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Lie Derivations in Prime Rings With Involution

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Abstract. Let R be a non-GPI prime ring with involution and characteristic $\neq 2, 3$. Let K denote the skew elements of R, and C denote the extended centroid of R. Let δ be a Lie derivation of K into itself. Then $\delta = \rho + \epsilon$ where ϵ is an additive map into the skew elements of the extended centroid of R which is zero on $[K, K]$, and ρ can be extended to an ordinary derivation of $\langle K\rangle$ into RC, the central closure.

1 Introduction

Let R be a prime ring with involution $*$. The *skew* elements are the subset $K = \{x \in R \mid x^* = -x\}$. A *Lie derivation* is an additive mapping δ which satisfies $[x, y]^\delta = [x^\delta, y] + [x, y^\delta]$. For W any subset of R, we write $\langle W\rangle$ for the associative subring of R generated by W. Let C be the extended centroid of R, then $*$ can be extended to an involution of C, which we also denote by $*$. Let C_* be the symmetric elements in C, then for any nonzero skew $\beta \in C$, βC_* are the skew elements of C. A subset W of R is said to satisfy a *GPI* (or *is* GPI) over C if there exists a nonzero generalized polynomial $f(x_1, x_2, \ldots, x_m)$ in the free product $R_C\langle x_1, x_2, \ldots\rangle$ such that $f(w_1, w_2, \ldots, w_m) = 0$ for all $w_i \in W$.

In a recent paper of Swain [6] it was shown that in prime rings with involution of the first kind (identity on the extended centroid) a Lie derivation of K can be extended to an ordinary derivation of $\langle K\rangle$. The key ingredient in the proof when the ring is non-GPI was a result on the nature of trilinear mappings of the skew elements. A recent result of Blau [2] on triadditive mappings of the skew elements will allow the authors to extend those results to non-GPI prime rings with involutions of the second kind. Indeed, we show:

Theorem 1.1 *Let R be a non-GPI prime ring with involution and characteristic $\neq 2, 3$. Let K denote the skew elements of R, and C denote the extended centroid of R. Let δ be a Lie derivation of K into itself. Then $\delta = \rho + \epsilon$ where ϵ is an additive map into the skew elements of the extended centroid of R which is zero on $[K, K]$, and ρ is a Lie derivation which can be extended to an ordinary derivation of $\langle K\rangle$ into RC.*

The authors believe that the proof in the case when R is GPI and the involution of the second kind is tractable, using the socle to take advantage of results on prime rings without involution.

We include at this point several remarks which will be useful later.

Remark 1.2 **(See [3, proof of Theorem 2.3])** $\langle K\rangle = K \oplus K \circ K$, where $K \circ K$ is the additive C-span of elements of the form $xy + yx$, $x, y \in K$, or, equivalently, of all squares x^2, $x \in K$.

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***Remark 1.3* (See [3, proofs of Theorems 2.2, 2.13, 1.6])** Each of $\langle K\rangle$, $\langle [K,K]\rangle$, and $\langle S\rangle$ (S is the set of symmetric elements of R) contains a nonzero ideal of R.

***Remark 1.4* (Follows from [4, Theorem 4.9])** K is GPI over C if and only if R is GPI over C.

***Remark 1.5* ([1, Lemma 1])** Suppose R is not GPI, and X is an indeterminate. Let $\{f_{ij}(X) \in R_C\langle X\rangle \mid j = 1,\dots,n_i\}$ for $i = 1,\dots,n$ be n sets of C-independent generalized monomials. Then there exists $x \in K$ such that $\{f_{ij}(x) \mid j = 1,\dots,n_i\}$ for $i = 1,\dots,n$ are C-independent subsets of R.

The proof of Theorem 1.1 will be accomplished in several steps. We will first describe a sufficient condition for extending a Lie derivation of K to an ordinary derivation of $\langle K\rangle$. Then we will show that any Lie derivation of K can be written as the sum of a Lie derivation satisfying this condition and an additive map from K into the extended centroid of R.

