

LEGENDRE TRANSFORMS AND APÉRY'S SEQUENCES

ASMUS L. SCHMIDT

(Received August 27, 1992)

Communicated by J. H. Loxton

Abstract

This article studies particular sequences satisfying polynomial recurrences, among those Apéry's sequence

$$a_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

which is shown to be the Legendre transform of the sequence

$$c_k = \sum_{j=0}^k \binom{k}{j}^3.$$

This results in the construction of simultaneous approximations of $\pi^2/8$ and $\zeta(3)$.

1991 *Mathematics subject classification* (Amer. Math. Soc.): 11B37, 11J13, 33C45.

1. Introduction

For a sequence (c_k) we shall consider its *Legendre transform* (a_n) defined by

$$a_n = \sum_{k=0}^n c_k \binom{n}{k} \binom{n+k}{k}.$$

It should be noticed that each sequence (a_n) is the Legendre transform of a unique sequence (c_k) (cf. Section 2). We shall also consider the sequence of *Legendre polynomials* belonging to (c_k) defined by

$$a_n(x) = \sum_{k=0}^n c_k \binom{n}{k} \binom{n+k}{k} x^k.$$

The classical Legendre polynomials orthogonal on $[-1, 0]$ belongs in this way to the sequence (c_k) with $c_k = 1$.

This article is motivated by the following conjecture (see [10]): For integral r , $r \geq 2$, numerical evidence indicates that each of the sequences

$$a_n^{(r)} = \sum_{k=0}^n \binom{n}{k}^r \binom{n+k}{k}^r$$

is the Legendre transform of an integral sequence $(c_k^{(r)})$. Challenged by this problem it was noticed by W. Deuber, W. Thumser and B. Voigt (University of Bielefeld) that the corresponding sequence (c_k) for $r = 2$ seemed to be

$$(1) \quad c_k = c_k^{(2)} = \sum_{j=0}^k \binom{k}{j}^3.$$

This was then proved independently by Strehl (University of Erlangen-Nürnberg, see [13]), and myself. Strehl obtained the more general formula

$$\sum_{k=0}^n \binom{n}{k}^2 \frac{(\alpha+\beta+n+k)^2}{\binom{\alpha+k}{k} \binom{\beta+k}{k}} = \sum_{k=0}^n \binom{n}{k} \frac{(\alpha+\beta+n+k)}{\binom{\beta+k}{k}} \sum_{j=0}^k \binom{k}{j}^2 \frac{\binom{\beta+k}{j}}{\binom{\alpha+j}{j}},$$

where α and β are parameters. The choice $\alpha = \beta = 0$ gives the formula (1) for (c_k) for $r = 2$. In [13] Strehl also proved that $c_k^{(3)}$ is integral by establishing the formula

$$c_k^{(3)} = \sum_{j=0}^k \binom{k}{j}^2 \binom{2j}{j}^2 \binom{2j}{k-j}.$$

It is well known (see [1, 6]) that Apéry's sequence

$$(2) \quad a_n = a_n^{(2)} = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfies the recurrence relation

$$(3) \quad (n+1)^3 a_{n+1} - ((n+1)^3 + n^3 + 4(2n+1)^3) a_n + n^3 a_{n-1} = 0 \quad \text{for } n \geq 0.$$

The sequence (1) has also long been known to satisfy the recurrence relation (see [1, 2, 3, 5, 6, 12])

$$(4) \quad (k+1)^2 c_{k+1} - (7k^2 + 7k + 2) c_k - 8k^2 c_{k-1} = 0 \quad \text{for } k \geq 0.$$

After presenting some simple properties of the Legendre transform in Section 2, we consider in Section 3 a class of three term recurrent sequences (c_k) such that the

corresponding sequence (a_n) is also three term recurrent. Simple examples of this kind are described in Section 4.

In Section 5 we consider the important recurrence (4) leading through Legendre transforms to Apéry’s sequences related to $\zeta(3)$. In addition to obtaining the formula

$$a_n^{(2)} = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^k \binom{k}{j}^3$$

we also get simultaneous approximations of $\pi^2/8$ and $\zeta(3)$.

In Section 6 we consider the simpler sequence

$$(5) \quad a_n = \sum_{k=0}^n \binom{2k}{k} \binom{n}{k} \binom{n+k}{k},$$

which is the Legendre transform of the sequence

$$(6) \quad c_k = \sum_{j=0}^k \binom{k}{j}^2 = \binom{2k}{k}.$$

This sequence is rather peculiar, namely

$$\begin{aligned} a_0 = 1^2, & \quad a_1 = 5 \cdot 1^2, & a_2 = 7^2, & \quad a_3 = 5 \cdot 11^2, & a_4 = 91^2, \\ a_5 = 5 \cdot 155^2, & a_6 = 1345^2, & a_7 = 5 \cdot 2365^2, & a_8 = 20995^2, & a_9 = 5 \cdot 37555^2, \\ \dots \end{aligned}$$

This will be explained by means of some particular sequences of orthogonal polynomials.

The final section contains a number of computer-aided results of recurrent sequences (c_k) such that the corresponding sequence (a_n) is also recurrent. We propose to continue this investigation by extending the class of recurrent sequences (c_k) for which the corresponding sequence (a_n) of Legendre transforms is known to be recurrent (see also [9]). Such insight might also prove the conjecture about $a_n^{(r)}$ for values of $r \geq 3$.

