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QUASI-ISOMETRIC EMBEDDINGS INAPPROXIMABLE BY ANOSOV REPRESENTATIONS

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Abstract We construct examples of quasi-isometric embeddings of word hyperbolic groups into $\mathsf{SL}(d,\mathbb{R})$ for $d \geq 4$ which are not limits of Anosov representations into $\mathsf{SL}(d,\mathbb{R})$. As a consequence, we conclude that an analogue of the density theorem for $\mathsf{PSL}(2,\mathbb{C})$ does not hold for $\mathsf{SL}(d,\mathbb{R})$ when $d \geq 4$.

Key words and phrases: hyperbolic groups; Anosov representations; quasi-isometric embeddings

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1. Introduction

Let $g \geq 1$ and Γ_g be the word hyperbolic group with presentation

$$\Gamma_g = \left\langle \begin{matrix} a_1,b_1,\dots,a_{2g},b_{2g}, \\ c_1,d_1,\dots,c_{2g},d_{2g} \end{matrix} \middle| \begin{matrix} [a_1,b_1]\cdots[a_{2g},b_{2g}],[c_1,d_1]\cdots[c_{2g},d_{2g}], \\ [a_1,b_1]\cdots[a_g,b_g]\cdot[c_1,d_1]\cdots[c_g,d_g] \end{matrix} \right\rangle.$$

The group Γ_g is the fundamental group of a book of I-bundles, and by Thurston's geometrization theorem [19] it admits a convex co-compact representation into $\mathsf{PSL}(2,\mathbb{C})$ and thus Anosov representations into $\mathsf{SL}(d,\mathbb{R})$ for every $d \geq 4$. In this paper, we construct the first examples of quasi-isometric embeddings of word hyperbolic groups into $\mathsf{SL}(d,\mathbb{R})$ for $d \geq 5$ which are not limits of Anosov representations into $\mathsf{SL}(d,\mathbb{R})$. More precisely, we prove the following:

Theorem 1.1. *Let $g \geq 1$ and Γ_g be the word hyperbolic group already defined.*

- (i) *For every $d \geq 5$ there exists a quasi-isometric embedding $\rho : \Gamma_g \to \mathsf{SL}(d,\mathbb{R})$ such that ρ is not a limit of Anosov representations of Γ_g into $\mathsf{SL}(d,\mathbb{R})$.*
- (ii) *For $g \geq 4$, there exists a strongly irreducible quasi-isometric embedding $\psi : \Gamma_g \to \mathsf{SL}(12,\mathbb{R})$ such that ψ is not a limit of Anosov representations of Γ into $\mathsf{SL}(12,\mathbb{R})$.*

We remark that for $d \geq 6$ in Theorem 1.1(i), we may replace Γ_g with any one-ended word hyperbolic convex co-compact Kleinian group which admits a retraction to a free subgroup of rank at least 8 and is not virtually a free group or a surface group (see Theorem 3.1). The density conjecture for Kleinian groups established by the work of Brock and Bromberg [5], Brock, Canary, and Minsky [6], Namazi and Souto [20], and Ohshika [21] implies that every discrete and faithful representation of a word hyperbolic group into $\mathsf{PSL}(2,\mathbb{C})$ is an algebraic limit of Anosov representations. The representations constructed in Theorem 1.1 demonstrate the failure of the density conjecture for the higher-rank Lie group $\mathsf{SL}(d,\mathbb{R})$ for $d \geq 5$.

In infinitely many dimensions, Theorem 4.1 produces examples similar to those in Theorem 1.1(ii) whose elements are all semiproximal (i.e., admit a real eigenvalue of maximum modulus). Moreover, in Proposition 4.4 we also provide examples of quasi-isometric embeddings of surface groups and of free groups into $\mathsf{SL}(4,\mathbb{R})$ and $\mathsf{SL}(6,\mathbb{R})$ which are not in the closure of the space of Anosov representations. In particular, the density conjecture fails for $\mathsf{SL}(d,\mathbb{R})$ when $d \geq 4$.

An example of a quasi-isometric embedding of the free group of rank 2 which is not Anosov was constructed by Guichard in [12] (see also [11, Proposition A.1, p. 67]. Moreover, Guichard's example is unstable – that is, it is a limit of nondiscrete representations but also a limit of P_2-Anosov representations (see Definition 2.1) of the free group of rank 2 into $\mathsf{SL}(4,\mathbb{R})$.

For our constructions we shall use the following fact: For a P_1-Anosov subgroup Γ of $\mathsf{SL}(d,\mathbb{R})$, $d \geq 4$, every quasiconvex infinite-index subgroup Δ of Γ with connected Gromov boundary contains a finite-index subgroup whose infinite-order elements are all positively proximal (see Corollary 2.4). It follows that if $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is a limit of P_i-Anosov representations, then $\wedge^i \rho : \Gamma \to \mathsf{SL}\left(\wedge^i \mathbb{R}^d\right)$ is a limit of P_1-Anosov representations and the group $\wedge^i \rho(\Delta)$ contains a finite-index subgroup consisting entirely of positively semiproximal elements.

It is unknown to us whether there exist nearby deformations of the examples in Theorem 1.1 which are discrete, faithful, and Zariski dense in $\mathsf{SL}(d,\mathbb{R})$. In particular, we ask the following:

Question. *Does there exist an open neighbourhood U in $\mathrm{Hom}\big(\Gamma_g, \mathsf{SL}(d,\mathbb{R})\big)$ of the examples in Theorem 1.1 consisting entirely of discrete and faithful representations?*

The paper is organized as follows. In §2 we provide the necessary background on Anosov representations and prove Lemma 2.3, which is essential for our construction. In §3 we prove Theorem 1.1, and in §4 we prove Theorem 4.1, providing strongly irreducible examples in infinitely many dimensions.

2. Background

In this section, we define Anosov representations and prove two lemmas required for our construction.

Let $d \geq 2$ and denote by $(e_1, \ldots, e_d)$ the canonical basis of $\mathbb{R}^d$ and by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}^d$ so that the basis $(e_1, \ldots, e_d)$ is orthonormal. For a transformation $g \in \mathsf{SL}(d,\mathbb{R})$ we denote by $\ell_1(g) \geq \cdots \geq \ell_d(g)$ and $\sigma_1(g) \geq \cdots \geq \sigma_d(g)$ the moduli of the eigenvalues and the singular values of g in nonincreasing order, respectively. We recall that $\sigma_i(g)=\sqrt{\ell_i(gg^t)}$ for $1\le i\le d$, where g^t denotes the transpose matrix of g. For $1\le i\le d-1$, the matrix $g\in\mathsf{SL}(d,\mathbb{R})$ is called *P_i-proximal* if $\ell_i(g)>\ell_{i+1}(g)$. If $i=1$, we will say that g is *proximal*. If g is proximal, we denote by $\lambda_1(g)$ the unique eigenvalue of g of maximum modulus in which case $\lambda_1(g)=\pm\ell_1(g)$. A matrix g is called *semiproximal* if either $\ell_1(g)$ or $-\ell_1(g)$ is an eigenvalue of g. Obviously, if g is proximal then it is also semiproximal. A matrix g is called *positively semiproximal* if $\ell_1(g)$ is an eigenvalue of g, and *positively proximal* if g is proximal and $\lambda_1(g)=\ell_1(g)$.

2.1. Amalgamated products

Let $\{\Gamma_i\}_{i\in I}$ be a family of groups and H be a group, and suppose that there exists a family of monomorphisms $\varphi_i:H\hookrightarrow\Gamma_i$. The *amalgamated product of* $\{\Gamma_i\}_{i\in I}$ *with respect to* $\{\varphi_i\}_{i\in I}$ is the group with presentation

$$*_H\Gamma_i=\left\langle \Gamma_i,\ i\in I\,\middle|\,\mathrm{rel}(\Gamma_i),\ \varphi_i(h)^{-1}\varphi_j(h),\ i,j\in I,\ h\in H\right\rangle.$$

For every $i\in I$, the natural map $\iota_i:\Gamma_i\to *_H\Gamma_i$ is a monomorphism. For more details on amalgamated products, we refer the reader to [22].

2.2. Anosov representations

For a finitely generated group Γ we fix a left invariant word metric d_Γ induced by a finite generating subset of Γ; and for $\gamma\in\Gamma$, $|\gamma|_\Gamma$ denotes the distance of γ from the identity element $e\in\Gamma$. If Γ is word hyperbolic, $\partial_\infty\Gamma$ denotes the Gromov boundary of Γ. Every infinite-order element $\gamma\in\Gamma$ has exactly two distinct fixed points γ^+ and γ^- in $\partial_\infty\Gamma$, called the attracting and repelling fixed points of γ, respectively. If Γ is furthermore not virtually cyclic, $\partial_\infty\Gamma$ is perfect, and for every $x\in\partial_\infty\Gamma\setminus\{\gamma^+,\gamma^-\}$, we have $\lim_{n\to\infty}\gamma^{\pm n}x=\gamma^{\pm}$.