2 A Sufficient Condition for δ to be Extended

Lemma 2.1 *Let R be a prime, non-GPI ring with involution and characteristic $\neq 2, 3$. A Lie derivation $\delta\colon K \to RC$ can be extended to an ordinary derivation $\mu\colon \langle K\rangle \to RC$ if and only if $(x^3)^\delta = x^\delta x^2 + xx^\delta x + x^2x^\delta$ for all $x \in K$.*

Proof Clearly, if μ is a derivation which extends δ then

$$(x^3)^\delta = (x^3)^\mu = x^\mu x^2 + xx^\mu x + x^2x^\mu = x^\delta x^2 + xx^\delta x + x^2x^\delta.$$

Conversely, assume that for each $x \in K$

$$(x^3)^\delta = x^\delta x^2 + xx^\delta x + x^2x^\delta. \tag{1}$$

By replacing x in (1) successively with $x + y$, and $x - y$, and adding, we get

$$\begin{aligned}&2(xy^2 + yxy + y^2x)^\delta \\ &\quad = 2(x^\delta y^2 + xy^\delta y + xyy^\delta + y^\delta xy + yx^\delta y + yxy^\delta + y^\delta yx + yy^\delta x + y^2x^\delta).\end{aligned} \tag{2}$$

On the other hand

$$\begin{aligned}(xy^2 - 2yxy + y^2x)^\delta &= \big[[x,y],y\big]^\delta \\ &= x^\delta y^2 + xy^\delta y + xyy^\delta + y^\delta yx + yy^\delta x + y^2x^\delta \\ &\qquad - 2y^\delta xy - 2yx^\delta y - 2yxy^\delta.\end{aligned} \tag{3}$$

Adding equations (2) and (3) and dividing by 3 we get

$$(xy^2 + y^2x)^\delta = x^\delta y^2 + xy^\delta y + xyy^\delta + y^\delta yx + yy^\delta x + y^2x^\delta \tag{4}$$

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for all $x, y \in K$.

By Remark 1.2, $\langle K\rangle = K \oplus K \circ K$ and $K \circ K$ is spanned additively by the x^2, $x \in K$. Define a map $\mu\colon \langle K\rangle \to RC$ by $x + \sum y_i^2 \mapsto x^\delta + \sum(y_i y_i^\delta + y_i^\delta y_i)$. To show that μ is well defined, it suffices to show that $\sum y_i^2 = 0$ implies $\sum(y_i y_i^\delta + y_i^\delta y_i) = 0$. Assume $\sum y_i^2 = 0$ and let $s = \sum(y_i^\delta y_i + y_i y_i^\delta)$. Then for each $x \in K$, $0 = \sum(xy_i^2 + y_i^2 x)$ and by equation (4)

$$\begin{aligned}0 &= \sum(xy_i^2 + y_i^2 x)^\delta = \sum(x^\delta y_i^2 + xy_i^\delta y_i + xy_i y_i^\delta + y_i^\delta y_i x + y_i y_i^\delta x + y_i^2 x^\delta)\\ &= x^\delta \sum y_i^2 + x\sum(y_i^\delta y_i + y_i y_i^\delta) + \sum(y_i^\delta y_i + y_i y_i^\delta)x + \sum y_i^2 x^\delta = xs + sx.\end{aligned}$$

Thus, if $s \neq 0$, $xs + sx$ is a GPI on K which, by Remark 1.4, contradicts the assumption that R is not GPI. Thus μ is well defined and it is left to show that μ is a derivation on $\langle K\rangle$.

Using the identity $2xy = \{(x+y)^2 - x^2 - y^2 + [x,y]\}$ for all $x, y \in K$

$$\begin{aligned}(xy)^\mu &= \frac{1}{2}\{(x+y)^\delta(x+y) + (x+y)(x+y)^\delta - x^\delta x - xx^\delta - y^\delta y - yy^\delta + [x,y]^\delta\}\\ &= \frac{1}{2}\{2x^\delta y + 2xy^\delta\} = x^\delta y + xy^\delta.\end{aligned}\tag{5}$$

Using the identity $2x^2 y = \{x \circ [x,y] + x^2 \circ y\}$ and equations (4) and (5), for all $x, y \in K$

$$\begin{aligned}(x^2 y)^\mu &= \frac{1}{2}\Big\{\big(x[x,y]\big)^\mu + \big([x,y]x\big)^\mu + (x^2 y + yx^2)^\mu\Big\}\\ &= \frac{1}{2}\{x^\delta[x,y] + x[x,y]^\delta + [x,y]^\delta x + [x,y]x^\delta + x^\delta xy + xx^\delta y + x^2 y^\delta\\ &\qquad + y^\delta x^2 + yx^\delta x + yxx^\delta\}\\ &= x^\delta xy + xx^\delta y + x^2 y^\delta.\end{aligned}\tag{6}$$

For $u = x + \sum y_i^2 \in \langle K\rangle$ and $y \in K$, by (5) and (6)

$$\begin{aligned}(uy)^\mu &= \Big(xy + \sum y_i^2 y\Big)^\mu = x^\delta y + xy^\delta + \sum(y_i^\delta y_i y + y_i y_i^\delta y + y_i^2 y^\delta)\\ &= \Big[x^\delta + \sum(y_i^\delta y_i + y_i y_i^\delta)\Big]y + (x + \sum y_i^2)y^\delta = u^\mu y + uy^\delta.\end{aligned}\tag{7}$$