2. Simple properties of Legendre transforms

We shall mention the following simple results:

(i) If (a_n) is the Legendre transform of (c_k) then the following *inversion formula* holds:

$$c_k = \sum_{j=0}^k (-1)^{k-j} \frac{2j+1}{k+j+1} \frac{\binom{k}{j}}{\binom{k+j}{j}} a_j.$$

(ii)

$$\sum_{k=0}^n \frac{(-1)^k}{k+1} \binom{n}{k} \binom{n+k}{k} = \begin{cases} 1 & n = 0, \\ 0 & n > 0. \end{cases}$$

(iii) For $m \in \mathbb{N}$ we have

$$\sum_{k=0}^n \frac{(-1)^k}{2k - 2m + 1} \binom{n}{k} \binom{n+k}{k} = \frac{(-1)^m (2n + 1 - 2(m - 1)) \cdots (2n - 1)(2n + 1)(2n + 3) \cdots (2n + 1 + 2(m - 1))}{(1 \cdot 3 \cdots (2m - 1))^2}.$$

(iv) For $m \in \mathbb{N}$ we let

$$c_k^{(m)} = \begin{cases} 1 & k = m, \\ 0 & k \neq m. \end{cases}$$

Then obviously

$$\sum_{k=0}^n c_k^{(m)} \binom{n}{k} \binom{n+k}{k} = \frac{(n - m + 1) \cdots (n - 1)n(n + 1) \cdots (n + m)}{(m!)^2}.$$

Since

$$\binom{n}{k} \binom{n+k}{k} = \binom{2k}{k} \binom{n+k}{n-k},$$

the relation (i) is an immediate consequence of the well-known relations for so-called *Legendre pairs* (cf. [8]):

$$a_n = \sum_k \binom{n+k}{n-k} b_k$$

if and only if

$$b_n = \sum_k (-1)^{n+k} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) a_k.$$

Notice that (ii) follows from (i), when applied to the sequence (a_n) with $a_0 = 1, a_n = 0$ for $n > 0$. For another derivation of (ii) see [9]. Notice also that the formulas in (iii) and (iv) together give the inverse Legendre transform of an arbitrary polynomial sequence (a_n) .

3. Three term recurrences and Legendre transforms

We shall prove the following result:

THEOREM 1. *Let $A, B, C, D, E \in \mathbb{R}, C \neq 0$ be constants. We consider polynomials*

$$\begin{aligned} P_0(k) &= Ak^2 + Bk + C, \\ P_2(k) &= Dk^2, \\ Q_1(k) &= k(Ak + (B - A)), \\ P_1(k) &= Dk(k + 1) - Q_1(k) + E, \end{aligned}$$

and polynomials

$$\begin{aligned} p_0(n) &= (n + 1)P_0(n), \\ p_2(n) &= n(P_0(n) - (B - A)(2n + 1)), \\ q_1(n) &= 2P_1(n) + 2Q_1(n) = 2Dn(n + 1) + 2E, \\ p_1(n) &= p_0(n) + p_2(n) + (2n + 1)q_1(n). \end{aligned}$$

(i) *Suppose the sequence (c_k) satisfies the recurrence*

$$(7) \quad P_0(k)c_{k+1} - P_1(k)c_k - P_2(k)c_{k-1} = 0 \quad \text{for } k \geq 1$$

with initial values $c_0 = 1, c_1 = E/C$. Then the Legendre transform (a_n) of (c_k) satisfies the recurrence

$$(8) \quad p_0(n)a_{n+1} - p_1(n)a_n + p_2(n)a_{n-1} = 0 \quad \text{for } n \geq 1$$

with initial values $a_0 = 1, a_1 = 1 + 2E/C$.

(ii) *Suppose the sequence (c_k) satisfies the recurrence (7) with initial values $c_0 = 0, c_1 = 1$. Then the Legendre transform (a_n) of (c_k) satisfies the recurrence*

$$(9) \quad p_0(n)a_{n+1} - p_1(n)a_n + p_2(n)a_{n-1} = C(4n + 2) \quad \text{for } n \geq 1$$

with initial values $a_0 = 0, a_1 = 2$.

(iii) *Suppose the sequence (c_k) satisfies the recurrence*

$$(10) \quad P_0(k)c_{k+1} - P_1(k)c_k - P_2(k)c_{k-1} = \frac{(-1)^k}{k + 1} \quad \text{for } k \geq 1$$

with initial values $c_0 = 0, c_1 = 1/C$. Then the Legendre transform (a_n) of (c_k) satisfies the recurrence (8) with initial values $a_0 = 0, a_1 = 2/C$.