Let $\rho:\Gamma\to\mathsf{SL}(d,\mathbb{R})$ be a representation. Since Γ is finitely generated, there exist constants $A,a>0$ such that

$$\max\left\{\sigma_1(\rho(\gamma)),\sigma_d(\rho(\gamma))^{-1}\right\}\le\frac{\sigma_1(\rho(\gamma))}{\sigma_d(\rho(\gamma))}\le Ae^{a|\gamma|_\Gamma}$$

for every $\gamma\in\Gamma$. The representation ρ is called a *quasi-isometric embedding* if there exist constants $J,K>0$ such that

$$\frac{\sigma_1(\rho(\gamma))}{\sigma_d(\rho(\gamma))}\ge Ke^{J|\gamma|_\Gamma}$$

for every $\gamma\in\Gamma$. Equivalently, if we equip the symmetric space $\mathsf{X}_d=\mathsf{SL}(d,\mathbb{R})/K_d$, where $K_d=\mathsf{SO}(d)$, with the distance function

$$\mathsf{d}(gK_d,hK_d)=\left(\sum_{i=1}^{d}\left(\log\sigma_i\left(g^{-1}h\right)\right)^2\right)^{\frac{1}{2}},\quad g,h\in\mathsf{SL}(d,\mathbb{R}),$$

then ρ is a quasi-isometric embedding if and only if the orbit map of ρ, $\tau_\rho:(\Gamma,d_\Gamma)\to(\mathsf{X}_d,\mathsf{d})$, $\tau_\rho(\gamma)=\rho(\gamma)K_d$ for $\gamma\in\Gamma$, is a quasi-isometric embedding.

For a representation of a finitely generated group, a much stronger property than being a quasi-isometric embedding is being *Anosov*. Anosov representations were introduced by

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Labourie [17] in his study of Hitchin representations and further developed by Guichard and Wienhard in [13]. We define Anosov representations by using a characterization in terms of gaps between singular values of elements, established by Kapovich, Leeb, and Porti in [15] and Bochi, Potrie, and Sambarino in [4].

Definition 2.1. Let Γ be a finitely generated group and $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ be a representation. For $1 \leq i \leq d-1$, the representation ρ is called P_i-Anosov if there exist constants $C, a > 0$ with the property

$$\frac{\sigma_i(\rho(\gamma))}{\sigma_{i+1}(\rho(\gamma))} \geq Ce^{a|\gamma|_\Gamma}$$

for every $\gamma \in \Gamma$.

In addition, it was proved in [15] and [4] that a finitely generated group which admits an Anosov representation into $\mathsf{SL}(d,\mathbb{R})$ is necessarily word hyperbolic. We shall (a little abusively) call a representation $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ *Anosov into* $\mathsf{SL}(d,\mathbb{R})$ if it is P_i-Anosov for some $1 \leq i \leq d-1$. Note that ρ is P_i-Anosov if and only if the exterior power $\wedge^i \rho$ is P_1-Anosov. Moreover, since $\sigma_j\left(g^{-1}\right) = \sigma_{d-j+1}(g)^{-1}$ for $g \in \mathsf{SL}(d,\mathbb{R})$ and $j \in \{i, i+1\}$, the representation ρ is P_i-Anosov if and only if ρ is P_{d-i}-Anosov. The property of being Anosov is stable – that is, for every P_i-Anosov representation ρ there exists an open neighbourhood U of ρ in $\mathrm{Hom}(\Gamma, \mathsf{SL}(d,\mathbb{R}))$ consisting entirely of P_i-Anosov representations (see [17] and [13, Theorem 5.14]). Examples of Anosov representations include quasi-isometrically embedded subgroups of simple real rank 1 Lie groups and their small deformations into higher-rank Lie groups, Hitchin representations, and holonomies of strictly convex projective structures on closed manifolds.

For $1 \leq m \leq d-1$, denote by $\mathsf{Gr}_m\left(\mathbb{R}^d\right)$ the Grassmannian of m-planes in $\mathbb{R}^d$. Every P_i-Anosov representation $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ admits a unique pair of continuous, ρ-equivariant maps $\xi_\rho^i : \partial_\infty \Gamma \to \mathsf{Gr}_i\left(\mathbb{R}^d\right)$ and $\xi_\rho^{d-i} : \partial_\infty \Gamma \to \mathsf{Gr}_{d-i}\left(\mathbb{R}^d\right)$ called the *Anosov limit maps*. We refer the reader to [13] and [11] for a careful discussion of Anosov limit maps and their properties. We mention here some of their main properties:

(i) The maps ξ_ρ^i and ξ_ρ^{d-i} are *compatible* – that is, $\xi_\rho^i(x) \subset \xi_\rho^{d-i}(x)$ for every $x \in \partial_\infty \Gamma$.

(ii) For every $\gamma \in \Gamma$ of infinite order, $\rho(\gamma)$ is P_i- and P_{d-i}-proximal where $\xi_\rho^i(\gamma^+)$ and $\xi_\rho^{d-i}(\gamma^+)$ are the attracting fixed points of $\rho(\gamma)$ in $\mathsf{Gr}_i\left(\mathbb{R}^d\right)$ and $\mathsf{Gr}_{d-i}\left(\mathbb{R}^d\right)$, respectively.

(iii) The maps ξ_ρ^i and ξ_ρ^{d-i} are *transverse* – that is, for every $x, y \in \partial_\infty \Gamma$ with $x \neq y$, $\mathbb{R}^d = \xi_\rho^i(x) \oplus \xi_\rho^{d-i}(y)$.

For a finitely generated group Γ, we denote by $\Gamma(2)$ the intersection of all finite-index subgroups of Γ of index at most 2. Note that since Γ is finitely generated, it has finitely many subgroups of index at most 2, and hence $\Gamma(2)$ is a finite-index subgroup of Γ.

An open subset Ω of $\mathbb{P}\left(\mathbb{R}^d\right)$ is called *properly convex* if it is bounded and convex in an affine chart of $\mathbb{P}\left(\mathbb{R}^d\right)$.

We shall use the following observation:

Observation 2.2. *Suppose that Γ is a finitely generated subgroup of $\mathsf{GL}(d,\mathbb{R})$ which preserves a properly convex domain Ω of $\mathbb{P}(\mathbb{R}^d)$. Then the finite-index subgroup $\Gamma(2)$ of Γ preserves a properly convex open cone C in $\mathbb{R}^d$.*

Proof. Let $\pi:\mathbb{R}^d\setminus\{(0,\dots,0)\}\to\mathbb{P}(\mathbb{R}^d)$ be the natural projection. There exists a properly convex open cone $C\subset\mathbb{R}^d$ such that $\pi^{-1}(\Omega)=C\cup(-C)$ and $C\cap(-C)$ is empty. Note that $H:=\{g\in\Gamma: gC=C\}$ is a subgroup of Γ. If H is a proper subgroup of Γ, given $w\in\Gamma\setminus H$ we have $wC=-C$ and hence $\Gamma=H\cup wH$. It follows that H is a finite-index subgroup of Γ of index at most 2. In particular, $\Gamma(2)$ is a subgroup of H. □

The key property of Anosov representations that we use for our construction is that when Γ is neither a free group nor a surface group, then for every P_1-Anosov representation $\rho:\Gamma\to\mathsf{SL}(d,\mathbb{R})$, the image $\rho(\Gamma)$ contains a quasiconvex subgroup with connected Gromov boundary, whose elements are all positively proximal. Given a representation $\psi:\Gamma\to\mathsf{GL}(d,\mathbb{R})$ of a word hyperbolic group Γ, a continuous ψ-equivariant map $\xi:\partial_\infty\Gamma\to\mathsf{Gr}_i(\mathbb{R}^d)$ (if it exists) is called *dynamics-preserving* if for every infinite-order element $\gamma\in\Gamma$, $\psi(\gamma)$ is P_i-proximal and $\xi(\gamma^+)$ is the attracting fixed point of $\psi(\gamma)$ in $\mathsf{Gr}_i(\mathbb{R}^d)$.

The following lemma is essential for the construction of our examples:

Lemma 2.3. *Let Γ be a word hyperbolic group and Δ be a quasiconvex and infinite-index subgroup of Γ such that $\partial_\infty\Delta$ is connected. Let $d\geq 4$ and $\rho:\Gamma\to\mathsf{SL}(d,\mathbb{R})$ be a representation. Suppose that there exists a sequence $\{\rho_n:\Gamma\to\mathsf{SL}(d,\mathbb{R})\}_{n\in\mathbb{N}}$ of representations such that the following hold:*

(i) *For every $n\in\mathbb{N}$, ρ_n admits a continuous, ρ_n-equivariant, dynamics-preserving map $\xi_{\rho_n}:\partial_\infty\Gamma\to\mathbb{P}(\mathbb{R}^d)$.*

(ii) $\lim_n\rho_n=\rho$.

Then for every $\delta\in\Delta(2)$, $\rho(\delta)$ is positively semiproximal.

Proof. We first show that for every $n\in\mathbb{N}$ and $\delta\in\Delta(2)$, $\rho_n(\delta)$ is positively proximal. Let us fix $n\in\mathbb{N}$. Since in Γ, Δ has infinite index and is quasiconvex, we may find $w\in\Gamma$ such that w^+ and w^- are not in $\partial_\infty\Delta$. By definition, $\rho_n(w)$ is proximal with attracting fixed point in $\mathbb{P}(\mathbb{R}^d)$ the line $\xi_{\rho_n}(w^+)$. Let $V^-_{\rho_n(w)}\subset\mathbb{R}^d$ be the repelling hyperplane of $\rho_n(w)$. We claim that the connected compact set $\mathcal{C}_{\rho_n(\Delta)}:=\xi_{\rho_n}(\partial_\infty\Delta)$ is contained in the affine chart $A_n=\mathbb{P}(\mathbb{R}^d)\setminus\mathbb{P}\left(V^-_{\rho_n(w)}\right)$. If not, there exists $x\in\partial_\infty\Delta$ with $\xi_{\rho_n}(x)\in\mathbb{P}\left(V^-_{\rho_n(w)}\right)$ and hence $\rho_n(w^m)\xi_{\rho_n}(x)=\xi_{\rho_n}(w^mx)$ is contained in $\mathbb{P}\left(V^-_{\rho_n(w)}\right)$ for every $m\in\mathbb{N}$. However, $\lim_m\xi_{\rho_n}(w^mx)=\xi_{\rho_n}(\lim_m w^mx)=\xi_{\rho_n}(w^+)$, and $\xi_{\rho_n}(w^+)$ is not in $\mathbb{P}\left(V^-_{\rho_n(w)}\right)$. The claim follows, and $\xi_{\rho_n}(\partial_\infty\Delta)\subset A_n$.

