Since the width of M is 2 we must have $n = 2$. We may assume that W consists of all vectors $a = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ where $\xi \in \mathbf{R}$ and $\eta + \bar{\eta} = 0$. If a is normalized so that $a^*a = 1$ we can write $\xi = \cos t$,

$\eta = \alpha \sin t$, where $\alpha^2 = -1$. Then $(\phi \circ \rho)\ (a)$ is the reflection

$$\begin{pmatrix} -\cos\theta & \alpha\sin\theta \\ -\alpha\sin\theta & \cos\theta \end{pmatrix}, \quad \theta = 2t.$$

Thus M consists of all such matrices. Each of these reflections interchanges the subspaces $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \mathbf{H}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \mathbf{H}$. Since M is normalized by G_0, these subspaces must be G_0-invariant. This is a contradiction, and so this case cannot occur.

The proof is now completed.

Note added in proof. The connected totally geodesic submanifolds of $P(\mathbf{H}^n)$, which we have classified in Theorem 2, have been determined earlier by J. A. Wolf in his paper "Elliptic spaces in Grassmann manifolds", Illinois J. Math. *7* (1963), 447–462. His proof is different and is based on the classification of compact Riemannian symmetric spaces of rank one.

Finite subgroups of the quaternionic unitary group $Sp(n)$ which are generated by pseudo-reflections have been classified recently by A. M. Cohen in his paper "Finite quaternionic reflection groups", Memorandum Nr. *229* (1978), Technische Hogeschool Twente, Enschede, Netherlands.

T. Yu. Sysoeva has classified in her paper "Reductive linear algebraic groups generated by quasi-reflections," Serdika *1* (1975), No. 3–4, 337–345, all complex groups mentioned in the title. (A quasi-reflection is an invertible linear transformation u such that $\text{rank}(u - 1) = 1$.)

References

1. K. Abe, *Application of a Riccati type differential equation to Riemannian manifolds with totally geodesic distributions*, Tôhoku Math. J. *25* (1973), 425–444.

2. A. M. Cohen, *Finite complex reflection groups*, Ann. Scient, Ecole Norm. Sup. *9* (1976), 379–436.

3. H. S. M. Coxeter, *Discrete groups generated by reflections*, Annals of Math. *35* (1934), 588–621.

——— *Groups generated by unitary reflections of period two*, Can. J. Math. *9* (1957), 243–272.

4. D. Ž. Djoković and J. Malzan, *Products of reflections in the quaternionic unitary group*, J. of Algebra *59* (1979), 399–411.

5. M. L. Eaton and M. Perlman, *Generating $O(n)$ with reflections*, Pacific J. Math. *73* (1977), 73–80.

6. ——— *Reflection groups, generalized Schur functions and the geometry of majorization*, Annals of Probability *5* (1977), 829–860.

7. S. Helgason, *Differential geometry and symmetric spaces* (Academic Press, New York, 1969).

8. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. *II* (Interscience, New York, 1969).

9. G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Can. J. Math. *6* (1954), 274–304.

10. J. A. Wolf, *Spaces of constant curvature* (McGraw-Hill, New York, 1967).

University of Waterloo,
Waterloo, Ontario