PROOF. For abbreviation we let

$$a_{n,k} = c_k \binom{n}{k} \binom{n+k}{k},$$

where (c_k) is any sequence. The Legendre transform (a_n) of (c_k) is then given by

$$a_n = \sum_{k=0}^n a_{n,k}.$$

We notice first that

$$(11) \quad p_0(n)a_{n+1,k} - p_1(n)a_{n,k} + p_2(n)a_{n-1,k} = \\ \left(2p_0(n) \binom{n}{k-1} \binom{n+k}{k} - 2p_2(n) \binom{n}{k} \binom{n+k-1}{k-1} \right) \\ - (2n+1)q_1(n) \binom{n}{k} \binom{n+k}{k} \Big) c_k.$$

Using the method of *creative telescoping* (cf. [5, 6]) we let

$$A_{n,k} = - \binom{n}{k} \binom{n+k}{k} (2n+1) \left((q_1(n) - 2Q_1(k))c_k + 2P_2(k)c_{k-1} \right)$$

for $0 \leq k \leq n$, and with the proviso that $A_{n,k} = 0$ for $k < 0$ or $k > n$. An easy rearrangement shows that identically for $0 < k < n + 1$:

$$(12) \quad A_{n,k} - A_{n,k-1} = \left(2p_0(n) \binom{n}{k-1} \binom{n+k}{k} - 2p_2(n) \binom{n}{k} \binom{n+k-1}{k-1} \right) \\ - (2n+1)q_1(n) \binom{n}{k} \binom{n+k}{k} \Big) c_k \\ - 2 \binom{n}{k-1} \binom{n+k-1}{k-1} (2n+1) \times \\ (P_0(k-1)c_k - P_1(k-1)c_{k-1} - P_2(k-1)c_{k-2}).$$

In particular for $k = 1$, and using (11), we also obtain

$$(13) \quad A_{n,1} - A_{n,0} = \\ p_0(n)a_{n+1,1} - p_1(n)a_{n,1} + p_2(n)a_{n-1,1} - (4n+2)(Cc_1 - Ec_0).$$

We also notice that

$$(14) \quad A_{n,0} = -(2n+1)q_1(n)c_0.$$

Case 1. Assume first that (c_k) satisfies (7) for $k \geq 1$. Then by (11) and (12)

$$(15.1) \quad A_{n,k} - A_{n,k-1} = p_0(n)a_{n+1,k} - p_1(n)a_{n,k} + p_2(n)a_{n-1,k}$$

for $1 < k < n + 1$. By (14) the relation (15.1) also holds for $k = 0$. Using (7) we get

$$(16.1) \quad -A_{n,n} = \binom{2n}{n} (2n+1) (2P_1(n)c_n + 2P_2(n)c_{n-1}) = 2 \binom{2n}{n} (2n+1) P_0(n) c_{n+1} \\ = \binom{2n+2}{n+1} (n+1) P_0(n) c_{n+1} = p_0(n) a_{n+1,n+1},$$

so that relation (15.1) also holds for $k = n + 1$. Consequently by (13) and (15.1)

$$\begin{aligned} & p_0(n)a_{n+1} - p_1(n)a_n + p_0(n)a_{n-1} \\ &= \sum_{k=0}^{n+1} (A_{n,k} - A_{n,k-1}) + (4n + 2)(Cc_1 - Ec_0) \\ &= (4n + 2)(Cc_1 - Ec_0), \end{aligned}$$

which proves the two first claims of the theorem.

Case 2. Assume next that (c_k) satisfies (10) for $k \geq 1$ with $c_0 = 0, c_1 = 1/C$. Then (15.1) is replaced by

$$(15.2) \quad \begin{aligned} A_{n,k} - A_{n,k-1} &= p_0(n)a_{n+1,k} - p_1(n)a_{n,k} + p_2(n)a_{n-1,k} \\ &\quad - (4n + 2) \binom{n}{k-1} \binom{n+k-1}{k-1} \frac{(-1)^{k-1}}{k} \end{aligned}$$

for $1 < k < n + 1$, and (16.1) is replaced by

$$(16.2) \quad -A_{n,n} = p_0(n)a_{n+1,n+1} - (4n + 2) \binom{2n}{n} \frac{(-1)^n}{n+1}.$$

By (14) and (16.2) it follows that the relation (15.2) also holds for $k = n + 1$ and also for $k = 0$ when omitting the last term in (15.2). Since $Cc_1 - Ec_0 = 1$ it follows by (13), (15.2) and Section 2(ii) that

$$\begin{aligned} & p_0(n)a_{n+1} - p_1(n)a_n + p_2(n)a_{n-1} \\ &= \sum_{k=0}^{n+1} (A_{n,k} - A_{n,k-1}) + (4n + 2) \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \frac{(-1)^k}{k+1} \\ &= 0 \quad \text{for } n > 0. \end{aligned}$$

This proves the last claim of the theorem.

REMARK 1. Assume that $B = 2A$, which implies that $p_2(n + 1) = p_0(n)$. Assume further that $p_0(n) \neq 0$ for $n \geq 0$, and that $a_n \neq 0$ for $n \geq 0$. To distinguish the three sequences (a_n) in (i) – (iii) they will here be denoted $(a_n), (a'_n), (a''_n)$, respectively. The following formulas are easily deduced:

$$\begin{aligned} a''_n &:= \begin{vmatrix} a_n & a''_n \\ a_{n+1} & a''_{n+1} \end{vmatrix} = \frac{2C}{p_0(n)}, \\ D_n &:= \begin{vmatrix} a_{n-1} & a'_{n-1} & a''_{n-1} \\ a_n & a'_n & a''_n \\ a_{n+1} & a'_{n+1} & a''_{n+1} \end{vmatrix} = \frac{-4C^2(2n + 1)}{p_0(n - 1)p_0(n)}, \end{aligned}$$