Let $V_n=\langle u:[u]\in\mathcal{C}_{\rho_n(\Delta)}\rangle$. The connected set $\mathcal{C}_{\rho_n(\Delta)}$ also lies in the affine chart $A_n\cap\mathbb{P}(V_n)$ of $\mathbb{P}(V_n)$, and the convex hull $\mathrm{Conv}_{A_n\cap\mathbb{P}(V_n)}\left(\mathcal{C}_{\rho_n(\Delta)}\right)$ is preserved by $\rho_n|_{V_n}(\Delta)$. By definition, $\mathcal{C}_{\rho_n(\Delta)}$ spans V_n, so the interior $\mathrm{Int}\left(\mathrm{Conv}_{A_n\cap\mathbb{P}(V_n)}\left(\mathcal{C}_{\rho_n(\Delta)}\right)\right)$ of the convex hull of $\mathcal{C}_{\rho_n(\Delta)}$ in $A_n\cap\mathbb{P}(V_n)$ is a well-defined properly convex subset of $A_n\cap\mathbb{P}(V_n)$. In particular, the properly convex subset $\mathrm{Int}\left(\mathrm{Conv}_{A_n\cap\mathbb{P}(V_n)}\left(\mathcal{C}_{\rho_n(\Delta)}\right)\right)$ is preserved by

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$\rho_n|_{V_n}(\Delta)$. By Observation 2.2, there exists a properly convex open cone $C_n \subset V_n$ such that $\rho_n|_{V_n}(\delta)C_n = C_n$ for every $\delta \in \Delta(2)$. Note that for every $\delta \in \Delta(2)$, the attracting fixed point $\xi_{\rho_n}(\delta^+)$ of $\rho_n(\delta)$ is always in V_n, and $\rho_n|_{V_n}(\delta)$ is proximal. Thus, $\lambda_1(\rho_n(\delta)) = \lambda_1(\rho_n|_{V_n}(\delta))$ for every $\delta \in \Delta(2)$. By [2, Lemma 3.2], we have $\lambda_1(\rho_n(\delta)) > 0$ and hence $\rho_n(\delta)$ is positively proximal for every $\delta \in \Delta(2)$.

Now set $\delta \in \Delta(2)$. By the previous arguments, for every $n \in \mathbb{N}$, we have $\lambda_1(\rho_n(\delta)) > 0$ and there exists a unit vector $u_n \in \mathbb{R}^d$ such that $\rho_n(\delta)u_n = \lambda_1(\rho_n(\delta))u_n$. Up to passing to a subsequence, we may assume that $\lambda := \lim_n \lambda_1(\rho_n(\delta))$ exists. The number $\lambda > 0$ has to be an eigenvalue (not necessarily of multiplicity 1) of $\lim_n \rho_n(\delta) = \rho(\delta)$ of maximum modulus. The conclusion follows. □

We immediately deduce the following corollary:

Corollary 2.4. *Let Γ be a word hyperbolic group and Δ be a quasiconvex and infinite-index subgroup of Γ such that $\partial_\infty \Delta$ is connected. Let $d \geq 4$ and $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ be a representation. Suppose that there exists a sequence of P_i-Anosov representations $\{\rho_n : \Gamma \to \mathsf{SL}(d,\mathbb{R})\}_{n\in\mathbb{N}}$ such that $\lim_n \rho_n = \rho$. Then for every $\delta \in \Delta(2)$, $\wedge^i \rho(\delta)$ is positively semiproximal.*

On the other hand, the images of Anosov representations might contain elements which are not positively proximal. In fact, this is the case for all Fuchsian representations into $\mathsf{SL}(2,\mathbb{R})$.

For a group H, denote by $H^{(2)} = \left\langle \left\{ghg^{-1}h^{-1} : g,h \in H\right\}\right\rangle$ the commutator subgroup of H.

Lemma 2.5. *Let F_k denote the free group on $k \geq 2$ generators. Let $j : F_k \to \mathsf{SL}(2,\mathbb{R})$ be a quasi-isometric embedding and H be a free subgroup of F_k of rank at least 2. Then for every $a \in F_k \setminus H$, there exists $w \in H^{(2)}$ such that $\lambda_1(j(wa)) < 0$.*

Proof. Note that $j\left(H^{(2)}\right)$ is discrete in $\mathsf{SL}(2,\mathbb{R})$; hence by [9, Lemma 2] (see also [1, Theorem 1.6]), there exists $w_0 \in H^{(2)}$ such that $\lambda_1(j(w_0)) < 0$. Then there exists $h \in \mathsf{GL}(2,\mathbb{R})$ such that

$$j(w_0) = h \begin{bmatrix} \lambda_1(j(w_0)) & 0 \\ 0 & \frac{1}{\lambda_1(j(w_0))} \end{bmatrix} h^{-1}.$$

Since $\left\{w_0^+, w_0^-\right\} \cap \{a^+, a^-\}$ is empty and j is P_1-Anosov, by transversality we have that the line $j(w_0)\xi_1^j(a^\pm) = \xi_1^j(w_0 a^\pm)$ is different from $\xi_1^j(a^+)$ and $\xi_1^j(a^-)$ and hence $\left\langle h^{-1}j(a)he_1, e_1\right\rangle$ is not zero. Then we notice that

$$\lim_{n\to\infty} \frac{\lambda_1\left(j\left(w_0^n a\right)\right)}{\lambda_1\left(j\left(w_0^n\right)\right)} = \left\langle h^{-1}j(a)he_1, e_1\right\rangle$$

and hence we have

$$\lim_{n\to\infty} \frac{\lambda_1\left(j\left(w_0^{2n+1} a\right)\right)}{\lambda_1\left(j\left(w_0^{2n} a\right)\right)} = \lambda_1(j(w_0)) < 0.$$

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For large enough $n \in \mathbb{N}$, the numbers $\lambda_1\left(j\left(w_0^{2n}a\right)\right)$ and $\lambda_1\left(j\left(w_0^{2n+1}a\right)\right)$ have opposite signs, and the conclusion follows. □

We also need the following observation:

Observation 2.6. *Let F_2 be the free group on $\{a,b\}$ and $\rho : F_2 \to \mathsf{SL}(2,\mathbb{R})$ be a quasi-isometric embedding. Set $k \in \mathbb{N}$ and let $\phi_k : F_2 \to F_2$ be the monomorphism defined by $\phi_k(a) = b^k a b^k$ and $\phi_k(b) = a^k b a^k$. Note that $|\phi_k(\gamma)|_{F_2} \geq (2k+1)|\gamma|_{F_2}$ for every $\gamma \in F_2$. Therefore, there exists a constant $C > 0$ depending only on ρ, such that*

$$\ell_1(\rho(\phi_k(\gamma)) \geq \ell_1(\rho(\gamma))^{Ck}$$

for every $\gamma \in F_2$ and $k \in \mathbb{N}$.

We end this section with the following remark showing the necessity of the connectedness of the Gromov boundary $\partial_\infty \Delta$ in Lemma 2.3.

We denote by S_g the closed orientable hyperbolic surface of genus $g \geq 2$.

Remark 2.7. Let $\tau_{2d} : \mathsf{SL}(2,\mathbb{R}) \to \mathsf{SL}(2d,\mathbb{R})$, $d \geq 1$, be the unique (up to conjugation) irreducible representation, and fix $j : \pi_1(S_g) \to \mathsf{SL}(2,\mathbb{R})$ a quasi-isometric embedding. For every deformation ρ of $\tau_{2d} \circ j$ and every noncyclic free subgroup F of $\pi_1(S_g)$, $\rho(F)$ contains an element whose eigenvalues are all negative. Indeed, by [9, Lemma 2], we may find $w \in F$ with $\lambda_1(j(w)) < 0$. Suppose that $\{\rho_t\}_{t \in [0,1]}$ is a continuous path of representations with $\rho_0 = \tau_{2d} \circ j$ and $\rho_1 = \rho$. It follows by Labourie's work [17] that ρ_t is P_i-Anosov for every $0 \leq t \leq 1$ and $1 \leq i \leq d-1$. In particular, for every i, the map $t \mapsto \lambda_1\left(\wedge^i \rho_t(w)\right)$ is continuous and nonzero, and hence $\lambda_1\left(\wedge^i \rho_t(w)\right) \lambda_1\left(\wedge^i \tau_{2d}(j(w))\right) > 0$. Note that

$$\lambda_1\left(\wedge^i \tau_{2d}(j(w))\right) = \lambda_1(j(w))^{i(2d-i)}$$

for every $1 \leq i \leq d$, and hence $\lambda_1\left(\wedge^i \rho_t(j(w))\right) > 0$ if and only if i is even. We deduce that $\rho_t(w)$ has all of its eigenvalues negative for every $0 \leq t \leq 1$.

3. The construction

By using Lemmas 2.3 and 2.5, we construct representations of the fundamental group Γ_g of a book of I-bundles of Theorem 1.1, which are not limits of Anosov representations of Γ_g in $\mathsf{SL}(d,\mathbb{R})$ for $d \geq 5$. We recall that given a group K and a subgroup H of K, a homomorphism $r : K \to H$ is called a *retraction* if $r(h) = h$ for every $h \in H$.