Finally, for $u \in \langle K\rangle$, $v = z + \sum w_i^2 \in \langle K\rangle$, using (7)

$$\begin{aligned}(uv)^\mu &= (uz)^\mu + \sum(uw_i^2)^\mu = u^\mu z + uz^\delta + \sum\big[(uw_i)^\mu w_i + (uw_i)w_i^\delta\big]\\ &= u^\mu z + uz^\delta + \sum(u^\mu w_i^2 + uw_i^\delta w_i + uw_i w_i^\delta)\\ &= u^\mu\Big(z + \sum w_i^2\Big) + u\Big[z^\delta + \sum(w_i^\delta w_i + w_i w_i^\delta)\Big] = u^\mu v + uv^\mu.\end{aligned}$$

Thus μ is a derivation of $\langle K\rangle$.

3 The Triadditive Mapping B

Define a mapping $B\colon K^3 \to K$ by

$$B(x,y,z) = \frac{1}{6}(xyz + xzy + yxz + yzx + zxy + zyx)^\delta$$
$$- \frac{1}{6}(x^\delta yz + xy^\delta z + xyz^\delta + x^\delta zy + xz^\delta y + xzy^\delta + y^\delta xz$$
$$+ yx^\delta z + yxz^\delta + y^\delta zx + yz^\delta x + yzx^\delta + z^\delta xy + zx^\delta y + zxy^\delta$$
$$+ z^\delta yx + zy^\delta x + zyx^\delta).$$

We note that B is additive in each component (*i.e.*, $B(x+w,y,z) = B(x,y,z) + B(w,y,z)$, *etc.*) and invariant under any permutation of x, y, z. We call any mapping with these properties *triadditive*.

Define the trace of B by $T(x) = B(x,x,x) = (x^3)^\delta - (x^\delta x^2 + xx^\delta x + x^2 x^\delta)$ and note that B has *commuting trace*. That is,

$$\big[T(x), x\big] = (x^3)^\delta x - x^\delta x^3 - xx^\delta x^2 - x^2x^\delta x - x(x^3)^\delta + xx^\delta x^2 + x^2x^\delta x + x^3x^\delta$$
$$= \big[(x^3)^\delta, x\big] + [x^3, x^\delta] = [x^3, x]^\delta = 0.$$

The following critical result of Blau [2, Theorem 5.3] applies to B.

Theorem 3.1 *Suppose R is a prime ring with involution, characteristic $\neq 2, 3$, and R is not GPI. Let K denote the skew elements of R, C the extended centroid of R, and C_* the symmetric elements of C. If $B\colon K^3 \to K$ is a triadditive mapping with commuting trace, then there exist β skew in C, $\lambda \in C_*$, an additive map $\gamma\colon K \to \beta C_*$, a biadditive map $\mu\colon K^2 \to C_*$, and a triadditive map $\nu\colon K^3 \to \beta C_*$ such that for all $x, y, z \in K$,*

$$6B(x,y,z) = \lambda(xyz + xzy + yxz + yzx + zxy + zyx)$$
$$+ \gamma(z)(xy + yx) + \gamma(y)(xz + zx) + \gamma(x)(yz + zy)$$
$$+ \mu(y,z)x + \mu(x,z)y + \mu(x,y)z + \nu(x,y,z).$$

Applying Theorem 3.1 to the map defined previously, we have

$$\begin{aligned}(xyz + \cdots + zyx)^\delta - (x^\delta yz + \cdots + zyx^\delta)\\ = \lambda(xyz + \cdots + zyx) + \gamma(z)(xy + yx) + \gamma(y)(xz + zx)\\ + \gamma(x)(yz + zy) + \mu(y,z)x + \mu(x,z)y + \mu(x,y)z + \nu(x,y,z)\end{aligned} \tag{8}$$

for all $x, y, z \in K$.

We proceed to investigate the natures of the constant λ and the maps γ, μ, and ν.