$$d'_n := \left| \begin{matrix} a_n & a'_n \\ a_{n+1} & a'_{n+1} \end{matrix} \right| = \frac{2C}{p_0(n)} \sum_{\nu=0}^n (2\nu + 1)a_\nu ,$$

$$\alpha'' := \lim \frac{a''_n}{a_n} = 2C \sum_{n=0}^{\infty} \frac{1}{p_0(n)a_n a_{n+1}} ,$$

$$\alpha' := \lim \frac{a'_n}{a_n} = 2C \sum_{n=0}^{\infty} \frac{1}{p_0(n)a_n a_{n+1}} \sum_{\nu=0}^n (2\nu + 1)a_\nu ,$$

$$\alpha'' - \frac{a''_n}{a_n} = 2C \sum_{\nu=n}^{\infty} \frac{1}{p_0(\nu)a_\nu a_{\nu+1}} ,$$

$$\alpha' = \sum_{n=0}^{\infty} (2n + 1)(a_n \alpha'' - a''_n) .$$

The formulas concerning infinite series are purely formal, and convergence must therefore be ascertained when applied.

4. Examples

EXAMPLE 1. (Classical and generalized Legendre polynomials.) For $A = B = D = 0, C = 1, E = x$ the recurrence

$$c_{k+1} - xc_k - 0 \cdot c_{k-1} = 0$$

has the solution $c_k = x^k$. The corresponding sequence

$$a_n = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} x^k$$

then satisfies the recurrence

$$(n + 1)a_{n+1} - (2n + 1)(1 + 2x)a_n + na_{n-1} = 0.$$

The polynomials $a_n = a_n(x)$ (Legendre polynomials) are orthogonal with respect to Lebesgue measure on $[-1, 0]$.

For $A = B = 0, C = 1, D = x_1, E = x_0$ we get the recurrence

$$c_{k+1} - (x_0 + k(k + 1)x_1)c_k - k^2 x_1 c_{k-1} = 0.$$

The corresponding sequence of generalized Legendre polynomials a_n then satisfies the recurrence

$$(n + 1)a_{n+1} - (2n + 1)(1 + 2(x_0 + n(n + 1)x_1))a_n + na_{n-1} = 0.$$

Compare [9] for a wider class of generalized Legendre polynomials.

EXAMPLE 2. (Orthogonal polynomials related to Bernoulli numbers.) For $A = D = 0, B = C = 1, E = x$ the recurrence

$$(k + 1)c_{k+1} - (x - k)c_k - 0 \cdot c_{k-1} = 0$$

has the solution $c_k = \binom{x}{k}$. The corresponding sequence

$$a_n = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \binom{x}{k}$$

therefore satisfies the recurrence

$$(n + 1)^2 a_{n+1} - (2n + 1)(1 + 2x)a_n - n^2 a_{n-1} = 0.$$

When defining a linear functional s on $\mathbb{R}[x]$ by

$$s\left(\binom{x}{k}\right) = \frac{(-1)^k}{k + 1},$$

it follows easily by Section 2 (ii) and the recurrence relation that the polynomials $a_n = a_n(x)$ are orthogonal with respect to the functional s , and that

$$s(a_n(x)^2) = \frac{(-1)^n}{2n + 1}.$$

Since (compare [7])

$$x^n = \sum_{k=0}^n A_{nk} \binom{x}{k},$$

where

$$A_{nk} = \sum_{j=0}^k (-1)^j \binom{k}{j} (k - j)^n = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n,$$

it follows that

$$s_n = s(x^n) = \sum_{k=0}^n A_{nk} \frac{(-1)^k}{k + 1} = \sum_{k=0}^n \frac{1}{k + 1} \sum_{j=0}^k (-1)^j \binom{k}{j} j^n = B_n.$$

EXAMPLE 3. (Recurrent sequences related to Pell’s equation.) Let p be a prime number and $m \in \mathbb{N}$. Suppose that $(x_1, x_2) \in \mathbb{Z}^2$ is an arbitrary solution to Pell’s equation

$$x_1^2 - px_2^2 = \varepsilon, \quad \varepsilon = \pm 1.$$

Let $A = 1, B = 0, C = -p^{2m-1}$ be fixed. We consider two cases of values for D and E corresponding to a specific solution to Pell's equation:

Case (i). For $D = \varepsilon p x_2^2, E = -p^m x_1 x_2 - \varepsilon p^{2m} x_2^2$, the recurrence (7) has the integral solution $c_k = D^k(x_1 k + \varepsilon p^m x_2), k \geq 0$.

Case (ii). For $D = -\varepsilon x_1^2, E = -p^m x_1 x_2 + \varepsilon p^{2m-1} x_1^2$, the recurrence (7) has the integral solution $c_k = D^k(x_2 k - \varepsilon p^{m-1} x_1), k \geq 0$.