Proof of Theorem 1.1. The subgroup $\Delta = \langle a_1, b_1, \ldots, a_{2g}, b_{2g} \rangle$ of Γ_g is isomorphic to the fundamental group $\pi_1(S_{2g})$:

$$\langle a_1, b_1, \ldots, a_{2g}, b_{2g} | [a_1, b_1] \cdots [a_{2g}, b_{2g}] \rangle .$$

The subgroup $F = \langle a_1, b_1, \ldots, a_g, b_g \rangle$ of Δ is free on $2g$ generators. Note that there exists a retraction of Γ_g onto the surface subgroup Δ. Moreover, there is a retraction $r : \Delta \to F$ which sends $a_i \mapsto a_i$, $b_i \mapsto b_i$, $a_{g+i} \mapsto b_{g-i+1}$, and $b_{g+i} \mapsto a_{g-i+1}$ for $1 \leq i \leq g$. Note that the retraction r is induced by the topological retraction of S_{2g} onto a compact subsurface homeomorphic to S_g minus an open disk. We finally obtain a retraction $R : \Gamma_g \to F$.

We first construct reducible examples in all dimensions greater than or equal to 5. By [8, §4, p. 26], there exists a convex co-compact representation $i:\Gamma_g \hookrightarrow \mathsf{SL}(2,\mathbb{C})$ such that $i(\Delta)$ is a subgroup of $\mathsf{SL}(2,\mathbb{R})$. Let $S:\mathsf{SL}(2,\mathbb{C}) \to \mathsf{SO}_0(3,1)$ be the covering epimorphism whose kernel is $\{\pm I_2\}$ so that $S\left(\operatorname{diag}\left(a,\frac{1}{a}\right)\right)$ is conjugate to $\operatorname{diag}\left(a^2,1,1,\frac{1}{a^2}\right)$ for every $a \in \mathbb{R}^*$. We consider the following representations:

(a) $\rho_0 := S \circ i : \Gamma_g \to \mathsf{SL}(4,\mathbb{R})$ is P_1-Anosov, and for every $\gamma \in \Delta$, the matrix $\rho_0(\gamma)$ is positively proximal.

(b) Let $\tau_2 : \mathsf{SL}(2,\mathbb{C}) \to \mathsf{SL}(4,\mathbb{R})$ be the irreducible representation

$$\tau_2(h) = \begin{bmatrix} \mathrm{Re}(h) & -\mathrm{Im}(h) \\ \mathrm{Im}(h) & \mathrm{Re}(h) \end{bmatrix}, \quad h \in \mathsf{SL}(2,\mathbb{C}),$$

and define $\rho_1 := \tau_2 \circ i : \Gamma \to \mathsf{SL}(4,\mathbb{R})$. Note that ρ_1 is P_2-Anosov.

(i) Suppose that $d=5$. Note that $\rho_1(\Delta)$ is a subgroup of $\tau_2(\mathsf{SL}(2,\mathbb{R}))$. By Lemma 2.5, we can find $w \in F^{(2)} \subset \Delta(2)$ such that $\lambda_1\left(i\left(wa_1^2\right)\right) < 0$. Now we consider a group homomorphism $\varepsilon : F \to \mathbb{R}^+$ such that $\varepsilon\left(wa_1^2\right) = \varepsilon\left(a_1^2\right) = x$ with $x^{5/4} > \ell_1\left(i\left(wa_1^2\right)\right) = \ell_1\left(\rho_1\left(wa_1^2\right)\right)$. We consider the representation $\rho:\Gamma_g \to \mathsf{SL}(5,\mathbb{R})$ defined as follows:

$$\rho(\gamma) = \begin{bmatrix} \frac{1}{\sqrt[4]{\varepsilon(R(\gamma))}}\rho_1(\gamma) & 0 \\ 0 & \varepsilon(R(\gamma)) \end{bmatrix}, \quad \gamma \in \Gamma.$$

Notice that the eigenvalues of $\rho\left(wa_1^2\right)$ in decreasing order are

$$x, \quad x^{-1/4}\lambda_1\left(\rho_1\left(wa_1^2\right)\right), \quad x^{-1/4}\lambda_1\left(\rho_1\left(wa_1^2\right)\right), \quad \frac{1}{x^{1/4}\lambda_1\left(\rho_1\left(wa_1^2\right)\right)}, \quad \frac{1}{x^{1/4}\lambda_1\left(\rho_1\left(wa_1^2\right)\right)}.$$

The matrix $\wedge^2\rho\left(wa_1^2\right)$ is not positively semiproximal. Since $wa_1^2 \in \Delta(2)$, by Corollary 2.4 the representation ρ cannot be a limit of P_2-Anosov representations of Γ into $\mathsf{SL}(5,\mathbb{R})$. Note also that $\ker(\varepsilon) \cap \Delta(2)$ contains a free subgroup and $i(\ker(\varepsilon)\cap\Delta(2))$ is a discrete subgroup of $\mathsf{SL}(2,\mathbb{R})$. Hence, by Lemma 2.5, there exists $h \in \Delta(2)$ with $\varepsilon(h) = 1$ and $\lambda_1(\rho(h)) = \lambda_1(\rho_1(h)) = \lambda_1(i(h)) < 0$. Therefore, by Corollary 2.4, ρ is not a limit of P_1-Anosov representations of Γ into $\mathsf{SL}(5,\mathbb{R})$.

We now assume that $d=6$. By Observation 2.6 we can find a quasi-isometric embedding $j:F \to \mathsf{SL}(2,\mathbb{R})$ such that

$$\ell_1(j(\gamma)) \geq \ell_1(\rho_0(\gamma))^2$$

for every $\gamma \in F$. Now we consider the representation $\rho:\Gamma_g \to \mathsf{SL}(6,\mathbb{R})$, defined as follows:

$$\rho(\gamma) = \begin{bmatrix} \rho_0(\gamma) & 0 \\ 0 & j(R(\gamma)) \end{bmatrix}, \quad \gamma \in \Gamma_g.$$

By Lemma 2.5, we can find $w \in F \cap \Delta(2)$ such that $\lambda_1(j(w)) < 0$. Since $\ell_1(j(w)) > \ell_1(\rho_0(w))$, we have $\lambda_1(\rho(w)) = \lambda_1(j(w))$ and

$$\lambda_1\left(\wedge^2\rho(w)\right) = \lambda_1(j(w))\lambda_1(S(i(w))) = \lambda_1(j(w))\lambda_1(i(w))^2 < 0.$$

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Moreover, the matrix $\wedge^3\rho(w)$ has the number $\lambda_1(j(w))\ell_1(\rho_0(w)) < 0$ as an eigenvalue of maximum modulus and multiplicity 2. It follows by Corollary 2.4 that ρ, $\wedge^2\rho$, and $\wedge^3\rho$ cannot be limits of P_1-Anosov representations. This completes the proof of this case.

Now suppose $d \geq 7$. Again, by Observation 2.6, there exists a quasi-isometric embedding $j_0 : F \to \mathsf{SL}(2,\mathbb{R})$ with the property

$$\ell_1(j_0(\gamma)) \geq \ell_1(\rho_0(\gamma))^2$$

for every $\gamma \in \langle a_1, b_1\rangle$. There exists $w \in \langle a_1, b_1\rangle^{(2)}$ such that $\lambda_1\left(j_0\left(wa_1^2\right)\right) < 0$. We consider group homomorphisms $\varepsilon_1, \ldots, \varepsilon_{d-6} : \langle a_1, b_1\rangle \to \mathbb{R}^+$ such that $\ell_1\left(j_0\left(wa_1^2\right)\right) > \varepsilon_1\left(a_1^2\right) > \cdots > \varepsilon_{d-6}\left(a_1^2\right) > \ell_1\left(\rho_0\left(wa_1^2\right)\right)$. Then the representation $\rho : \Gamma_g \to \mathsf{GL}(d,\mathbb{R})$ defined by the blocks

$$\rho(\gamma) = \mathrm{diag}(\rho_0(\gamma), j_0(R(\gamma)), \varepsilon_1(R(\gamma)), \ldots, \varepsilon_{d-6}(R(\gamma)))$$

has the property that $\wedge^i\rho\left(wa_1^2\right)$ is proximal but not positively proximal for every $1 \leq i \leq d-4$. Corollary 2.4 shows that for every $1 \leq i \leq \frac{d}{2}$, the representation

$$\hat{\rho}(\gamma) = \frac{1}{\sqrt[d]{\det(\rho(\gamma))}}\rho(\gamma), \quad \gamma \in \Gamma_g,$$

is not a limit of P_i-Anosov representations into $\mathsf{SL}(d,\mathbb{R})$.

(ii) Now we construct a strongly irreducible quasi-isometric embedding of Γ_g into $\mathsf{SL}(12,\mathbb{R})$ which is not a limit of P_i-Anosov representations for $1 \leq i \leq 6$. We assume that $g \geq 4$. By Observation 2.6, we can find quasi-isometric embeddings $\iota_1 : \langle a_1, b_1, a_2\rangle \to \mathsf{SL}(2,\mathbb{R})$ and $\iota_2 : \langle b_2, a_3, b_3\rangle \to \mathsf{SL}(2,\mathbb{R})$ such that

$$\ell_1(\iota_1(h_1)) \geq \ell_1(\rho_0(h_1))^6 \qquad \text{and} \qquad \ell_1(\iota_2(h_2)) \geq \ell_1(\rho_0(h_2))^5$$

for every $h_1 \in \langle a_1, b_1, a_2\rangle$ and $h_2 \in \langle b_2, a_3, b_3\rangle$. By Lemma 2.5, we can find an element $w \in \Gamma_g(2) \cap \langle a_1, b_1\rangle$ such that $\lambda := \lambda_1\left(\iota_1\left(wa_2^2\right)\right) < 0$. Let $\mu = \lambda_1\left(\rho_0\left(wa_2^2\right)\right) > 0$. Now consider a homomorphism $\varepsilon : \langle a_1, b_1, a_2\rangle \to \mathbb{R}^+$ such that $\varepsilon(a_2) = x$ with

$$|\lambda| > x^3 > \frac{|\lambda|}{\mu^2} > 1 > \frac{\mu^2}{|\lambda|}$$

and the representation $\iota_1' : \langle a_1, b_1, a_2\rangle \to \mathsf{SL}(3,\mathbb{R})$ defined as follows:

$$\iota_1'(\gamma) = \begin{bmatrix} \frac{1}{\sqrt{\varepsilon(\gamma)}}\iota_1(\gamma) & 0 \\ 0 & \varepsilon(\gamma) \end{bmatrix}, \quad \gamma \in \langle a_1, b_1, a_2\rangle.$$

