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ON KY FAN'S COVERING THEOREMS FOR SIMPLEXES

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New covering properties of simplexes are obtained which extend and complement some covering theorems for simplexes of Fan.

1. Introduction

The covering theorems for simplexes of Sperner [**7**] and Knaster-Kuratowski-Mazurkiewicz [**3**] are well-known. These classical results were later generalized in Fan [**1, 2**]. In this paper further covering properties of simplexes are proved which extend and complement those obtained by Fan. Our main result is Theorem 2 in Section 3. For its proof a key step is provided by Lemma 1 in Section 2 which is based on a basic congruence relation in integer labelling (see (4) below) given in Fan [**2**, Theorem 2] and an idea suggested in Sies [**5, 6**].

2. A Lemma on Integer Labelling

We first recall some results of Fan on the integer labellings of pseudomanifolds which are needed to prove Lemma 1. For further references we refer to Fan [**2**].

For $x \in \mathbb{R}^{n+1}$, let $x_1, \dots, x_{n+1}$ be the components of x, and let $I_+(x) = \{i : x_i > 0\}$, $I_0(x) = \{i : x_i = 0\}$. We denote an n-simplex by $\Sigma^n = \{x \in \mathbb{R}^{n+1} : x_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant n+1,\ \sum_{i=1}^{n+1} x_i = 1\}$, and the boundary of Σ^n by $\partial\Sigma^n$. Let M^n be a triangulation of Σ^n. An admissible labelling ϕ of M^n assigns to each vertex v of M^n an integer $\phi(v)$ satisfying

$$\phi(v) \in \{\pm 1, \dots, \pm(n+1)\}; \tag{1}$$

$$\phi(v) + \phi(w) \neq 0 \quad \text{if } v, w \text{ are adjacent vertices;} \tag{2}$$

$$\phi(v) > 0 \quad \text{if } v \text{ is a vertex in } \partial\Sigma^n. \tag{3}$$

For any combination $(\varepsilon_1, \dots, \varepsilon_{n+1})$ of the signs $\varepsilon_i = \pm 1$, we denote by $\alpha(\varepsilon_1 1, \dots, \varepsilon_{n+1}(n+1))$ the number of those n-simplexes of M^n, each of which is labelled under ϕ by $\varepsilon_1 1, \dots, \varepsilon_{n+1}(n+1)$ at its vertices, and denote by $\beta(1, \dots, n)$ the number of those

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boundary $(n-1)$-simplexes of M^n, each of which is labelled under ϕ by $1, \ldots, n$ at its vertices. The congruence relation of Fan referred to in Section 1 asserts that for any combination $(\varepsilon_1, \ldots, \varepsilon_{n+1})$ of the signs $\varepsilon_i = \pm 1$ with at least one $\varepsilon_i = -1$,

$$(4) \qquad \alpha(\varepsilon_1 1, \ldots, \varepsilon_{n+1}(n+1)) \equiv \alpha(1, \ldots, n+1) + \beta(1, \ldots, n) \qquad \text{mod } 2.$$

We suppose further that ϕ satisfies one of the following boundary conditions:

$$(5) \qquad \phi(v) \in I_+(v) \qquad \text{for all vertices } v \text{ of } M^n \text{ in } \partial\Sigma^n;$$

$$(6) \qquad \phi(v) \in I_0(v) \qquad \text{for all vertices } v \text{ of } M^n \text{ in } \partial\Sigma^n.$$

If (5) holds, then it follows from the well-known Sperner's lemma [**7**] that

$$(7) \qquad \beta(1, \ldots, n) \equiv 1 \qquad \text{mod } 2.$$

If we assume in addition to (6) that M^n restricted to $\partial\Sigma^n$ is a further subdivision of the barycentric subdivision of $\partial\Sigma^n$, then as shown in Fan [**2**, Corollary 2], (7) also holds. In what follows, we shall make this additional assumption whenever (6) is assumed. Thus if ϕ satisfies either (5) or (6), then

$$(8) \qquad \alpha(-1, \ldots, -(n+1)) + \alpha(1, \ldots, n+1) \equiv 1 \qquad \text{mod } 2.$$