In particular for $C = -2$ (that is $p = 2, m = 1$) the fundamental solution $(x_1, x_2) = (1, 1)$ with $\varepsilon = -1$ gives in case (i) $(D, E) = (-2, 2)$, and thus the recurrence $(k^2 - 2)c_{k+1} + (3k^2 + k - 2)c_k + 2k^2 c_{k-1} = 0$ has integral solution $c_k = (-2)^k(k - 2)$. The corresponding sequence

$$a_n = \sum_{k=0}^n (-2)^k(k - 2) \binom{n}{k} \binom{n+k}{k}$$

therefore satisfies the recurrence

$$(n + 1)(n^2 - 2)a_{n+1} + (2n + 1)(3n^2 + 3n - 2)a_n + n(n^2 + 2n - 1)a_{n-1} = 0.$$

In case (ii), $(D, E) = (1, -4)$, and thus the recurrence

$$(k^2 - 2)c_{k+1} - (2k - 4)c_k - k^2 c_{k-1} = 0$$

has integral solution $c_k = k + 1$. The corresponding sequence

$$a_n = \sum_{k=0}^n (k + 1) \binom{n}{k} \binom{n+k}{k}$$

therefore satisfies the recurrence

$$(n + 1)(n^2 - 2)a_{n+1} - (2n + 1)(3n^2 + 3n - 10)a_n + n(n^2 + 2n - 1)a_{n-1} = 0.$$

Analogously for $C = -2$ the solution $(x_1, x_2) = (3, 2)$ with $\varepsilon = 1$ gives in case (i) $(D, E) = (8, -28)$, and thus the recurrence

$$(k^2 - 2)c_{k+1} - (7k^2 + 9k - 28)c_k - 8k^2 c_{k-1} = 0$$

has integral solution $c_k = 8^k(3k + 4)$. The corresponding sequence

$$a_n = \sum_{k=0}^n 8^k(3k + 4) \binom{n}{k} \binom{n+k}{k}$$

therefore satisfies the recurrence

$$(n + 1)(n^2 - 2)a_{n+1} - (2n + 1)(17n^2 + 17n - 58)a_n + n(n^2 + 2n - 1)a_{n-1} = 0.$$

In case (ii), $(D, E) = (-9, 6)$, and thus the recurrence

$$(k^2 - 2)c_{k+1} + (10k^2 + 8k - 6)c_k + 9k^2c_{k-1} = 0$$

has integral solution $c_k = (-9)^k(2k - 3)$. The corresponding sequence

$$a_n = \sum_{k=0}^n (-9)^k(2k - 3) \binom{n}{k} \binom{n+k}{k}$$

therefore satisfies the recurrence

$$(n + 1)(n^2 - 2)a_{n+1} + (2n + 1)(17n^2 + 17n - 10)a_n + n(n^2 + 2n - 1)a_{n-1} = 0.$$

EXAMPLE 4. For $A = D = 2, B = 3, C = -2, E = -\frac{3}{2}x$ the recurrence is

$$(k + 2)(2k - 1)c_{k+1} - (k - 3x/2)c_k - 2k^2c_{k-1} = 0.$$

The corresponding sequence (a_n) then satisfies the recurrence

$$(n + 1)(n + 2)(2n - 1)a_{n+1} - (2n + 1)(2(n^2 + n - 1) + (2n - 1)(2n + 3)x)a_n + n(n - 1)(2n + 3)a_{n-1} = 0.$$

The polynomials $a_n = a_n(x)$ are orthogonal with respect to a measure μ concentrated in the single point $x = -2/3$.

5. Apéry's sequences

By applying Theorem 1 for $A = C = 1, B = E = 2, D = 8$, that is for

$$P_0 = (k + 1)^2, \quad P_1 = 7k^2 + 7k + 2, \quad P_2 = 8k^2,$$

we get

THEOREM 2. (i) *The Legendre transform (a_n) of the sequence (c_k) in (1) (which has $c_0 = 1, c_1 = 2$) satisfies the recurrence (3) with initial values $a_0 = 1, a_1 = 5$.*

(ii) *The Legendre transform (a_n) of the sequence (c_k) satisfying the recurrence (4) and having initial values $c_0 = 0, c_1 = 3$ satisfies the following recurrence*

$$(17) \quad (n + 1)^3 a_{n+1} - ((n + 1)^3 + n^3 + 4(2n + 1)^3) a_n + n^3 a_{n-1} = 3(4n + 2) \quad \text{for } n \geq 0$$

with initial values $a_0 = 0, a_1 = 6$.

(iii) The Legendre transform (a_n) of the sequence (c_k) satisfying the recurrence

$$(18) \quad (k + 1)^2 c_{k+1} - (7k^2 + 7k + 2)c_k - 8k^2 c_{k-1} = \frac{(-1)^k 3}{k + 1} \quad \text{for } k \geq 1$$

and having initial values $c_0 = 0, c_1 = 3$ satisfies the recurrence (3) for $n \geq 1$ with initial values $a_0 = 0, a_1 = 6$.

REMARK 2. . As in Remark 1 we denote the three sequences (a_n) in (i)–(iii) by $(a_n), (a'_n), (a''_n)$, respectively, and similarly for the sequences (c_k) .

