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Free Frequencies for a Three Layered Earth Model

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Abstract. The Hamiltonian formalism is applied to the treatment of the free motion of a three layered Earth model, where the effects of the pressure coupling, centrifugal deformation as well as gravitational and viscous–electromagnetic torques, are considered. Analytical expressions of the four normal modes of this dynamical system are derived.

1. Introduction

The Hamiltonian formalism of Kinoshita (1977) for the rotation of the rigid Earth is being extended step by step by Getino and Ferrándiz in order to account for non–rigid models. Up to now, the deformation of the mantle produced by both rotational and tidal perturbations, the delay in the response due to the anelasticity, the effect of the fluid outer core (FOC) and the dissipative effects at the core–mantle boundary (CMB) have been formulated. The first accurate analytical nutation series deduced from this Hamiltonian theory can be found in Getino and Ferrándiz (1999). They provide the highest accuracy ever obtained by any analytical nutation series, since the deviation in CEP (celestial ephemeris pole) offsets with respect to IERS 96 is kept below 1 mas in the time domain.

At present, we are working on the improvement of the theory by considering a three–layer Earth model, in which different interactions between each layer are treated. This paper is devoted to the study of the free nutations corresponding to such a model. However, unlike previous papers, which were formulated in the classical Andoyer–like variables (Getino and Ferrándiz, 1998), we introduce here a new non–singular set of canonical variables. This set presents several advantages with respect to the former one, since the equations of motion can be linearized directly under the Hamiltonian framework.

2. Earth model

In this work we consider a three–layer Earth model composed of:

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- A deformable axial–symmetrical mantle,
- A fluid outer core (FOC),
- A deformable axial–symmetrical solid inner core (SIC).

We assume that the three layers perform rigid rotations about the same fixed point without any disturbing force external to the Earth. However, there are different effects which must be taken into account in order to describe properly the dynamical behaviour of this system. Namely, we have modelled the following ones.

1. Pressure (inertial) coupling: This torque is due to pressure forces at FOC-mantle and FOC-SIC boundaries. In a Hamiltonian framework (variational approach) it can be obtained without using the equations of hydrodynamics. We only need to build the kinetic energy of the system.

   With our hypothesis the procedure reduces to computing the inertia tensors of each layer. The expressions of the tensors of inertia of the mantle and SIC are straightforwardly obtained since they are rigid bodies. The tensor of inertia of the FOC is more difficult since its shape changes during the evolution of the system. Nevertheless, its expression is achieved by means of a suitable construction (see Escapa, Getino and Ferrándiz 1999).

2. Gravitational coupling: A torque of gravitational origin is exerted on SIC by the regions of the Earth outside the inner core. This torque is derived from a potential, V_g, which is a function of the angle between the revolution axes of the mantle and SIC.

3. Rotational deformation: As a consequence of rotation, the Earth suffers a deformation. This effect is taken into account by adding new terms to the tensors of inertia. These terms are linear combinations of the angular velocities, following a generalization of Sasao, Okubo and Saito (1980) for a two–layer Earth model. These expressions are also followed by Mathews *et al.* (1991).

4. Viscous–electromagnetic torques: Dissipative torques of electromagnetic and viscous origin are considered in the mantle-FOC and FOC-SIC boundaries. These torques are linear combinations of the difference of angular velocities between each layer. Let us remark that these torques are not derivable from a potential function.

3. Canonical Variables

Once we have modelled, from a mechanical point of view, all the considered effects, we formulate the problem by means of the Hamiltonian formalism. At this stage the choice of the canonical variables is crucial.

Up to now, the Hamiltonian nonrigid Earth theory (by Getino and Ferrándiz [1995, 1998,...], among others) was formulated by means of the Andoyer–like variables, see Figure 1. These variables present the inconvenience that they have virtual singularities for the short-axis mode motion, and therefore the

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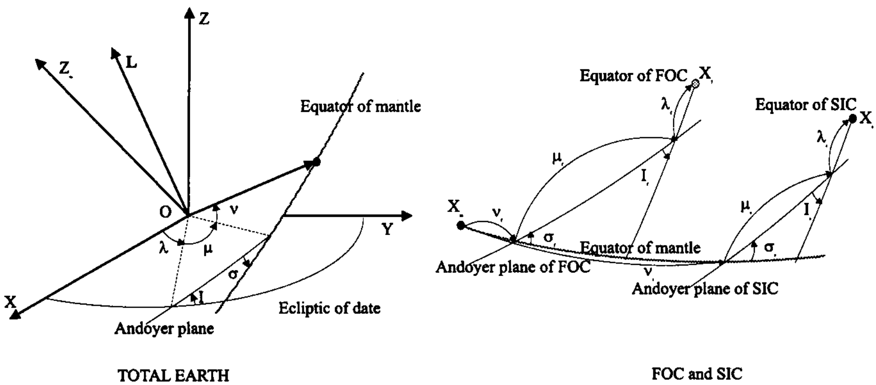

Figure 1. Andoyer variables

free motion cannot be solved in this set. Instead of using Andoyer variables we have employed a non-singular set of canonical variables. This set is well-defined in a short-axis mode motion, whence the free motion of the Earth can be solved without leaving the Hamiltonian framework. On the other hand, this set has the disadvantage that it does not have a clear kinematical or geometrical interpretation, a situation which does not hold in the case of the Andoyer set.