Lemma 3.2 *For all $x, y \in K$,*

$$(xyx)^\delta = x^\delta yx + xy^\delta x + xyx^\delta + \frac{\lambda}{3}(x^2y + xyx + yx^2) + \frac{1}{3}\gamma(x)(xy + yx)$$
$$+ \frac{1}{3}\gamma(y)x^2 + \frac{1}{3}\mu(x,y)x + \frac{1}{6}\mu(x,x)y + \frac{1}{6}\nu(x,y,x).$$

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Proof Setting $z = x$ in (8), rearranging terms, and dividing by 2 yields

$$
\begin{aligned}
(9)\qquad &(x^2y + xyx + yx^2)^\delta \\
&\quad = x^\delta xy + xx^\delta y + x^2y^\delta + x^\delta yx + xy^\delta x + xyx^\delta + y^\delta x^2 \\
&\qquad + yx^\delta x + yxx^\delta + \lambda(x^2y + xyx + yx^2) + \gamma(x)(xy + yx) \\
&\qquad + \gamma(y)x^2 + \mu(x, y)x + \frac{1}{2}\mu(x, x)y + \frac{1}{2}\nu(x, y, x).
\end{aligned}
$$

Using the identity $3(xyx) = x^2y + xyx + yx^2 - \big[[y, x], x\big]$ and (9)

$$
\begin{aligned}
(xyx)^\delta = \frac{1}{3}\Big[&x^\delta xy + xx^\delta y + x^2y^\delta + x^\delta yx + xy^\delta x + xyx^\delta + y^\delta x^2 + yx^\delta x + yxx^\delta \\
&+ \lambda(x^2y + xyx + yx^2) + \gamma(x)(xy + yx) + \gamma(y)x^2 + \mu(x, y)x \\
&+ \frac{1}{2}\mu(x, x)y + \frac{1}{2}\nu(x, y, x) - y^\delta x^2 + 2xy^\delta x - x^2y^\delta \\
&- yx^\delta x + xyx^\delta + x^\delta yx - xx^\delta y - yxx^\delta + xyx^\delta + x^\delta yx - x^\delta xy\Big] \\
= x^\delta yx &+ xy^\delta x + xyx^\delta + \frac{\lambda}{3}(x^2y + xyx + yx^2) + \frac{1}{3}\gamma(x)(xy + yx) + \frac{1}{3}\gamma(y)x^2 \\
&+ \frac{1}{3}\mu(x, y)x + \frac{1}{6}\mu(x, x)y + \frac{1}{6}\nu(x, y, x).
\end{aligned}
$$

Lemma 3.3 *In (8), $\lambda = 0$.*

Proof Replacing y with x in Lemma 3.2 yields

$$
(10)\qquad (x^3)^\delta = x^\delta x^2 + xx^\delta x + x^2x^\delta + \lambda x^3 + \gamma(x)x^2 + \frac{1}{2}\mu(x, x)x + \frac{1}{6}\nu(x, x, x).
$$

Replacing x with x^3 in Lemma 3.2 and using (10) yields

$$
\begin{aligned}
(x^3yx^3)^\delta &= \big[\text{ right-hand side of (10) }\big]yx^3 + x^3y^\delta x^3 + x^3y\big[\text{ rhs. (10) }\big] \\
&\quad + \frac{\lambda}{3}(x^6y + x^3yx^3 + yx^6) + \frac{1}{3}\gamma(x^3)(x^3y + yx^3) + \frac{1}{3}\gamma(y)x^6 \\
&\quad + \frac{1}{3}\mu(x^3, y)x^3 + \frac{1}{6}\mu(x^3, x^3)y + \frac{1}{6}\nu(x^3, y, x^3) \\
(11)\qquad &= x^\delta x^2yx^3 + xx^\delta xyx^3 + x^2x^\delta yx^3 + x^3y^\delta x^3 + x^3yx^\delta x^2 + x^3yxx^\delta x \\
&\quad + x^3yx^2x^\delta + \frac{\lambda}{3}(x^6y + 7x^3yx^3 + yx^6) + \gamma(x)(x^3yx^2 + x^2yx^3) \\
&\quad + \frac{1}{3}\gamma(x^3)(x^3y + yx^3) + \frac{1}{3}\gamma(y)x^6 + \frac{1}{2}\mu(x, x)(x^3yx + xyx^3) \\
&\quad + \frac{1}{3}\mu(x^3, y)x^3 + \frac{1}{6}\mu(x^3, x^3)y + \frac{1}{6}\nu(x, x, x)(x^3y + yx^3) \\
&\quad + \frac{1}{6}\nu(x^3, y, x^3).
\end{aligned}
$$