LEMMA 1. *Let ϕ, ψ be two admissible labellings of M^n such that*

(9) $\quad \psi(v) \neq \phi(w)$ *if v, w are vertices of M^n in $\partial\Sigma^n$ such that either $v = w$ or v, w are adjacent;*

(10) $\quad \psi(v) = -\phi(v)$ *if v is a vertex of M^n in $\Sigma^n \setminus \partial\Sigma^n$.*

If either ϕ or ψ satisfies one of the boundary conditions (5) and (6), then

$$(11) \qquad \alpha^{\phi}(-1, \ldots, -(n+1)) + \alpha^{\psi}(-1, \ldots, -(n+1)) \equiv 1 \qquad \text{mod } 2,$$

where the superscript ϕ or ψ indicates the labelling with respect to which the counting is performed.

PROOF: Since the roles of ϕ and ψ in both (9) and (10) are exchangable, it suffices to prove the theorem assuming that ϕ satisfies either (5) or (6). Then (8) with α specified to α^{ϕ} holds true. We shall show more precisely that

$$(12) \qquad \alpha^{\phi}(1, \ldots, n+1) = \alpha^{\psi}(-1, \ldots, -(n+1)).$$

Clearly (12) holds if the equality sign is replaced by the sign $\geqslant$. It remains to show that if σ is an n-simplex of M^n labelled under ϕ by $1, \ldots, n+1$ at its vertices, then none of the vertices of σ is in $\partial\Sigma^n$. Indeed, if σ has a vertex v in $\partial\Sigma^n$, then by (9) $\psi(v) \neq \phi(w)$ for any vertex w of σ in $\partial\Sigma^n$. Moreover, by (2) and (10) $\psi(v) \neq \phi(w)$ for any vertex w of σ in $\Sigma^n \setminus \partial\Sigma^n$. It follows that $\psi(v) \notin \{1, \ldots, n+1\}$ which is absurd. This proves (11). ∎

3. Covering Theorems for Simplexes

We denote by I a subset of $\{1,\ldots,n+1\}$ and by I' the complement of I in $\{1,\ldots,n+1\}$. For $A \subset \Sigma^n$, we denote $A' = \Sigma^n \setminus A$. Our main result is the following

Theorem 2. *Let A_i, A_{-i} be $2(n+1)$ closed subsets of the n-simplex Σ^n. Consider the following statements:*

$$\bigcup_{i=1}^{n+1}(A_i \cup A_{-i}) = \Sigma^n, \quad A_i \cap A_{-i} = \emptyset \quad \text{for } 1 \leqslant i \leqslant n+1 \tag{13}$$

$$\partial\Sigma^n \subset \bigcup_{i=1}^{n+1} A_{-i}; \tag{14}$$

$$x \in \bigcup_{i \in I_+(x)} A_i \quad \text{for all } x \in \partial\Sigma^n; \tag{15}$$

$$x \in \bigcup_{i \in I_0(x)} A_i \quad \text{for all } x \in \partial\Sigma^n. \tag{16}$$

If (13), (14) and either (15) or (16) are satisfied, then

$$\left(\bigcap_{i\in I'} A_i\right) \cap \left(\bigcap_{i \in I} A_{-i}\right) \neq \emptyset \quad \text{if } \emptyset \neq I \neq \{1,\ldots,n+1\}. \tag{17}$$

Moreover, at least one of the following

$$\bigcap_{i=1}^{n+1} A_i \neq \emptyset; \tag{18}$$

$$\bigcap_{i=1}^{n+1} A_{-i} \neq \emptyset \tag{19}$$

holds.