Notice that $\varepsilon\left(wa_2^2\right) = x^2$, and by the choice of $x > 0$, the matrix $\iota_1'\left(wa_2^2\right)$ is proximal with eigenvalues (in decreasing order) $\frac{\lambda}{x}, x^2, \frac{1}{\lambda x}$. By Lemma 2.5 we can also find $z \in \langle b_2, a_3\rangle^{(2)}$ such that $s := \lambda_1\left(\iota_2\left(zb_3^2\right)\right) < 0$. We consider the representations $\iota_2' : \langle b_2, a_3, b_3\rangle \to \mathsf{SL}(3,\mathbb{R})$, defined as

$$\iota_2'(\delta) = \begin{bmatrix} \iota_2(\delta) & 0 \\ 0 & 1 \end{bmatrix}, \quad \delta \in \langle b_2, a_3, b_3\rangle,$$

and $A : F \to \mathsf{SL}(3,\mathbb{R})$, defined as

$$A(\gamma) = \iota_1'(\gamma), \quad \gamma \in \langle a_1, b_1, a_2\rangle,$$
$$A(\delta) = \iota_2'(\delta), \quad \delta \in \langle b_2, a_3, b_3\rangle,$$
$$A(\langle a_4, b_4, \ldots, a_g, b_g\rangle) \text{ is chosen to be Zariski dense in } \mathsf{SL}(3,\mathbb{R}).$$

We obtain a Zariski dense representation $A \circ R : \Gamma_g \to \mathsf{SL}(3,\mathbb{R})$.

We first observe that $\rho_0 \otimes (A \circ R)$ is strongly irreducible and a quasi-isometric embedding. For every finite-index subgroup H of Γ_g, the restriction of the product $\rho_0 \times (A \circ R) : H \to \mathsf{SO}(3,1) \times \mathsf{SL}(3,\mathbb{R})$ is Zariski dense (see, e.g., [10]). Note that the tensor product representation $\otimes : \mathsf{SO}(3,1) \times \mathsf{SL}(3,\mathbb{R}) \to \mathsf{SL}(12,\mathbb{R})$, $(\alpha,\beta) \mapsto \alpha \otimes \beta$, is irreducible. Hence any proper $(\rho_0 \otimes (A \circ R))(H)$-invariant subspace V of $\mathbb{R}^{12} = \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^3$ has to be invariant under $\alpha \otimes \beta$ for every $\alpha \in \mathsf{SO}(3,1)$ and $\beta \in \mathsf{SL}(3,\mathbb{R})$. Therefore, V is trivial and $\rho_0 \otimes (A \circ R)$ is strongly irreducible. Moreover, since ρ_0 is P_1-Anosov, there exist $C, a > 0$ such that $\frac{\sigma_1(\rho_0(\gamma))}{\sigma_2(\rho_0(\gamma))} \geq C e^{a|\gamma|_\Gamma}$ and hence $\sigma_1(\rho_0(\gamma))^4 = \frac{\sigma_1(\rho_0(\gamma))}{\sigma_2(\rho_0(\gamma))}\frac{\sigma_1(\rho_0(\gamma))}{\sigma_3(\rho_0(\gamma))}\frac{\sigma_1(\rho_0(\gamma))}{\sigma_4(\rho_0(\gamma))} \geq C^3 e^{3a|\gamma|_\Gamma}$ for every $\gamma \in \Gamma_g$. Note that

$$\sigma_1(\rho_0(\gamma) \otimes A(R(\gamma))) = \sigma_1(\rho_0(\gamma))\sigma_1(A(R(\gamma))) \geq \sigma_1(\rho_0(\gamma)) \geq C^{\frac{3}{4}} e^{\frac{3a}{4}|\gamma|_\Gamma}$$

for every $\gamma \in \Gamma_g$. It follows that $\rho_0 \otimes (A \circ R)$ is a quasi-isometric embedding.

We claim that the tensor product representation $\rho_0 \otimes (A \circ R) : \Gamma_g \to \mathsf{SL}(12,\mathbb{R})$ is not a limit of Anosov representations. We consider the element $wa_2^2 \in \Gamma_g(2)$. We have $A\left(R\left(wa_2^2\right)\right) = A\left(wa_2^2\right) = \iota_1'\left(wa_2^2\right)$ and the matrix $\rho_0\left(wa_2^2\right) \otimes A\left(wa_2^2\right)$ is conjugate to the matrix

$$c = \operatorname{diag}\left(\mu^2, 1, 1, \frac{1}{\mu^2}\right) \otimes \operatorname{diag}\left(\frac{\lambda}{x}, x^2, \frac{1}{\lambda x}\right), \quad \lambda = \lambda_1\left(\iota_1'\left(wa_2^2\right)\right) < 0.$$

By the choice of $x > 0$, since $|\lambda| > x^3 > \frac{|\lambda|}{\mu^2} > 1$, the first seven eigenvalues, in decreasing order of their moduli, are

$$\frac{\lambda}{x}\mu^2, \quad x^2\mu^2, \quad \frac{\lambda}{x}, \quad \frac{\lambda}{x}, \quad x^2, \quad x^2, \quad \frac{\lambda}{x\mu^2}.$$

The matrix $\wedge^i c$ is proximal for $i = 1,2,4,6$ but not positively proximal. Thus, by Corollary 2.4, $\rho_0 \otimes (A \circ R)$ is not a limit of P_i-Anosov representations for $i = 1,2,4,6$. The matrix $\wedge^5 c$ has the number $\lambda^3\mu^4 x < 0$ as an eigenvalue of maximum modulus and multiplicity 2. Therefore, $\wedge^5 c$ is not positively semiproximal and $\rho_0 \otimes (A \circ R)$ cannot be a limit of P_5-Anosov representations (again by Corollary 2.4). Now we consider the element $zb_3^2 \in \Gamma(2)$. Note that $A\left(R\left(zb_3^2\right)\right) = A\left(zb_3^2\right) = \iota_2'\left(zb_3^2\right)$ and $\rho_0\left(zb_3^2\right) \otimes A\left(zb_3^2\right)$ is conjugate to the matrix

$$h = \operatorname{diag}\left(\nu^2, 1, 1, \frac{1}{\nu^2}\right) \otimes \operatorname{diag}\left(s, 1, \frac{1}{s}\right), \quad s = \lambda_1\left(\iota_2\left(zb_3^2\right)\right) < 0, \ \nu = \lambda_1\left(\rho_0\left(zb_3^2\right)\right).$$

Since $|s| > \nu^4$, the first five eigenvalues of h, in decreasing order of their moduli, are

$$s\nu^2, \quad s, \quad s, \quad \frac{s}{\nu^2}, \quad \nu^2.$$

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We notice that $\wedge^3 h$ is proximal, with first eigenvalue $s^3\nu^2 < 0$. It follows that $\rho_0 \otimes (A \circ R)$ is not a limit of P_3-Anosov representations.

We obtain the following generalization of Theorem 1.1 for $d \geq 6$:

Theorem 3.1. *Let Γ be an one-ended word hyperbolic group which admits a convex cocompact representation into $\mathsf{SO}(3,1)$ and is not virtually isomorphic to a free group or a surface group. Suppose that Γ retracts onto a free subgroup of rank at least 8. Then for every $d \geq 6$, there exists a quasi-isometric embedding $\psi : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ which is not a limit of Anosov representations of Γ into $\mathsf{SL}(d,\mathbb{R})$. Moreover, if $d = 12$, ψ can be chosen to be strongly irreducible.*

The proof of this theorem is similar to the proof of Theorem 1.1 for $d \geq 6$.

4. Additional examples

By following similar arguments as in Theorem 1.1(ii) and increasing the number of surface groups of the fundamental group of the I-bundle, it is possible to obtain strongly irreducible quasi-isometric embeddings for infinitely many odd dimensions such that every nontrivial element is semiproximal.

Theorem 4.1. *Let $g \geq 3$ and $n \geq 5$. For $0 \leq i \leq 2^{n-3}$, let*

$$\Gamma_{gi} = \left\langle \begin{matrix} a_{1i}, b_{1i}, \ldots, a_{(2g)i}, b_{(2g)i}, \\ c_{1i}, d_{1i}, \ldots, c_{(2g)i}, d_{(2g)i} \end{matrix} \middle| \begin{matrix} [a_{1i}, b_{1i}] \cdots [a_{(2g)i}, b_{(2g)i}], [c_{1i}, d_{1i}] \cdots [c_{(2g)i}, d_{(2g)i}], \\ [a_{1i}, b_{1i}] \cdots [a_{gi}, b_{gi}] \cdot [c_{1i}, d_{1i}] \cdots [c_{gi}, d_{gi}] \end{matrix} \right\rangle$$

be a copy of the word hyperbolic group Γ_g from Theorem 1.1, and consider the word hyperbolic group

$$\Delta_n = \left\langle \Gamma_{g0}, \Gamma_{g1}, \ldots, \Gamma_{g2^{n-3}} \middle| \begin{matrix} \mathrm{rel}(\Gamma_{gj}), [a_{10}, b_{10}] \cdots [a_{g0}, b_{g0}] \cdot ([a_{1j}, b_{1j}] \cdots [a_{gj}, b_{gj}])^{-1} \\ j = 0, 1, \ldots, 2^{n-3} \end{matrix} \right\rangle.$$

For every odd $n \geq 5$, there exists a strongly irreducible quasi-isometric embedding $\tau_n : \Delta_n \to \mathsf{SL}(3n,\mathbb{R})$ which is not a limit of Anosov representations of Δ_n into $\mathsf{SL}(3n,\mathbb{R})$, and for every $\gamma \in \Delta_n$, $\tau_n(\gamma)$ has all of its eigenvalues of maximum modulus real.