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Replacing y with xyx in Lemma 3.2 yields

$$
\begin{aligned}
(x^2yx^2)^\delta &= x^\delta xyx^2 + x\Big[x^\delta yx + xy^\delta x + xyx^\delta + \frac{\lambda}{3}(x^2y + xyx + yx^2) \\
&\qquad + \frac{1}{3}\gamma(x)(xy + yx) + \frac{1}{3}\gamma(y)x^2 + \frac{1}{3}\mu(x,y)x + \frac{1}{6}\mu(x,x)y \\
&\qquad + \frac{1}{6}\nu(x,y,x)\Big]x + x^2yxx^\delta + \frac{\lambda}{3}(x^3yx + x^2yx^2 + xyx^3) \\
&\quad + \frac{1}{3}\gamma(x)(x^2yx + xyx^2) + \frac{1}{3}\gamma(xyx)x^2 + \frac{1}{3}\mu(x,xyx)x \\
&\quad + \frac{1}{6}\mu(x,x)xyx + \frac{1}{6}\nu(x,xyx,x) \\
&= x^\delta xyx^2 + xx^\delta yx^2 + x^2y^\delta x^2 + x^2yx^\delta x + x^2yxx^\delta \\
&\quad + \frac{2\lambda}{3}(x^3yx + x^2yx^2 + xyx^3) + \frac{2}{3}\gamma(x)(x^2yx + xyx^2) \\
&\quad + \frac{1}{3}\gamma(y)x^4 + \frac{1}{3}\gamma(xyx)x^2 + \frac{1}{3}\mu(x,y)x^3 + \frac{1}{3}\mu(x,x)xyx \\
&\quad + \frac{1}{3}\mu(x,xyx)x + \frac{1}{6}\nu(x,y,x)x^2 + \frac{1}{6}\nu(x,xyx,x).
\end{aligned} \tag{12}
$$

Finally, replacing y with x^2yx^2 in Lemma 3.2 and using (12) yields

$$
\begin{aligned}
(x^3yx^3)^\delta &= x^\delta x^2yx^3 + x\big[\text{ rhs. (12) }\big]x + x^3yx^2x^\delta \\
&\quad + \frac{\lambda}{3}(x^4yx^2 + x^3yx^3 + x^2yx^4) + \frac{1}{3}\gamma(x)(x^3yx^2 + x^2yx^3) \\
&\quad + \frac{1}{3}\gamma(x^2yx^2)x^2 + \frac{1}{3}\mu(x,x^2yx^2)x + \frac{1}{6}\mu(x,x)x^2yx^2 \\
&\quad + \frac{1}{6}\nu(x,x^2yx^2,x) \\
&= x^\delta x^2yx^3 + xx^\delta xyx^3 + \cdots + x^3yx^2x^\delta \\
&\quad + \lambda(x^4yx^2 + x^3yx^3 + x^2yx^4) + \gamma(x)(x^3yx^2 + x^2yx^3) \\
&\quad + \frac{1}{3}\gamma(y)x^6 + \frac{1}{3}\gamma(xyx)x^4 + \frac{1}{3}\gamma(x^2yx^2)x^2 \\
&\quad + \frac{1}{3}\mu(x,y)x^5 + \frac{1}{2}\mu(x,x)x^2yx^2 + \frac{1}{3}\mu(x,xyx)x^3 + \frac{1}{3}\mu(x,x^2yx^2)x \\
&\quad + \frac{1}{6}\nu(x,y,x)x^4 + \frac{1}{6}\nu(x,xyx,x)x^2 + \frac{1}{6}\nu(x,x^2yx^2,x).
\end{aligned} \tag{13}
$$

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Subtracting (13) from (11) yields

$$
\begin{aligned}
0 = {} & \frac{\lambda}{3}x^6y - \lambda x^4yx^2 + \frac{4\lambda}{3}x^3yx^3 - \lambda x^2yx^4 + \frac{\lambda}{3}yx^6 - \frac{1}{3}\mu(x,y)x^5 \\
& + \frac{1}{2}\mu(x,x)x^3yx - \frac{1}{2}\mu(x,x)x^2yx^2 + \frac{1}{2}\mu(x,x)xyx^3 \\
& + \frac{1}{6}\big[2\gamma(x^3) + \nu(x,x,x)\big]x^3y + \frac{1}{6}\big[2\gamma(x^3) + \nu(x,x,x)\big]yx^3 \\
& - \frac{1}{6}\big[2\gamma(xyx) + \nu(x,y,x)\big]x^4 + \frac{1}{3}\big[\mu(x^3,y) - \mu(x,xyx)\big]x^3 \\
& - \frac{1}{6}\big[2\gamma(x^2yx^2) + \nu(x,xyx,x)\big]x^2 + \frac{1}{6}\mu(x^3,x^3)y - \frac{1}{3}\mu(x,x^2yx^2)x \\
& + \frac{1}{6}\nu(x^3,y,x^3) - \frac{1}{6}\nu(x,x^2yx^2,x)
\end{aligned}
\tag{14}
$$

for all $x, y \in K$.

