A NOTE ON THE RADICAL OF ROW-FINITE MATRICES

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Let R be an associative ring, J an infinite index set, and R_J the ring of all $J \times J$ row-finite matrices over R. The Jacobson radical of R will be denoted by $\Gamma(R)$.

In [5] a *diagonalized* matrix is defined as follows:

DEFINITION. A row-finite matrix A over R is *diagonalized* provided that, if $\{a_{i_1j_1}, a_{i_2j_2}, a_{i_3j_3}, \ldots\}$ is a sequence of entries of A such that $\{j_1, j_2, j_3, \ldots\}$ contains infinitely many distinct elements, then there exists a positive integer p such that $a_{i_1j_1}a_{i_2j_2}\ldots a_{i_pj_p} = 0$.

It is shown in [5] that, if R is a commutative ring and A is a row-finite matrix over R, then A is in $\Gamma(R_J)$ if and only if A is in $[\Gamma(R)]_J$ and A is diagonalized. Utilizing the recent results of N. E. Sexauer and J. E. Warnock, we can now show that it is not necessary to assume that R is commutative.

THEOREM. *Let A be an element of R_J. Then A is in $\Gamma(R_J)$ if and only if A is in $[\Gamma(R)]_J$ and each element of the left ideal of R_J generated by A is diagonalized.*

Proof. Suppose that $A \in \Gamma(R_J)$. E. M. Patterson has shown in [2] that $\Gamma(R_J) \subseteq [\Gamma(R)]_J$. Let B be an element of the left ideal of R_J generated by A. Let $\{b_{i_1j_1}, b_{i_2j_2}, b_{i_3j_3}, \ldots\}$ be a set of entries of B such that $\{j_1, j_2, j_3, \ldots\}$ contains infinitely many distinct elements. Let $k(1) = 1$. Suppose that t is an integer greater than 1 and that $k(s)$ has been defined for each positive integer s less than t. Let $k(t)$ be a positive integer such that $j_{k(t)} \neq j_{k(s)}$ for each positive integer s less than t. For each positive integer s,

$$b_{i_{k(s)+1},j_{k(s)+1}}\, b_{i_{k(s)+2},j_{k(s)+2}} \ldots b_{i_{k(s+1)},j_{k(s+1)}} \in B_{j_{k(s+1)}},$$

the left ideal of R generated by the elements of the $j_{k(s+1)}$th column of B. Since $B \in \Gamma(R_J)$, by the main theorem of [4] there exists a positive integer p such that $b_{i_1j_1}b_{i_2j_2}\ldots b_{i_pj_p} = 0$.

Conversely, suppose that A is an element of $[\Gamma(R)]_J$ that is not contained in $\Gamma(R_J)$. By [4, Main Theorem and Proposition 3], there exists a sequence $\{b_{i_1j_1}, b_{i_2j_2}, b_{i_3j_3}, \ldots\}$ such that, for each positive integer k, $b_{i_kj_k} = \sum_{h=1}^{s_k} x_{hk}a_{h'j_k}$, where s_k is a positive integer, each $x_{hk} \in R$, each h' is a positive integer which depends upon h, each $a_{h'j_k}$ is in the j_kth column of A, $j_k \neq j_m$ if $k \neq m$, and $b_{i_1j_1}b_{i_2j_2}\ldots b_{i_nj_n} \neq 0$ for each positive integer n. Thus

$$\sum_{h=1}^{s_1} x_{h1}a_{h'j_1}b_{i_2j_2}\ldots b_{i_nj_n} = b_{i_1j_1}b_{i_2j_2}\ldots b_{i_nj_n} \neq 0$$

for each positive integer n. Since there are only finitely many integers h such that $1 \leqq h \leqq s_1$, there exists a positive integer $h_1 \leqq s_1$ such that $x_{h_11}a_{h_1'j_1}b_{i_2j_2}\ldots b_{i_nj_n} \neq 0$ for infinitely many integers n greater than 1. Suppose that, for some positive integer k, there exist integers $h_1, h_2, \ldots, h_k$ such that $1 \leqq h_i \leqq s_i$ for $1 \leqq i \leqq k$ and $x_{h_11}a_{h_1'j_1}\ldots x_{h_kk}a_{h_k'j_k}b_{i_{k+1}j_{k+1}}\ldots b_{i_nj_n} \neq 0$ for infinitely many integers n greater than k. If h is a positive integer not greater than s_{k+1}

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and n is an integer greater than k, let $f_{hn} = x_{h_1 1} a_{h_1' j_1} \dots x_{h_k k} a_{h_k' j_k} x_{h,k+1} a_{h' j_{k+1}} b_{i_{k+2} j_{k+2}} \dots b_{i_n j_n}$. Then $\sum_{h=1}^{s_{k+1}} f_{hn} = x_{h_1 1} a_{h_1' j_1} \dots x_{h_k k} a_{h_k' j_k} b_{i_{k+1} j_{k+1}} b_{i_{k+2} j_{k+2}} \dots b_{i_n j_n} \neq 0$ for infinitely many integers n greater than k. Thus there exists a positive integer h_{k+1}, not greater than s_{k+1}, such that $f_{h_{k+1} n} \neq 0$ for infinitely many integers n greater than $k+1$. Therefore there exists a sequence $\{x_{h_1 1} a_{h_1' j_1}, x_{h_2 2} a_{h_2' j_2}, \dots\}$ such that $x_{h_1 1} a_{h_1' j_1} x_{h_2 2} a_{h_2' j_2} \dots x_{h_n n} a_{h_n' j_n} \neq 0$ for each positive integer n. Well-order J. Let $Y = (y_{ij})$ be the element of R_J defined in the following way. For each positive integer m, if k_m is the mth element of J, let $y_{k_m h_m'} = x_{h_m m}$. Let $y_{ij} = 0$ for all other members of $J \times J$. Let $Z = YA$. Then Z is an element in the left ideal of R_J generated by A and, if m is a positive integer and k_m is the mth element of J, then $x_{h_m m} a_{h_m' j_m}$ is the (k_m, j_m)th entry of Z. Since $x_{h_1 1} a_{h_1' j_1} x_{h_2 2} a_{h_2' j_2} \dots x_{h_n n} a_{h_n' j_n} \neq 0$ for each positive integer n, Z is not diagonalized.

REFERENCES

1. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloquium Publications **37** (Providence, R.I., 1956).

2. E. M. Patterson, On the radical of certain rings of infinite matrices, *Proc. Roy. Soc. Edinburgh Sect.* A **65** (1960), 263–271.

3. E. M. Patterson, On the radical of rings of row-finite matrices, *Proc. Roy. Soc. Edinburgh Sect.* A **66** (1961–62), 42–46.

4. N. E. Sexauer and J. E. Warnock, The radical of row-finite matrices over an arbitrary ring, *Trans. Amer. Math. Soc.* **139** (1969), 287–295.

5. R. Slover, The Jacobson radical of row-finite matrices, *J. of Algebra* **12** (1969), 345–359.

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