Let us recall some useful facts. The Lie algebra of $\mathsf{SO}(m+1,1)$ is

$$\mathfrak{so}(m+1,1) = \left\{ \begin{bmatrix} A & u \\ u^t & 0 \end{bmatrix} : A + A^t = 0_{m+1},\ u \in \mathbb{R}^{m+1} \right\}.$$

The subalgebra $\mathfrak{so}(m,1) \subset \mathfrak{so}(m+1,1)$ contains all matrices in $\mathfrak{so}(m+1,1)$ having zeros in the first row and column. For $1 \leq i,j \leq m+2$, let E^{ij} be the $(m+2) \times (m+2)$ matrix having 1 in the (i,j)-entry and 0 in the remaining entries. For an $(m+2) \times (m+2)$ matrix Y, Y_{ij} denotes the (i,j)-entry of Y. For two square matrices X and Y, their commutator is defined as $(X,Y) = XY - YX$.

We shall use the following fact:

Fact 4.2. *For $m \geq 3$, $\mathfrak{so}(m,1)$ is a self-normalizing maximal subalgebra of $\mathfrak{so}(m+1,1)$.*

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Proof. Suppose that $X \in \mathfrak{so}(m+1,1)$ such that $(X,\mathfrak{so}(m,1)) \subset \mathfrak{so}(m,1)$. Note that $E^{2(m+1)} + E^{(m+1)2} \in \mathfrak{so}(m,1)$ and $\left(X,E^{2(m+2)}+E^{(m+2)2}\right)_{12} = X_{1(m+2)}$. It follows that $X_{1(m+2)} = X_{(m+2)1} = 0$. Moreover, we have $\left(X,E^{2i}-E^{i2}\right)_{12} = -X_{1i}$ and $\left(X,E^{2i}-E^{i2}\right)_{1i} = X_{12}$ for every $3 \leq i \leq m+1$. We conclude that $X_{1i} = X_{i1} = 0$ for $2 \leq i \leq m+1$ and hence $X \in \mathfrak{so}(m,1)$.

Set $Y \in \mathfrak{so}(m+1,1) \setminus \mathfrak{so}(m,1)$ and let $\mathfrak{g}$ be the subalgebra generated by Y and $\mathfrak{so}(m,1)$. Looking at the commutators (Y,Z) and $\left(Y,E^{i(m+2)}+E^{(m+2)i}\right)$ for $Z \in \mathfrak{so}(m) \subset \mathfrak{so}(m,1)$ and $2 \leq i \leq m+1$, it is not hard to deduce that $\mathfrak{g}$ has to contain all matrices in $\mathfrak{so}(m+1,1)$ with nonzero first row. The conclusion follows. □

For our construction we shall use the following lemma, which follows from work of Johnson and Millson [14]:

Lemma 4.3. *Let $\Delta = \mathcal{G}_1 *_{g_1=g_2} \mathcal{G}_2$ be the amalgamated product of two torsion-free word hyperbolic groups $\mathcal{G}_1$ and $\mathcal{G}_2$ along the maximal cyclic subgroups $\langle g_1\rangle$ and $\langle g_2\rangle$ of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Suppose that $\rho:\Delta \to \mathsf{SO}(m,1)$ is a convex co-compact representation such that $\rho(\mathcal{G}_i)$ is Zariski dense in $\mathsf{SO}(m,1)$ for $i=1,2$ and $\rho(g_1)=\rho(g_2)$ lies in a copy of $\mathsf{SO}_0(2,1) \subset \mathsf{SO}(m,1)$. Then there exists a Zariski dense and convex co-compact representation $\rho':\Delta \to \mathsf{SO}(m+1,1)$.*

Proof. The representation ρ' is obtained by applying a Johnson–Millson deformation [14] for the representation $\mathrm{diag}(1,\rho)$. We briefly explain the construction (see also [16, Lemma 6.3]): Let X be a vector in $\mathfrak{so}(m+1,1) \setminus \mathfrak{so}(m,1)$ such that $\rho(g_1)X\rho(g_1)^{-1} = \rho(g_2)X\rho(g_2)^{-1} = X$. Then consider the family of representations $\rho_t : \Delta \to \mathsf{SO}(m+1,1)$ where

$$\rho_t(\gamma) = \rho(\gamma), \quad \gamma \in \mathcal{G}_1,$$
$$\rho_t(\gamma) = \exp(tX)\rho(\gamma)\exp(-tX), \quad \gamma \in \mathcal{G}_2.$$

For small enough $t>0$, the Lie algebra of the Zariski closure of ρ_t, $\mathfrak{g}_t$, strictly contains $\mathfrak{so}(m,1)$, and hence Fact 4.2 shows that $\mathfrak{g}_t = \mathfrak{so}(m+1,1)$. It follows that ρ_t is Zariski dense in $\mathsf{SO}(m+1,1)$. By the stability of convex co-compact representations into $\mathsf{SO}(m+1,1)$, established by Thurston [23, Proposition 8.3.3] (see also [7, Theorem 2.5.1]), ρ_t is convex co-compact for $t>0$ small enough. □

Proof of Theorem 4.1. The group Δ_n is isomorphic to the amalgamated product of $\{\Gamma_{gi}\}_{i=0}^{2^{n-3}}$ with respect to the monomorphisms $\varphi_i : \langle t\rangle \to \Gamma_{gi}$, $\varphi_i(t) = [a_{1i},b_{1i}]\cdots[a_{gi},b_{gi}]$. For every i, $\varphi_i(t)$ is a maximal cyclic subgroup of Γ_{gi}, and hence Δ_n is word hyperbolic by the Bestvina–Feighn combination theorem [3]. For the rest of the proof we identify Γ_{gi} with the subgroup $\langle\{a_{ji},b_{ji},c_{ji},d_{ji} : 1 \leq j \leq 2g\}\rangle$ of Δ_n. We set $b = \varphi_0(t) = [a_{10},b_{10}]\cdots[a_{g0},b_{g0}] \in \Gamma_i$ for every i.

For our construction of τ_n, we will first exhibit a strongly irreducible representation of Δ_n into $\mathsf{SL}(n,\mathbb{R})$ and then consider the tensor product with a representation of Δ_n into $\mathsf{SL}(3,\mathbb{R})$.

Notice that $\Delta_0 = \left\langle a_{10},b_{10},\ldots,a_{(2g)0},b_{(2g)0}\right\rangle \subset \Gamma_{g0}$ is isomorphic to $\pi_1(S_{2g})$. By [8], there exists a convex co-compact representation $\rho_1 : \Delta_n \to \mathsf{SO}(3,1)$ such that $\rho_1|_{\Delta_0}$ is Fuchsian

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– that is, $\rho_1|_{\Delta_0} = S \circ \rho_0$ for some convex co-compact representation $\rho_0 : \Delta_0 \to \mathsf{SL}(2,\mathbb{R})$, and $S : \mathsf{SL}(2,\mathbb{C}) \to \mathsf{SO}_0(3,1)$ is the covering epimorphism. Notice that since Γ_g is not a surface group, $\rho_1(\Gamma_{gi})$ is a Zariski dense subgroup of $\mathsf{SO}(3,1)$ for every $0 \le i \le 2^{n-3}$. By Lemma 4.3 we can find a convex co-compact representation $\rho_2 : \Delta_n \to \mathsf{SO}(4,1)$ such that for every $0 \le i \le 2^{n-4}-1$, $\rho_2\left(\left\langle \Gamma_{g(2i+1)}, \Gamma_{g(2i+2)} \right\rangle\right)$ is a Zariski dense subgroup of $\mathsf{SO}(4,1)$ and $\rho_2(\gamma) = \mathrm{diag}(1, \rho_1(\gamma))$ for all $\gamma \in \Delta_0$. Now we may view Δ_n as the amalgamated product of Γ_{g0} with $\langle \Gamma_{g1}, \Gamma_{g2} \rangle, \dots, \left\langle \Gamma_{g(2^{n-3}-1)}, \Gamma_{g2^{n-3}} \right\rangle$ (each of which is isomorphic to $\Gamma_g *_{\langle b \rangle} \Gamma_g$ along $\langle b \rangle$). Since $\rho_2\left(\left\langle \Gamma_{g(2i+1)}, \Gamma_{g(2i+2)} \right\rangle\right)$ is Zariski dense for every i, by Lemma 4.3 we can find a convex co-compact representation $\rho_3 : \Delta_n \to \mathsf{SO}(5,1)$ such that $\rho_3\left(\left\langle \Gamma_{g(4i+1)}, \Gamma_{g(4i+2)}, \Gamma_{g(4i+3)}, \Gamma_{g(4i+4)} \right\rangle\right)$ is Zariski dense in $\mathsf{SO}(5,1)$ for $0 \le i \le 2^{n-5}-1$ and $\rho_3(\gamma) = \mathrm{diag}(1,1,\rho_1(\gamma))$ for all $\gamma \in \Delta_0$. By continuing similarly, we obtain a Zariski dense convex co-compact representation $\rho_{n-3} : \Delta_n \to \mathsf{SO}(n-1,1)$ with $\rho_{n-3}(\gamma) = \mathrm{diag}(I_{n-4}, \rho_1(\gamma))$ for all $\gamma \in \Delta_0$.