Since $B = 2A$ the formulas in Remark 1 applies with $p_0(n) = (n + 1)^3$. In this case it is well known (cf. [6]) that

$$\lim \frac{c'_k}{c_k} = \pi^2/8, \quad \alpha'' := \lim \frac{a''_n}{a_n} = \zeta(3).$$

Therefore also

$$\alpha' := \lim \frac{a'_n}{a_n} = \pi^2/8, \quad \lim \frac{c''_k}{c_k} = \zeta(3).$$

The simultaneous approximation of $\pi^2/8$ and $\zeta(3)$ is illustrated in the following two tables:

k	c_k	c'_k	c''_k	$\pi^2/8 - c'_k/c_k$	$\zeta(3) - c''_k/c_k$
0	1	0	0	1.233700550	1.202056903
1	2	3	3	-0.266299450	-0.297943097
2	10	12	93/8	0.033700550	0.039556903
3	56	208/3	1217/18	-0.004394688	-0.005284367
4	346	1280/3	239429/576	0.000559895	0.000683067

n	a_n	a'_n	a''_n	$\pi^2/8 - a'_n/a_n$	$\zeta(3) - a''_n/a_n$
0	1	0	0	1.233700550	1.202056903
1	5	6	6	0.033700550	0.002056903
2	73	90	351/4	0.000823838	0.000002109
3	1445	5348/3	62531/36	0.000021196	0.000000002
4	33001	122140/3	11424695/288	0.000000561	0.000000000

6. A peculiar sequence

We consider now the sequence $a_n = \sum_{k=0}^n \binom{2k}{k} \binom{n}{k} \binom{n+k}{k} x^k$. To explain the properties mentioned in the introduction we consider the following sequences of polynomials

$$(19) \quad a_n(x) = \sum_{k=0}^n \binom{2k}{k} \binom{n}{k} \binom{n+k}{k} x^k,$$

$$(20) \quad P_n^-(x) = \sum_{j=0}^n \binom{n}{j} \binom{n+j-\frac{1}{2}}{j} x^j,$$

$$(21) \quad P_n^+(x) = \sum_{j=0}^n \binom{n}{j} \binom{n+j+\frac{1}{2}}{j} x^j.$$

Then we claim that

$$(22) \quad a_{2n}(x) = P_n^-(4x)^2,$$

$$(23) \quad a_{2n+1}(x) = (1 + 4x)P_n^+(4x)^2.$$

Since $a_n = a_n(1)$ we get in particular

$$(24) \quad a_{2n} = P_n^-(4)^2, \quad a_{2n+1} = 5P_n^+(4)^2,$$

which explains the peculiarities of the sequence a_n .

The polynomials $P_n^-(x)$ and $P_n^+(x)$ are expressible in terms of the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)_n}{n!} F[-n, n + \alpha + \beta + 1; \alpha + 1; (1 - x)/2].$$

In fact

$$P_n^-(x) = P_n^{(0,-\frac{1}{2})}(2x + 1) = F[-n, n + \frac{1}{2}; 1; -x],$$

$$P_n^+(x) = P_n^{(0,+\frac{1}{2})}(2x + 1) = F[-n, n + \frac{3}{2}; 1; -x].$$

Therefore formula (22) follows immediately from Clausen’s formula (cf. [11])

$$F^2[a, b; a + b + \frac{1}{2}; z] = {}_3F_2[2a, 2b, a + b; 2a + 2b, a + b + \frac{1}{2}; z]$$

with $a = -n, b = n + \frac{1}{2}, z = -4x$.

The polynomial sequence $u_n = P_n^-(x)$ satisfies the recurrence relation

$$(25) \quad (n + 1)(2n + 1)(4n - 1)u_{n+1} - (4n + 1)\left(\frac{1}{2}(4n - 1)(4n + 3)x + 4n^2 + 2n - 1\right)u_n + n(2n - 1)(4n + 3)u_{n-1} = 0 \quad \text{for } n \geq 0,$$

and is thus (cf. [14]) a sequence of orthogonal polynomials with respect to a normalized measure m^- on \mathbb{R} . The measure is given by

$$(26) \quad dm^-(t) = \begin{cases} dt/2\sqrt{1+t}, & t \in]-1, 0], \\ 0 & \text{otherwise,} \end{cases}$$

the corresponding moments being

$$(27) \quad \mu_n^- = \frac{(-1)^n 4^n}{(2n+1) \binom{2n}{n}}, \quad n \geq 0.$$

Also

$$(28) \quad \|P_n^-(x)\|^2 = \frac{1}{4n+1}, \quad n \geq 0.$$

Analogously formula (22) follows from a more general formula of Orr (see [11, Theorem III]), and contiguous relations.

Similarly the polynomial sequence $u_n = P_n^+(x)$ satisfies the recurrence relation

$$(29) \quad \begin{aligned} (n+1)(2n+3)(4n+1)u_{n+1} - (4n+3)\left(\frac{1}{2}(4n+1)(4n+5)x + 4n^2 + 6n + 1\right)u_n \\ + n(2n+1)(4n+5)u_{n-1} = 0 \quad \text{for } n \geq 0, \end{aligned}$$

and is a sequence of orthogonal polynomials with respect to a normalized measure m^+ on \mathbb{R} . The measure is given by

$$(30) \quad dm^+(t) = \begin{cases} \frac{3}{2}\sqrt{1+t} dt, & t \in]-1, 0] \\ 0 & \text{otherwise} \end{cases}$$

the corresponding moments being

$$(31) \quad \mu_n^+ = \frac{(-1)^n 3 \cdot 4^n}{(2n+1)(2n+3) \binom{2n}{n}}, \quad n \geq 0.$$

Also

$$(32) \quad \|P_n^+(x)\|^2 = \frac{3}{4n+3}, \quad n \geq 0.$$

7. Some computer-aided results

We shall mention some further examples of recurrences for Legendre transforms.