Since R is not GPI, then K is also not GPI (Remark 1.4) and hence not PI. Then there exists an $x \in K$ which is not algebraic of degree ≤ 6 over C (otherwise K would satisfy the symmetric polynomial S_6). Fix such an x; then the generalized monomials in (14) are C-independent in $R_C\langle Y\rangle$. By Remark 1.5, there exists $y \in K$ which makes (14) a linear combination of C-independent elements of R. Hence, each coefficient of (14) must be zero, in particular $\lambda = 0$. This completes the proof of the lemma.

The conclusion of Lemma 3.2 can now be rewritten as

$$
\begin{aligned}
(xyx)^\delta = {} & x^\delta yx + xy^\delta x + xyx^\delta + \frac{1}{3}\gamma(x)(xy+yx) + \frac{1}{3}\gamma(y)x^2 \\
& + \frac{1}{3}\mu(x,y)x + \frac{1}{6}\mu(x,x)y + \frac{1}{6}\nu(x,y,x).
\end{aligned}
\tag{15}
$$

Lemma 3.4 *In (8), μ is identically* 0.

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Proof Replacing x with xyx in (15) yields

$$\begin{aligned}(xyxyxyx)^{\delta} &= \big[\text{ rhs. } (15)\,\big]yxyx + xyxy^{\delta}xyx + xyxy\big[\text{ rhs. } (15)\,\big]\\ &\quad + \frac{1}{3}\gamma(xyx)(xyxy + yxyx) + \frac{1}{3}\gamma(y)xyx^2yx + \frac{1}{3}\mu(xyx, y)xyx\\ &\quad + \frac{1}{6}\mu(xyx, xyx)y + \frac{1}{6}\nu(xyx, y, xyx)\\ &= x^{\delta}yxyxyx + \cdots + xyxyxyx^{\delta}\\ &\quad + \frac{1}{3}\gamma(x)(xy^2xyx + xyxyxy + yxyxyx + xyxy^2x)\\ &\quad + \frac{1}{3}\gamma(y)(x^2yxyx + xyx^2yx + xyxyx^2)\\ &\quad + \frac{1}{3}\gamma(xyx)(xyxy + yxyx) + \frac{2}{3}\mu(x, y)xyxyx + \frac{1}{6}\mu(x, x)y^2xyx\\ &\quad + \frac{1}{6}\mu(x, x)xyxy^2 + \frac{1}{3}\mu(xyx, y)xyx + \frac{1}{6}\mu(xyx, xyx)y\\ &\quad + \frac{1}{6}\nu(x, y, x)(xyxy + yxyx) + \frac{1}{6}\nu(xyx, y, xyx).\end{aligned} \tag{16}$$

Replacing x and y with y and xyx respectively in (15) yields

$$\begin{aligned}(yxyxy)^{\delta} &= y^{\delta}xyxy + y\big[\text{ rhs. } (15)\,\big]y + yxyxy^{\delta} + \frac{1}{3}\gamma(y)(yxyx + xyxy)\\ &\quad + \frac{1}{3}\gamma(xyx)y^2 + \frac{1}{3}\mu(y, xyx)y + \frac{1}{6}\mu(y, y)xyx + \frac{1}{6}\nu(y, xyx, y)\\ &= y^{\delta}xyxy + \cdots + yxyxy^{\delta} + \frac{1}{3}\gamma(x)(y^2xy + yxy^2)\\ &\quad + \frac{1}{3}\gamma(y)(yxyx + yx^2y + xyxy) + \frac{1}{3}\gamma(xyx)y^2 + \frac{1}{3}\mu(x, y)yxy\\ &\quad + \frac{1}{6}\mu(x, x)y^3 + \frac{1}{6}\mu(y, y)xyx + \frac{1}{3}\mu(y, xyx)y\\ &\quad + \frac{1}{6}\nu(x, y, x)y^2 + \frac{1}{6}\nu(y, xyx, y).\end{aligned} \tag{17}$$

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Finally, replacing y with $yxyxy$ in (15) and using (17) yields

$$
\begin{aligned}
(xyxyxyx)^\delta &= x^\delta yxyxyx + x\big[\text{ rhs. (17) }\big]x + xyxyxyx^\delta \\
&\quad + \frac{1}{3}\gamma(x)(xyxyxy + yxyxyx) + \frac{1}{3}\gamma(yxyxy)x^2 \\
&\quad + \frac{1}{3}\mu(x, yxyxy)x + \frac{1}{6}\mu(x,x)yxyxy + \frac{1}{6}\nu(x, yxyxy, x) \\
&= x^\delta yxyxyx + \cdots + xyxyxyx^\delta \\
&\quad + \frac{1}{3}\gamma(x)(yxyxyx + xy^2xyx + xyxy^2x + xyxyxy) \\
&\quad + \frac{1}{3}\gamma(y)(x^2yxyx + xyx^2yx + xyxyx^2) + \frac{1}{3}\gamma(xyx)xy^2x \\
&\quad + \frac{1}{6}\mu(x,x)xy^3x + \frac{1}{6}\mu(y,y)x^2yx^2 + \frac{1}{6}\mu(x,x)yxyxy \\
&\quad + \frac{1}{3}\mu(x,y)xyxyx + \frac{1}{3}\mu(y, xyx)xyx \\
&\quad + \frac{1}{3}\mu(x, yxyxy)x + \frac{1}{6}\nu(x,y,x)xy^2x + \frac{1}{6}\nu(x, xyx, x)x^2 \\
&\quad + \frac{1}{6}\nu(x, yxyxy, x).
\end{aligned}
\tag{18}
$$

Subtracting (18) from (16) yields

$$
\begin{aligned}
0 &= \frac{1}{3}\mu(x,y)xyxyx + \frac{1}{6}\mu(x,x)y^2xyx + \frac{1}{6}\mu(x,x)xyxy^2 - \frac{1}{6}\mu(x,x)xy^3x \\
&\quad - \frac{1}{6}\mu(y,y)x^2yx^2 - \frac{1}{6}\mu(x,x)yxyxy + \frac{1}{6}\big[2\gamma(xyx) + \nu(x,y,x)\big]xyxy \\
&\quad + \frac{1}{6}\big[2\gamma(xyx) + \nu(x,y,x)\big]yxyx - \frac{1}{6}\big[2\gamma(xyx) + \nu(x,y,x)\big]xy^2x \\
&\quad - \frac{1}{6}\big[2\gamma(yxyxy) + \nu(y, xyx, y)\big]x^2 - \frac{1}{3}\mu(x, yxyxy)x + \frac{1}{6}\mu(xyx, xyx)y \\
&\quad + \frac{1}{6}\nu(xyx, y, xyx) - \frac{1}{6}\nu(x, yxyxy, x)
\end{aligned}
\tag{19}
$$

for all $x, y \in K$.

Now, fix an arbitrary $y \in K$. If y is not algebraic of degree ≤ 2 over C, then by Remark 1.5 there is an $x \in K$ such that the monomials in (19) are linearly independent over C. Thus, each coefficient must be zero and, in particular, $\mu(y,y) = 0$. If y is algebraic of degree ≤ 2 over C, then some of the monomials in (19) will change and combine, but the x^2yx^2 monomial is the only one with total degree in x of 4. Thus, for the same reason as before, $\mu(y,y) = 0$.

We have shown that $\mu(y,y) = 0$ for all $y \in K$. For x, y arbitrary in K, we have $\mu(x,y) = \frac{1}{2}[\mu(x+y, x+y) - \mu(x,x) - \mu(y,y)] = 0$ by the result just shown. Thus μ is identically 0. This completes the proof of the lemma.

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In equation (19), after eliminating all terms involving μ, fixing $x \in K$, looking at monomials with total degree 2 in y, and reasoning similar to the proof of Lemma 3.4, we have:

Lemma 3.5 $-2\gamma(xyx) = \nu(x, y, x)$ *for all* $x, y \in K$.

Using the previous lemmas with equation (15) we have

$$(xyx)^\delta = x^\delta yx + xy^\delta x + xyx^\delta + \frac{1}{3}\gamma(x)(xy + yx) + \frac{1}{3}\gamma(y)x^2 - \frac{1}{3}\gamma(xyx). \tag{20}$$

4 A New Derivation

Define $\epsilon\colon K \to \beta C_*$ (β as defined in Theorem 3.1) by $\epsilon(x) = x^\epsilon = -\frac{1}{3}\gamma(x)$.

Lemma 4.1 $[x, y]^\epsilon = 0$ *for all* $x, y \in K$.

Proof Let $x, y \in K$. We need only show that $\gamma([x, y]) = 0$. Using (20)

$$\begin{aligned}\big(x[x,y]x\big)^\delta &= x^\delta[x,y]x + x[x,y]^\delta x + x[x,y]x^\delta + \frac{1}{3}\gamma(x)\big(x[x,y] + [x,y]x\big)\\ &\quad + \frac{1}{3}\gamma\big([x,y]\big)x^2 - \frac{1}{3}\gamma\big(x[x,y]x\big)\\ &= x^\delta xyx - x^\delta yx^2 + xx^\delta yx - xyx^\delta x + x^2y^\delta x - xy^\delta x^2 + x^2yx^\delta\\ &\quad - xyxx^\delta + \frac{1}{3}\gamma(x)(x^2y - yx^2) + \frac{1}{3}\gamma\big([x,y]\big)x^2 - \frac{1}{3}\gamma\big(x[x,y]x\big).\end{aligned} \tag{21}$$

On the other hand, using (20)

$$\begin{aligned}\big(x[x,y]x\big)^\delta &= [x, xyx]^\delta = x^\delta xyx - xyxx^\delta + x(xyx)^\delta - (xyx)^\delta x\\ &= x^\delta xyx - xyxx^\delta + xx^\delta yx + x^2y^\delta x + x^2yx^\delta - x^\delta yx^2 - xy^\delta x^2\\ &\quad - xyx^\delta x + \frac{1}{3}\gamma(x)(x^2y - yx^2).\end{aligned} \tag{22}$$

Subtracting (22) from (21), and multiplying by 3 gives

$$0 = \gamma\big([x,y]\big)x^2 - \gamma\big(x[x,y]x\big). \tag{23}$$

Fix $x \in K$. If x is not algebraic of degree ≤ 2 over C, then by Remark 1.5, $\gamma([x, y]) = 0$. If x is algebraic of degree ≤ 2, then there exists $z \in K$ such that both z and $z + x$ are not algebraic of degree ≤ 2 (use Remark 1.5 with the C-independent sets $\{Z^2, Z, 1\}$ and $\{(Z + x)^2, Z + x, 1\}$). Then, as before, $\gamma([z, y]) = \gamma([z + x, y]) = 0$. So $\gamma([x, y]) = \gamma([z + x, y]) - \gamma([z, y]) = 0$. This completes the proof of the lemma.

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Define $\rho = \delta - \epsilon$. ρ is a Lie derivation of K since it agrees with δ on $[K, K]$. Also, replacing y with x in (20),

$$\begin{aligned}
x^\rho x^2 + xx^\rho x + x^2 x^\rho &= x^{\delta-\epsilon}x^2 + xx^{\delta-\epsilon}x + x^2x^{\delta-\epsilon} \\
&= x^\delta x^2 + xx^\delta x + x^2x^\delta - 3\epsilon(x)x^2 \\
&= x^\delta x^2 + xx^\delta x + x^2x^\delta + \gamma(x)x^2 \\
&= (x^3)^\delta + \frac{1}{3}\gamma(x^3) = (x^3)^{\delta-\epsilon} = (x^3)^\rho.
\end{aligned}$$

Thus ρ satisfies the criteria for Lemma 2.1 and can be extended to an ordinary derivation of $\langle K\rangle$ into RC. This completes the proof of Theorem 1.1.

If in Theorem 1.1, the involution is of the first kind, then $C_* = C$ and the map ϵ is identically 0. Thus δ itself can be extended to an ordinary derivation of $\langle K\rangle$. This result was previously shown by Swain in [6].

Corollary 4.2 *Let R be a simple non-GPI ring with involution and characteristic $\neq 2, 3$. Let δ be a Lie derivation of K. Then $\delta = \rho + \epsilon$ where ϵ is an additive map of K into the skew elements of the extended centroid of R which is zero on $[K, K]$, and ρ is a Lie derivation which can be extended to an ordinary derivation of R into RC.*

Proof By Remark 1.3, $\langle K\rangle$ contains a nonzero ideal of R; thus $\langle K\rangle = R$. Applying Theorem 1.1 completes the proof.

References

[1] K. I. Beidar, W. S. Martindale, 3rd, and A. V. Mikhalev, *Lie isomorphisms in prime rings with involution*. J. Algebra **169**(1994), 304–327.

[2] P. S. Blau, *Lie isomorphisms of non-GPI rings with involution*, to appear.

[3] I. N. Herstein, *Topics in Ring Theory*. University of Chicago Press, Chicago, 1969.

[4] W. S. Martindale, 3rd, *Prime rings with involution and generalized polynomial identities*. J. Algebra **22**(1972), 502–516.

[5] W. S. Martindale, 3rd and C. R. Miers, *Herstein's Lie theory revisited*. J. Algebra **98**(1986), 14–37.

[6] G. A. Swain, *Lie derivations of the skew elements of prime rings with involution*. J. Algebra **184**(1996), 679–704.

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