Let F be the free subgroup of Δ_0 generated by the elements $a_{10}, b_{10}, \dots, a_{g0}, b_{g0}$ and let $R : \Delta_n \to F$ be a retraction. We may choose a P_1-Anosov representation $A : F \to \mathsf{SL}(3,\mathbb{R})$ such that $A(\langle a_{30}, b_{30}, \dots, a_{g0}, b_{g0} \rangle)$ is Zariski dense in $\mathsf{SL}(3,\mathbb{R})$ and $A(\gamma) = \mathrm{diag}(\rho_0(\gamma), 1)$ for $\gamma \in \langle a_{10}, b_{10}, a_{20}, b_{20} \rangle$. Let $\phi_k : \langle a_{20}, b_{20} \rangle \to \langle a_{20}, b_{20} \rangle$ be the map defined as in Observation 2.6 and $k \in \mathbb{N}$ be large enough that

$$\ell_1(\rho_0(\phi_k(\gamma)) \ge \ell_1(\rho_1(\gamma))^{10}$$

for every $\gamma \in \langle a_{20}, b_{20} \rangle$. We modify A by considering $A_k : F \to \mathsf{SL}(3,\mathbb{R})$ such that

$$A_k(\gamma) = \mathrm{diag}(\rho_0(\phi_k(\gamma)), 1), \gamma \in \langle a_{20}, b_{20} \rangle$$
$$A_k(\gamma) = A(\gamma), \quad \gamma \in \langle a_{10}, b_{10}, a_{30}, b_{30}, \dots, a_{g0}, b_{g0} \rangle.$$

The image $A_k(F)$ is a P_1-Anosov subgroup of $\mathsf{SL}(3,\mathbb{R})$.

Now we consider the representation $\tau_n : \Delta_n \to \mathsf{SL}(3n,\mathbb{R})$ defined as follows:

$$\tau_n(\gamma) = \rho_{n-3}(\gamma) \otimes A_k(R(\gamma)), \quad \text{for all } \gamma \in \Delta_n.$$

Similarly as in the proof of Theorem 1.1(ii), τ_n is strongly irreducible, since the Zariski closures of ρ_{n-3} and $A_k \circ R$ are two nonlocally isomorphic simple Lie groups. Moreover, all elements of the group $\tau_n(\Delta_n)$ have all of their eigenvalues of maximum modulus real, since $A_k|_F$ and ρ_{n-3} are P_1-Anosov into $\mathsf{SL}(3,\mathbb{R})$ and $\mathsf{SL}(n,\mathbb{R})$, respectively. To see that τ_n is not a limit of Anosov representations, we may first find $w \in \Delta_n(2) \cap \langle a_{10}, b_{10} \rangle$ such that $s := \lambda_1(\rho_0(w)) < 0$. Then $c := \rho_{n-3}(w) \otimes A_k(w)$ is conjugate to $\mathrm{diag}\left(s^2, I_{n-2}, \frac{1}{s^2}\right) \otimes \mathrm{diag}\left(s, 1, \frac{1}{s}\right)$. The first $2n-1$ eigenvalues of the matrix c, in decreasing order, are

$$s^3, \quad s^2, \quad \underbrace{s, \dots, s}_{n-1}, \quad \underbrace{1, \dots, 1}_{n-2}.$$

Since n is odd and $s < 0$, we see that $\wedge^i \tau_n(w)$ is not positively semiproximal when i is even and $i \le n+1$ and when $n+1 \le i \le 2n-1$. We may also find $w' \in \Delta_n(2) \cap \langle a_{20}, b_{20} \rangle$ such that $q = \lambda_1(\rho_0(\phi_k(w')) < 0$. Let $p = \lambda_1(\rho(w'))$ and note that $|q| > p^{10}$. The matrix $h := \rho_{n-3}(w') \otimes A_k(w')$ is conjugate to the matrix $\mathrm{diag}\left(p^2, I_{n-2}, \frac{1}{p^2}\right) \otimes \mathrm{diag}\left(q, 1, \frac{1}{q}\right)$. The

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first $n+1$ eigenvalues of this matrix, in decreasing order, are

$$qp^2, \underbrace{q, q, \ldots, q}_{n-2}, \frac{q}{p^2}, p^2.$$

Since n is odd and $q < 0$, the matrix $\wedge^i \tau_n(w')$ is not positively semiproximal when i is odd and $i \leq n+1$. The conclusion follows by Corollary 2.4. □

In contrast with the previous examples, in order to construct quasi-isometric embeddings of surface groups which are not limits of Anosov representations we need to find elements whose eigenvalues are nonreal.

Proposition 4.4. *For every $g \geq 4$, there exist quasi-isometric embeddings $\psi : \pi_1(S_g) \to \mathsf{SL}(4,\mathbb{R})$ and $\rho : \pi_1(S_g) \to \mathsf{SL}(6,\mathbb{R})$ which are not limits of Anosov representations of $\pi_1(S_g)$ into $\mathsf{SL}(4,\mathbb{R})$ and $\mathsf{SL}(6,\mathbb{R})$, respectively. Moreover, ρ is strongly irreducible.*

Proof. Let $\rho_1 : \pi_1(S_g) \to \mathsf{SL}(2,\mathbb{R})$ be a quasi-isometric embedding and consider $\pi : \pi_1(S_g) \to \langle a_1, a_2, a_3, a_4 \rangle$ a retraction of $\pi_1(S_g)$ onto the free subgroup $\langle a_1, a_2, a_3, a_4 \rangle$ of rank 4. Define $\lambda := \lambda_1(\rho_1(a_1))$ and $\mu := \lambda_1(\rho_1(a_2))$ and fix $\theta \notin \pi\mathbb{Q}$.

We consider $x, y > 0$ such that $x^2 > |\lambda|$, $|\mu| > y^2 > \frac{1}{|\mu|}$, and a homomorphism $\varepsilon : \langle a_1, a_2, a_3, a_4 \rangle \to \mathbb{R}^+$ with $\varepsilon(a_1) = x$ and $\varepsilon(a_2) = y$. Let $R_\theta : \langle a_1, a_2, a_3, a_4 \rangle \to \mathsf{SL}(2,\mathbb{R})$ be a homomorphism such that $R_\theta(a_1)$ and $R_\theta(a_2)$ are conjugate to an irrational rotation of angle θ. We consider the representation ψ defined as follows:

$$\psi(\gamma) = \begin{bmatrix} \frac{1}{\varepsilon(\pi(\gamma))}\rho_1(\gamma) & 0 \\ 0 & \varepsilon(\pi(\gamma)) R_\theta(\pi(\gamma)) \end{bmatrix}, \quad \text{for every } \gamma \in \pi_1(S_g).$$

By the choice of $x > 0$ and $y > 0$, the matrices $\psi(a_1)$ and $\wedge^2 \psi(a_2)$ have the numbers $xe^{i\theta}, xe^{-i\theta}$, and $\mu e^{i\theta}, \mu e^{-i\theta}$, respectively, as their eigenvalues of maximum modulus. Corollary 2.4 implies that ψ is not a limit of Anosov representations of $\pi_1(S_g)$ into $\mathsf{SL}(4,\mathbb{R})$. Moreover, since ρ_1 is P_1-Anosov and $\frac{\sigma_1(\psi(\gamma))}{\sigma_4(\psi(\gamma))} \geq \frac{\sigma_1(\rho_1(\gamma))}{\sigma_2(\rho_1(\gamma))}$ for every $\gamma \in \pi_1(S_g)$, it follows that ψ is a quasi-isometric embedding. The claim follows.

Now we construct the representation ρ. We consider $s, t, \theta \in \mathbb{R}$ satisfying $s > |\lambda|^{2/3}$, $|\mu|^{-2/3} < t < 1$, and a representation $j_{s,t,\theta} : \langle a_1, a_2, a_3, a_4 \rangle \to \mathsf{SL}(3,\mathbb{R})$ such that

$$j_{s,t,\theta}(a_1) = \begin{bmatrix} s\cos\theta & -s\sin\theta & 0 \\ s\sin\theta & s\cos\theta & 0 \\ 0 & 0 & \frac{1}{s^2} \end{bmatrix}, \qquad j_{s,t,\theta}(a_2) = \begin{bmatrix} t\cos\theta & -t\sin\theta & 0 \\ t\sin\theta & t\cos\theta & 0 \\ 0 & 0 & \frac{1}{t^2} \end{bmatrix},$$

and $j_{s,t,\theta}(\langle a_3, a_4 \rangle)$ is Zariski dense in $\mathsf{SL}(3,\mathbb{R})$. By arguing as in the proof of Theorem 1.1(ii), the tensor product $\rho := \rho_1 \otimes (j_{s,t,\theta} \circ \pi)$ is a strongly irreducible quasi-isometric embedding of $\pi_1(S_g)$ into $\mathsf{SL}(6,\mathbb{R})$. By the choice of $s > 0$, the eigenvalues of the matrix $g := \rho_1(a_1) \otimes j_{s,t,\theta}(\pi(a_1))$, in decreasing order of their moduli, are

$$\lambda s e^{i\theta}, \quad \lambda s e^{-i\theta}, \quad \frac{s}{\lambda} e^{i\theta}, \quad \frac{s}{\lambda} e^{-i\theta}, \quad \frac{\lambda}{s^2}, \quad \frac{1}{\lambda s^2}.$$

The matrices g and $\wedge^3 g$ have their eigenvalues of maximum modulus nonreal, hence ρ is not a limit of P_1- or P_3-Anosov representations of $\pi_1(S_g)$ into $\mathsf{SL}(6,\mathbb{R})$. The eigenvalues of the matrix $h := \rho_1(a_2)\otimes j_{s,t,\theta}(\pi(a_2))$, in decreasing order of their moduli, are

$$\frac{\mu}{t^2},\quad \mu t e^{i\theta},\quad \mu t e^{-i\theta},\quad \frac{1}{\mu t^2},\quad \frac{t}{\mu}e^{i\theta},\quad \frac{t}{\mu}e^{-i\theta}.$$

The matrix $\wedge^2\rho(a_2)$ has its eigenvalues of maximum modulus nonreal, and therefore $\wedge^2\rho$ is not a limit of P_1-Anosov representations of $\pi_1(S_g)$. Moreover, we have $\frac{\sigma_1(\rho(\gamma))}{\sigma_6(\rho(\gamma))} \geq \frac{\sigma_1(\rho_1(\gamma))}{\sigma_2(\rho_1(\gamma))}$ for every $\gamma \in \pi_1(S_g)$, and hence ρ is a quasi-isometric embedding, since ρ_1 is. It follows that ρ has the required properties. □

Remark 4.5.

(i) The construction in Proposition 4.4 also works for finitely generated free groups of rank at least 4.

(ii) We note that it is possible to describe the proximal limit set of the irreducible examples we have constructed. For a proximal subgroup H of $\mathsf{SL}(n,\mathbb{R})$, the proximal limit set $\Lambda_H^{\mathbb{P}}$ is defined to be the closure of the attracting fixed points of proximal elements of H in $\mathbb{P}(\mathbb{R}^n)$. Let Δ be a nonelementary word hyperbolic group and suppose that $\phi_1 : \Delta \to \mathsf{SL}(n,\mathbb{R})$ and $\phi_2 : \Delta \to \mathsf{SL}(m,\mathbb{R})$ are two irreducible representations such that $\phi_1\otimes\phi_2$ is irreducible, ϕ_1 is P_1-Anosov, and ϕ_2 is either nonfaithful or nondiscrete. We claim that $\Lambda^{\mathbb{P}}_{(\phi_1\otimes\phi_2)(\Delta)}$ is homeomorphic to $\Lambda^{\mathbb{P}}_{\phi_1(\Delta)}\times\Lambda^{\mathbb{P}}_{\phi_2(\Delta)}$. We may assume that $e_1\otimes e_1$ is in $\Lambda^{\mathbb{P}}_{(\phi_1\otimes\phi_2)(\Delta)}$ and $[e_1]\in\mathbb{P}(\mathbb{R}^n)$ is the attracting eigenline of $\phi_1(w_1)$. Let $w_0\in\Delta$ be a nontrivial element such that $\phi_2(w_0) = I_m$. For $x,y\in\partial_\infty\Delta$ with $\{x,y\}\cap\{w_0^+,w_0^-,w_1^+,w_1^-\}$ empty, we may find a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Δ with $x = \lim_n\gamma_n$ and $y = \lim_n\gamma_n^{-1}$. Then $\lim_n\left(\gamma_n w_0\gamma_n^{-1}\right)w_1^+ = x$, and hence $\lim_n(\phi_1\otimes\phi_2)\left(\gamma_n w_0\gamma_n^{-1}\right)[e_1\otimes e_1] = [u_x\otimes e_1]$, where $\xi^1_{\phi_1}(x) = [u_x]$. It follows that $\Lambda^{\mathbb{P}}_{(\phi_1\otimes\phi_2)(\Delta)}$ contains the set $\left\{[u_z\otimes e_1] : \xi^1_{\phi_1}(z) = [u_z],\ z\in\partial_\infty\Delta\right\}$. Now since $(\phi_1\otimes\phi_2)(\Delta)$ and $\phi_2(\Delta)$ act minimally on $\Lambda^{\mathbb{P}}_{(\phi_1\otimes\phi_2)(\Delta)}$ and $\Lambda^{\mathbb{P}}_{\phi_2(\Delta)}$, respectively (see [1, Lemma 2.5]), we conclude that $\Lambda^{\mathbb{P}}_{(\phi_1\otimes\phi_2)(\Delta)} = \left\{[u_1\otimes u_2] : [u_i]\in\Lambda^{\mathbb{P}}_{\phi_i(\Delta)},\ i = 1,2\right\}$. We work similarly when ϕ_2 is nondiscrete. In particular, we deduce the following:

(a) In the construction of ρ in Theorem 1.1(ii), the representation $A : F \to \mathsf{SL}(3,\mathbb{R})$ can be chosen to be nondiscrete, and hence the proximal limit set of $\rho(\Gamma)$ in $\mathbb{P}\left(\mathbb{R}^{12}\right)$ is homeomorphic to $\partial_\infty\Gamma\times\mathbb{P}\left(\mathbb{R}^3\right)$.

(b) In Theorem 4.1, the proximal limit set of $\tau_n(\Delta_n)$ in $\mathbb{P}\left(\mathbb{R}^{3n}\right)$ is homeomorphic to $\partial_\infty\Delta_n\times C$, where C is a Cantor set.

(c) In Proposition 4.4, for $s,t>0$ generic, the representation $j_{s,t,\theta}$ is nondiscrete with dense image in $\mathsf{SL}(3,\mathbb{R})$. The proximal limit set of $\rho(\pi_1(S_g))$ in $\mathbb{P}\left(\mathbb{R}^6\right)$ is homeomorphic to $S^1\times\mathbb{P}\left(\mathbb{R}^3\right)$.

5. Concluding remarks

Let G and G' be two semisimple real algebraic Lie groups of real rank at least 2 and $\iota : G\hookrightarrow G'$ be an injective Lie group homomorphism. For an Anosov representation $\rho:\Gamma\to G$, the composition $\iota\circ\rho$ need not be Anosov into G' with respect to any pair of opposite

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parabolic subgroups of G'. The failure to be Anosov under composition with a Lie group embedding has already been exhibited by Guichard and Wienhard [13, §4, p. 22]. Our examples are not limits of Anosov representations of their domain group into the bigger special linear group $\mathsf{SL}(d,\mathbb{R})$, but are Anosov (with respect to a suitable pair of opposite parabolic subgroups) when considered as representations into their Zariski closure G.

Following the lines of the proof of Theorem 4.1, we construct examples of discrete and faithful representations which are not quasi-isometric embeddings (and hence cannot be Anosov into their Zariski closure) and are not a limit of Anosov representations of their domain group into $\mathsf{SL}(15,\mathbb{R})$. Let M^3 be a closed orientable hyperbolic 3-manifold which is a surface bundle over the circle (with fibers homeomorphic to S) and contains a totally geodesic closed surface S'. By using the Klein combination theorem, we may find a convex co-compact representation $\rho:\pi_1\left(M^3\right)*F_2\to\mathsf{SO}(4,1)$ whose restriction to the free factor F_2 is Zariski dense and $\rho|_{\pi_1(M^3)}=\mathrm{diag}(1,\rho_0)$; here $\rho_0:\pi_1\left(M^3\right)\to\mathsf{SO}(3,1)$ denotes the holonomy representation associated to M^3. Since the quotient of $\pi_1\left(M^3\right)$ by the normal subgroup $\pi_1(S)$ is cyclic, the intersection $H=\pi_1(S)\cap\pi_1(S')$ is a noncyclic, normal free subgroup of $\pi_1(S')$. Let $F=\langle a_1,\dots,a_r\rangle$ be a free subgroup of H of rank $r\geq 4$. We may find a finite cover $\hat{S}$ of S such that $F\subset\pi_1\left(\hat{S}\right)$ and there exists a retraction $R:\pi_1\left(\hat{S}\right)\to F$ [18, Theorem 1.6], which we extend to a retraction $R:\pi_1\left(\hat{S}\right)*F_2\to F$. Since S' is totally geodesic in M^3 and F is quasiconvex in $\pi_1(S')$, there exists a convex co-compact representation $\rho_1:F\to\mathsf{SL}(2,\mathbb{R})$ such that $\rho_0|_F=S\circ\rho_1$. As in Theorem 4.1, we consider k very large and $A_k:F\to\mathsf{SL}(3,\mathbb{R})$ a Zariski dense representation, such that $A_k(\gamma)=\mathrm{diag}(1,\rho_1(\gamma))$ for $\gamma\in\langle a_1,a_2\rangle$ and $A_k(\gamma)=\mathrm{diag}(1,\rho_1(\phi_k(\gamma)))$ for $\gamma\in\langle a_3,a_4\rangle$. The representation $\rho':\pi_1\left(\hat{S}\right)*F_2\to\mathsf{SL}(15,\mathbb{R})$,

$$\rho'(\gamma)=\rho(\gamma)\otimes A_k(R(\gamma)),\quad \gamma\in\pi_1\left(\hat{S}\right)*F_2,$$

is discrete and faithful (since ρ is) and not in the closure of Anosov representations of $\pi_1\left(\hat{S}\right)*F_2$ into $\mathsf{SL}(15,\mathbb{R})$. We note that since $A_k\circ R$ is not faithful and $\pi_1(S)$ is normal and of infinite index in $\pi_1\left(M^3\right)$, the representations $A_k\circ R$ and $\rho|_{\pi_1(\hat{S})}$ are not quasi-isometric embedings into $\mathsf{SO}(4,1)$ and $\mathsf{SL}(3,\mathbb{R})$, respectively. In particular, $\rho\times(A_k\circ R)$ is not Anosov with respect to any pair of opposite parabolic subgroups of $\mathsf{SO}(4,1)\times\mathsf{SL}(3,\mathbb{R})$. The Zariski closure of ρ' is $\mathsf{SO}(4,1)\otimes\mathsf{SL}(3,\mathbb{R})=\{g_1\otimes g_2:g_1\in\mathsf{SO}(4,1),\ g_2\in\mathsf{SL}(3,\mathbb{R})\}$ and it follows (see, e.g., [13, Corollary 3.6]) that $\rho\otimes(A_k\circ R)$ is not Anosov in its Zariski closure in $\mathsf{SL}(15,\mathbb{R})$.

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