Proof: Let d be a metric of $\mathbb{R}^{n+1}$, $d_i = d(A_i, A_{-i})$ for $1 \leqslant i \leqslant n+1$. We denote by d_0 the Lebesgue number of the family $\{A_{\pm i} : 1 \leqslant i \leqslant n+1\}$ of 2(n+1) closed sets. Let M^n be a triangulation of Σ^n with mesh less than $\min\{d_i : 0 \leqslant i \leqslant n+1\}$. For a vertex v of M^n, we define

$$\phi(v) = i,\ \psi(v) = j \quad \text{if } v \in \partial\Sigma^n,\ v \in A_i \cap A_{-j},$$

where $i, j \in \{1,\ldots,n+1\}$ and $i \in I_+(x)$ or $i \in I_0(x)$ according to whether (15) or (16) is assumed, and define

$$\phi(v) = k,\ \psi(v) = -k \quad \text{if } v \in \Sigma^n \setminus \partial\Sigma^n,\ v \in A_k,$$

where $k \in \{\pm 1, \ldots, \pm(n+1)\}$. It is straightforward to verify that ϕ, ψ are admissible labellings of Σ^n satisfying (9) and (10).

It follows from Lemma 1 that either $\alpha^\phi(-1, \ldots, -(n+1)) \equiv 1 \bmod 2$ or $\alpha^\psi(-1, \ldots, -(n+1)) \equiv 1 \bmod 2$. For $I \neq \emptyset$, let $(\varepsilon_1, \ldots \varepsilon_{n+1})$ be so chosen that $\varepsilon_i = -1$ if $i \in I$ and $\varepsilon_i = 1$ if $i \in I'$. If the first case is true, then by (4) $\alpha^\phi(\varepsilon_1 1, \ldots, \varepsilon_{n+1}(n+1)) \equiv 1 \bmod 2$. Thus there exists an n-simplex $\sigma = v_1, \ldots, v_{n+1}$ of M^n with $\phi(v_i) = \varepsilon_i i$, that is, $v_i \in A_{\varepsilon_i i}$, for $1 \leqslant i \leqslant n+1$. Since the mesh of M^n is less than the Lebesgue number d_0, it follows that (17), (19) hold. If the second case is true, then by (4) again $\alpha^\psi(\varepsilon_1 1, \ldots, \varepsilon_{n+1}(n+1)) \equiv 1 \bmod 2$. It follows from a similar argument that (17), (18) hold. This completes the proof of the theorem. $\square$

If we assume $\bigcap_{i=1}^{n+1} A_i = \emptyset$ in Theorem 2, then the condition (14) is superfluous, since in this case obviously $\alpha^\phi(1, \ldots, n+1) = 0$ and so by (8) $\alpha^\phi(-1, \ldots, -(n+1)) \equiv 1 \bmod 2$, where ϕ is the admissible labelling defined in the proof of Theorem 2. Thus we obtain (17), (19) without assuming (14). This is precisely the assertion of Fan [**2**, Theorems 3 or 4] according to whether (15) or (16) is assumed.

As an immediate consequence of Theorem 2, we obtain in Theorem 4 below another covering property of Σ^n. We first give a simple lemma which is also useful in another situation as noted at the end of Section 3 (see also Shih [**4**, Theorem 1]).

LEMMA 3. *Let X be a compact metric space, A_i $(1 \leqslant i \leqslant m)$ be open subsets of X. Let Γ be a family of nonempty subsets of $\{1, \ldots, m\}$ and for $\gamma \in \Gamma$ let X_γ be a closed subset of X such that for all $\gamma \in \Gamma$*

$$X_\gamma \subset \bigcup \{A_i : i \in \gamma\}.$$

Then there exist closed subsets B_i $(1 \leqslant i \leqslant m)$ such that $B_i \subset A_i$ for $1 \leqslant i \leqslant m$ and for all $\gamma \in \Gamma$

$$X_\gamma \subset \bigcup \{B_i : i \in \gamma\}.$$

PROOF: Let d be the metric of X. We define $f_i(x) = d(x, A_i')$ for $1 \leqslant i \leqslant m$, $f(x) = \max_{1 \leqslant i \leqslant m} f_i(x)$. For each $\gamma \in \Gamma$, $f(x) > 0$ on the compact set X_γ and so there exists $\varepsilon_\gamma > 0$ such that $f(x) \geqslant \varepsilon_\gamma$ for $x \in X_\gamma$. Let $\varepsilon = \min\{\varepsilon_\gamma : \gamma \in \Gamma\}$. Then the sets $B_i = \{x \in A_i : f_i(x) \geqslant \varepsilon\}$ $(1 \leqslant i \leqslant m)$ have the property required. $\square$

THEOREM 4. *Let A_i $(1 \leqslant i \leqslant n+1)$ be $n+1$ closed subsets of Σ^n and let*

$C, D \subset \Sigma^n$ such that D is closed and $C \subset D$. Consider the following statements:

$$C \cup \bigcup_{i=1}^{n+1} A_i = \Sigma^n, \quad D \cap \bigcap_{i=1}^{n+1} A_i = \emptyset; \tag{20}$$

$$\partial\Sigma^n \subset \bigcup_{i=1}^{n+1} A_i'. \tag{21}$$

If (20), (21) and either (15) or (16) are satisfied, then

$$\left(\bigcap_{i\in I'} A_i\right) \cap \left(\bigcap_{i\in I} A_i'\right) \neq \emptyset \quad \text{if } \emptyset \neq I \neq \{1, \dots, n+1\}$$

Moreover, at least one of (18) and

$$\bigcap_{i=1}^{n+1} A_i' \neq \emptyset$$

holds.

PROOF: It follows from the the second equality in (20) that $\Sigma^n = D' \cup \bigcup_{i=1}^{n+1} A_i'$. By Lemma 3 there exist closed subsets A_{-i} $(1 \leqslant i \leqslant n+1)$ of Σ^n such that $A_{-i} \subset A_i'$ for $1 \leqslant i \leqslant n+1$ and

$$\Sigma^n = D' \cup \bigcup_{i=1}^{n+1} A_{-i}, \quad \partial\Sigma^n \subset \bigcup_{i=1}^{n+1} A_{-i}.$$

It follows from the first equality in (20) that $\bigcup_{i=1}^{n+1} A_i \supset C' \supset D'$ and so

$$\Sigma^n = \bigcup_{i=1}^{n+1} (A_i \cup A_{-i}). \tag{22}$$

The result follows by applying Theorem 2 to the $2(n+1)$ closed sets A_i, A_{-i} $(1 \leqslant i \leqslant n+1)$. ∎

Some interesting special cases of Theorem 4 are obtained by properly specifying the sets C and D. Note that (21) is trivially satisfied if $\bigcap_{i=1}^{n+1} A_i = \emptyset$. Thus if we assume $\bigcup_{i=1}^{n+1} A_i = \Sigma^n$ in addition to either (15) or (16), then it follows that $\bigcap_{i=1}^{n+1} A_i \neq \emptyset$. This is the covering property of Knaster-Kuratowski-Mazurkiewicz or Sperner according to whether (15) or (16) is assumed. On the other hand, let us assume further in Theorem 4 that C is closed and $\bigcap_{i=1}^{n+1} A_i = \emptyset$, and write

$$\Sigma^n = \bigcup_{i=1}^{n+1} (A_i \cup (C \cap A_{-i}))$$

in place of (22). Since in this case the condition (14) is superfluous for Theorem 2 to hold as remarked following the proof of Theorem 2, we obtain

$$C \cap \left(\bigcap_{i \in I'} A_i \right) \cap \left(\bigcap_{i \in I} A'_i \right) \neq \emptyset \quad \text{if } I \neq \emptyset, \tag{23}$$

a conclusion stronger than Theorem 4 would imply. This is precisely the assertion of Fan [**2**, Theorem 5] or Fan [**1**, Theorem 1] according to whether (15) or (16) is assumed. We note that the condition (20) is equivalent to the condition

$$\bigcap_{i=1}^{n+1} A_i \subset \operatorname{int} \bigcup_{i=1}^{n+1} A_i$$

not involving the sets C, D, where int denotes the interior relative to Σ^n.

Finally we see by using Lemma 3 that Theorems 2 and 4 remain valid when the sets A_i, A_{-i} $(1 \leqslant i \leqslant n+1)$ in Theorem 2 and the sets A_i $(1 \leqslant i \leqslant n+1)$, C in Theorem 4 are assumed to be open. Moreover, (23) also holds if we assume further that $\bigcap_{i=1}^{n+1} A_i = \emptyset$.

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