EXAMPLE 5. For the sequence (cf. Section 6) $c_k = \binom{2k}{k} x^k$ and satisfying the recurrence

$$(k+1)c_{k+1} - (4k+2)xc_k = 0,$$

the corresponding Legendre transform (a_n) satisfies the four term recurrence

$$(2n + 1)(n + 2)^2 a_{n+2} - (2n + 3)(3n^2 + 6n + 2 + 4(2n + 1)(2n + 3)x) a_{n+1} + (2n + 1)(3n^2 + 6n + 2 + 4(2n + 1)(2n + 3)x) a_n - (2n + 3)n^2 a_{n-1} = 0.$$

This follows by a telescopic argument using

$$A_{n,k} = -4(2n + 1)(2n + 3) \binom{n + 1}{k} \binom{n + k}{k} (4k + 2)x c_k.$$

EXAMPLE 6. For the sequence (c_k) of Fibonacci numbers with $c_0 = c_1 = 1$ and satisfying the recurrence

$$c_{k+1} - c_k - c_{k-1} = 0,$$

the corresponding Legendre transform (a_n) satisfies the five term recurrence

$$(2n - 1)(n + 1)(n + 2)a_{n+2} - 4(2n - 1)(2n + 3)(n + 1)a_{n+1} - 2(2n + 1)(n^2 + n - 1)a_n - 4(2n - 1)(2n + 3)na_{n-1} + (2n + 3)(n - 1)na_{n-2} = 0.$$

This follows by a telescopic argument using

$$A_{n,k} = A_{n,0} \binom{n}{k-1} \binom{n+k-1}{k-1} \frac{1}{k^2} ((n^2 + n - k(k-1))c_k + k^2 c_{k-1})$$

with

$$A_{n,0} = -4(2n - 1)(2n + 1)(2n + 3).$$

EXAMPLE 7. For the sequence (c_k) of Legendre polynomials satisfying the recurrence

$$(k + 1)c_{k+1} - (2k + 1)(2x + 1)c_k + kc_{k-1} = 0,$$

the corresponding Legendre transform (a_n) satisfies the five term recurrence

$$(2n - 1)(n + 2)^2 a_{n+2} - (3 + 4x)(2n - 1)(2n + 3)^2 a_{n+1} + (2n + 1)(38n^2 + 38n - 29 + 8(2n - 1)(2n + 3)x) a_n - (3 + 4x)(2n - 1)^2 (2n + 3) a_{n-1} + (2n + 3)(n - 1)^2 a_{n-2} = 0.$$

This follows by a telescopic argument using

$$A_{n,k} = A_{n,0} \binom{n}{k-1} \binom{n+k-1}{k-1} \frac{1}{k^2} ((n^2 + n - k(k-1) - k(2k+1)(2x+1))c_k + k^2 c_{k-1})$$

with

$$A_{n,0} = 4(2n - 1)(2n + 1)(2n + 3).$$

EXAMPLE 8. For a sequence (c_k) satisfying a recurrence

$$P_0(k)c_{k+1} - P_1(k)xc_k - P_2(k)x^2c_{k-1} = 0,$$

for $k \geq 0$, where

$$P_0(k) = (k + 1)^2, \quad P_1(k) = \alpha k^2 + \alpha k + \beta, \quad P_2(k) = \gamma k^2,$$

the corresponding Legendre transform (a_n) satisfies the five term recurrence

$$\begin{aligned} &(2n - 1)n(n + 2)^3 a_{n+2} \\ &- (2n - 1)(2n + 3)(2n^3 + 6n^2 + 4n - 1 + 2nP_1(n + 1)x) a_{n+1} \\ &- (2n + 1)(4\gamma(2n - 1)(2n + 3)n(n + 1)x^2 \\ &- 2(2n - 1)(2n + 3)(P_1(n) - 2P_1(0))x - (6n^4 + 12n^3 - 2n^2 - 8n + 3)) a_n \\ &- (2n - 1)(2n + 3)(2n^3 - 2n + 1 + 2(n + 1)P_1(n - 1)x) a_{n-1} \\ &+ (2n + 3)(n + 1)(n - 1)^3 a_{n-2} = 0. \end{aligned}$$

This follows by a telescopic argument using

$$\begin{aligned} A_{n,k} &= -4(2n - 1)(2n + 1)(2n + 3) \binom{n + 1}{k} \binom{n + k - 1}{k} \\ &\times ((\gamma(n^2 + n - k(k - 1)))x^2 + P_1(k)x)c_k + P_2(k)x^2c_{k-1}. \end{aligned}$$

Important examples are

$$c_k = \sum_{j=0}^k \binom{k}{j}^3 x^k$$

satisfying the above recurrence with $(\alpha, \beta, \gamma) = (7, 2, 8)$, and

$$c_k = \sum_{j=0}^k \binom{k}{j}^2 \binom{k + j}{j} x^k$$

satisfying the above recurrence with $(\alpha, \beta, \gamma) = (11, 3, 1)$ (cf. [6]).