For the three–layer Earth model the non-singular set is composed of three groups of six variables:

- Y_1, Y_2, Λ, y_1, y_2 and $\lambda \longrightarrow$ for the total Earth
- $Y_{1f,s}$, $Y_{2f,s}$, $Y_{3f,s}$, $y_{1f,s}$, $y_{2f,s}$ and $y_{3f,s} \longrightarrow$ for FOC and SIC.

Λ and λ being the same variables as for the Andoyer set.

4. Equations of motion

The first order differential equations which govern the dynamical behaviour of the system are

$$\dot{Y} = -\frac{\partial H}{\partial y} + Q_y, \; \dot{y} = \frac{\partial H}{\partial Y} - Q_Y. \tag{1}$$

In these expressions:

- $H = T + V_g$ is the Hamiltonian of the system. T is the kinetic energy, which includes the effects of the pressure coupling and elasticity, and V_g is the potential derived from the gravitational torque.

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- Q_y, Q_Y are the generalized forces associated with the dissipative torques. The construction of these forces in a Hamiltonian framework are rather involved though straightforward (see Getino, González and Escapa 1999).

In order to solve analytically the differential system (1) we perform a linearization in the variables Y_2, y_2, Y_{2f}, y_{2f}, Y_{2s}, y_{2s}, Y_{3s} and y_{3s}. As usual, Poincaré (1910), it is stated that the third component of the angular velocity of each layer is a constant equal to the mean angular velocity of the Earth, Ω. In this way, a differential linear system of constant coefficients is obtained for the variables in which the linearization has been performed. This system provides us the four frequencies of the polar motion.

5. Normal modes

The analytical expressions of these free frequencies turn out to be

$$\begin{aligned} m_1 &= \Omega\,(A/A_m)(e-\kappa) && \rightarrow \ \textbf{CW} \\ m_2 &= \Omega\left[-1-(1+A_f/A_m)\left(e_f-\kappa_f+\tilde{R}_f\right)\right] && \rightarrow \ \textbf{RFCN} \\ m_3 &= \Omega\left[-1+\left(\delta+\kappa_s-\tilde{R}_f-k_g\,(e_s-\delta)\right)\right] && \rightarrow \ \textbf{PFCN} \\ m_4 &= \Omega\left(1+k_g\right)(e_s-\delta) && \rightarrow \ \textbf{ICW} \end{aligned} \tag{2}$$

These are the Chandler Wobble, Retrograde Free Core Nutation, Prograde Free Core Nutation and Inner Core Wobble modes. As it could be observed from (2), the modes depend on several Earth parameters. Namely

- Ellipticities and moments of inertia of the total Earth, FOC and SIC: e, e_f, e_s, A, A_m, A_f and A_s.
- A new parameter, δ:

  It is related to the change of shape of FOC during the evolution of the system (Escapa, Getino and Ferrándiz 1999).
- Elasticity coefficients: κ, κ_f and κ_s.

  These parameters characterize the elastic response of the total Earth, FOC and SIC.
- Complex dissipative coefficients: $\tilde{R}_f$ and $\tilde{R}_s$.

  They depend on the dissipative coupling constants over FOC boundaries.
- Gravitational coupling coefficient: k_g.

  It is determined by the density and ellipticity outside the inner core.

6. Conclusions

The inclusion of the dissipative torques in this Earth model has provided a damping in the **RFCN** and **PFCN** modes. Note that **CW** does not experience, in this order of approximation, any change in its classical frequency expression.

Therefore, any damping of the **CW** mode is due to other mechanisms different from those considered here, probably to processes within the mantle such as the anelasticity. On the other hand, the most relevant consequence of the inclusion of the gravitational coupling is to shorten the **ICW** period by about 70 per cent.

Finally, let us point out that using the non-singular set, the free motion problem has been notably simplified, though, as it was remarked, these variables do not have clear kinematical meaning. We are developing a Fukushima set for a nonrigid Earth model, based on the canonical variables of Fukushima (1994), which is hoped to join the advantages of the Andoyer and non-singular canonical sets.

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