EXAMPLE 9. For the sequence

$$c_k = \sum_{j=0}^k \binom{k}{j}^4 x^k,$$

satisfying the three term recurrence (see [1, 2, 4, 5, 6, 12])

$$P_0(k)c_{k+1} - P_1(k)xc_k - P_2(k)x^2c_{k-1} = 0,$$

where

$$P_0(k) = (k + 1)^3, \quad P_1(k) = 2(2k + 1)(3k^2 + 3k + 1), \quad P_2(k) = (4k - 1)4k(4k + 1),$$

the corresponding Legendre transform (a_n) satisfies the seven term recurrence

$$\begin{aligned} &(2n - 3)(2n - 1)n(n + 3)^4 a_{n+3} \\ &- (2n - 1)(2n - 3)(2n + 5)(3n^4 + 22n^3 + 52n^2 + 33n - 16 \\ &+ 2nP_1(n + 2)x)a_{n+2} \\ &- (2n - 3)(2n + 3)((1024n^5 + 6144n^4 + 10944n^3 + 2944n^2 - 5040n)x^2 \\ &- (192n^5 + 1104n^4 + 1728n^3 - 180n^2 - 1332n + 420)x \\ &- (15n^5 + 85n^4 + 120n^3 - 60n^2 - 126n + 80))a_{n+1} \\ &+ 2(2n - 3)(2n + 1)(2n + 5)(8(2n - 1)(2n + 3)(16n^2 + 16n - 15)x^2 \\ &- (72n^4 + 144n^3 - 98n^2 - 170n + 126)x - (5n^4 + 10n^3 - 10n^2 - 15n + 16))a_n \\ &- (2n - 1)(2n + 5)((1024n^5 - 1024n^4 - 3392n^3 + 3264n^2 + 2448n - 2160)x^2 \\ &- (192n^5 - 144n^4 - 768n^3 + 660n^2 + 756n - 756)x \\ &- (15n^5 - 10n^4 - 70n^3 + 60n^2 + 89n - 96))a_{n-1} \\ &- (2n - 3)(2n + 3)(2n + 5)(3n^4 - 10n^3 + 4n^2 + 17n - 16 \\ &+ 2(n + 1)P_1(n - 2)x)a_{n-2} \\ &+ (2n + 3)(2n + 5)(n + 1)(n - 2)^4 a_{n-3} = 0. \end{aligned}$$

This follows by a telescopic argument using

$$\begin{aligned} A_{n,k} = &-8(2n - 3)(2n - 1)(2n + 1)(2n + 3)(2n + 5) \binom{n + 1}{k - 1} \binom{n + k - 2}{k - 1} \frac{1}{k^2} \\ &\times (4(4k + 3)(4k + 5)(n^2 + n - (k - 1)(k - 2))x^2 c_k \\ &+ k(P_1(k)xc_k + P_2(k)x^2 c_{k-1})) \end{aligned}$$

with

$$A_{n,0} = -480(2n - 3)(2n - 1)(2n + 1)(2n + 3)(2n + 5).$$

The computations were performed by means of the GP-PARI system using the methods in [5].

Added in proof. It has been pointed out to me by Michael Stoll (University of Bonn) that arguments taken from R. P. Stanley, ‘Differentiably finite power series’, *European J. Combin.* **7** (1980), 175–188, lead to the result (illustrated by the examples above) that the set of polynomially recursive sequences is invariant under the Legendre transform and the inverse Legendre transform.

References

- [1] R. Askey and J. A. Wilson, 'A recurrence relation generalizing those of Apéry', *J. Austral. Math. Soc. (Series A)* **36** (1984), 267–278.
- [2] T. Cusick, 'Recurrences for sums of powers of binomial coefficients', *J. Combin. Theory Ser. A* **52** (1989), 77–83.
- [3] J. Franel, *L'Intermédiaire des Mathématiciens vol. 1* (1894).
- [4] ———, *L'Intermédiaire des Mathématiciens vol. 2* (1895).
- [5] M. A. Perlstadt, 'Some recurrences for sums of powers of binomial coefficients', *J. Number Theory* **27** (1987), 304–309.
- [6] A. J. van der Poorten, 'A proof that Euler missed . . . Apéry's proof of the irrationality of $\zeta(3)$ ', *Math. Intelligencer* **1** (1978/79), 195–203.
- [7] H. Rademacher, *Topics in analytic number theory* (Springer, Berlin, 1973).
- [8] J. Riordan, *Combinatorial identities* (Wiley, New York, 1968).
- [9] A. L. Schmidt, 'Generalized Legendre polynomials', *J. Reine Angew. Math.* **404** (1990), 192–202.
- [10] ———, 'Generalized q-Legendre polynomials', *J. Comput. Appl. Math.* **49** (1993), 243–249.
- [11] L. J. Slater, *Generalized hypergeometric functions* (Cambridge Univ. Press, London, 1966).
- [12] T. B. Staver, 'Om summasjon av potenser av binomialkoefficientene', *Norsk Matematisk Tidsskrift* **29** (1947), 97–103.
- [13] V. Strehl, 'Binomial identities — combinatorial and algorithmic aspects', *Discrete Math.* **136** (1994), 309–346.
- [14] G. Szegő, *Orthogonal polynomials* (Amer. Math. Soc., Providence, 1975).

Matematisk Institut
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark