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JACOBIAN ELLIPTIC FUNCTIONS IN SIGNATURE FOUR

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Abstract

The signature four elliptic theory of Ramanujan is provided with a counterpart to the Jacobian modular sine; this counterpart yields natural direct proofs of several hypergeometric identities recorded by Ramanujan, bypassing the signature four transfer principle of Berndt *et al*. [‘Ramanujan’s theories of elliptic functions to alternative bases’, *Trans. Amer. Math. Soc.* **347** (1995), 4163–4244].

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1. Introduction

The classical theory of Jacobian elliptic functions is intimately associated with the hypergeometric function $F(\frac{1}{2}, \frac{1}{2}; 1; -)$: thus, the Jacobian modular sine function $\mathrm{sn} = \mathrm{sn}(-, k)$ to modulus $k \in (0, 1)$ has fundamental periods $4K = 2\pi F(\frac{1}{2}, \frac{1}{2}; 1; k^2)$ and $2iK' = i\pi F(\frac{1}{2}, \frac{1}{2}; 1; 1 - k^2)$.

In his paper on modular equations and approximations to π, Ramanujan asserted that there are alternative elliptic theories in which the ‘classical’ hypergeometric function $F(\frac{1}{2}, \frac{1}{2}; 1; -)$ is replaced by $F(\frac{1}{4}, \frac{3}{4}; 1; -)$, $F(\frac{1}{3}, \frac{2}{3}; 1; -)$ or $F(\frac{1}{6}, \frac{5}{6}; 1; -)$. In his second notebook, he assembled many results pertaining to these alternative theories, but he does not seem to have made explicit the elliptic functions to which they are attached. All of these results were proved in [2] by Berndt, Bhargava and Garvan, who developed transfer principles by means of which classical elliptic results generate corresponding results in the alternative elliptic theories. The ‘signature three’ theory associated to $F(\frac{1}{3}, \frac{2}{3}; 1; -)$ is the most substantial of these alternative theories: its transfer principle is the most elaborate and it is closely related to the famous cubic theta-function identity of the Borwein brothers. In this signature alone, the authors of [2] brought to light an actual elliptic function that underlies the theory, remarking that the results in the other alternative theories are more easily extracted from the classical theory.

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Our interest in this paper centres on the 'signature four' theory associated to the hypergeometric function $F(\frac{1}{4}, \frac{3}{4}; 1; -)$. Shen, in [7], discovered an explicit elliptic function dn_2 that underlies this theory; he constructed this elliptic function by naturally modifying a standard construction of the Jacobian 'delta amplitude' dn. As in the classical theory, dn_2 is an elliptic function of order two; in contrast to the classical situation, dn_2 has double poles and is, in this sense, more Weierstrassian than Jacobian. We propose a variant of the construction in [7], producing a natural odd elliptic function rn_2. This function is not only of Jacobian type: up to elementary rescalings, rn_2 is, in fact, the Jacobian modular sine function to an appropriate modulus. We recover from rn_2 the full set of functions in [7]; we also extract directly from rn_2 several identities of Ramanujan that were hitherto established by the signature four transfer principle and otherwise, thereby further establishing rn_2 within the signature four theory.

2. Root function

Fix as modulus $\kappa \in (0, 1)$. The rule

$$f(T) = \int_0^T F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt$$

defines an odd strictly increasing bijection from $\mathbb{R}$ to $\mathbb{R}$. The evaluation

$$L := f\,(\tfrac{1}{2}\pi) = \tfrac{1}{2}\,\pi\, F\,(\tfrac{1}{4}, \tfrac{3}{4}; 1; \kappa^2)$$

is performed by termwise integration after expanding the hypergeometric series in the integrand.

THEOREM 2.1. $f(T + \pi) = f(T) + 2L$.

PROOF. Integrate the function $t \mapsto F(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \kappa^2 \sin^2 t)$ over the interval $[0, \pi + T]$ split at π: the integral over $[0, \pi]$ is exactly $2L$, while the integral over $[\pi, \pi + T]$ is seen to be exactly $f(T)$ after a π shift. □

Write $\phi : \mathbb{R} \to \mathbb{R}$ for the inverse to f; this inverse therefore satisfies

$$\phi(u + 2L) = \phi(u) + \pi.$$

As an auxiliary function, introduce the composite $\psi := \arcsin(\kappa \sin \phi)$. We now define the function $\sigma : \mathbb{R} \to \mathbb{R}$ by $\sigma = \sin \frac{1}{2}\psi$ with the warning that this is not a 'sigma function' in the traditional sense.

THEOREM 2.2. *The function σ has period $4L$.*

PROOF. From $\phi(u + 2L) = \phi(u) + \pi$, it follows that $\sin \phi(u + 2L) = -\sin \phi(u)$ and therefore that $\sigma(u + 2L) = -\sigma(u)$; thus, $\sigma(u + 4L) = \sigma(u)$. It is readily checked that the period $4L$ of σ is least positive. □

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Recall (say, from [3, page 101]) the hypergeometric evaluation

$$F\Big(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \sin^2\psi\Big) = \frac{\cos\frac{1}{2}\psi}{\cos\psi}.$$

THEOREM 2.3. *The function $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies the initial condition $\sigma(0) = 0$ and the differential equation $(\sigma^\circ)^2 = \sigma^4 - \sigma^2 + \frac{1}{4}\kappa^2$.*

PROOF. Here, σ° denotes the derivative of σ; the traditional prime $'$ is used later for another (quite traditional) purpose. The initial condition is trivial; for the differential equation, differentiate. From the definition $\sigma = \sin\frac{1}{2}\psi$, it follows that

$$\sigma^\circ = \tfrac{1}{2}(\cos\tfrac{1}{2}\psi)\,\psi^\circ;$$

from $\sin\psi = \kappa\sin\phi$, it follows that

$$\psi^\circ = \kappa\frac{\cos\phi}{\cos\psi}\,\phi^\circ;$$

while from

$$u = \int_0^{\phi(u)} F\Big(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \kappa^2\sin^2 t\Big)\,dt$$

together with the hypergeometric evaluation recalled before the theorem, it follows that

$$\phi^\circ = \frac{\cos\psi}{\cos\frac{1}{2}\psi}.$$

Mass cancellation produces $\sigma^\circ = \frac{1}{2}\kappa\cos\phi$; finally, squaring and trigonometric duplication result in

$$4(\sigma^\circ)^2 = \kappa^2 - \sin^2\psi = \kappa^2 - 4\sin^2\tfrac{1}{2}\psi\,\cos^2\tfrac{1}{2}\psi = \kappa^2 - 4\sigma^2(1-\sigma^2). \qquad \square$$

This initial value problem has a second solution: namely, $-\sigma$. The fact that the initial value problem has just these two solutions is evident upon further differentiation, leading to $\sigma^{\circ\circ} = 2\sigma^3 - \sigma$: as the right-hand side here is polynomial, specification of $\sigma(0)$ as zero and of $\sigma^\circ(0)$ as a square root of $\frac{1}{4}\kappa^2$ picks out a unique solution. The solution $\sigma = \sin\frac{1}{2}\psi$ is singled out by the specification $\sigma^\circ(0) = \frac{1}{2}\kappa$; equivalently, by the requirement that $\sigma(t) > 0$ when $t > 0$ is small.

As the right-hand side of the differential equation in Theorem 2.3 is a quartic with simple zeros, its solutions are—or rather, extend to be—elliptic on the plane. We shall identify the specific elliptic extension of $\sigma = \sin\frac{1}{2}\psi$ in two alternative forms, each of which has its uses: in Section 3, we identify σ in Weierstrassian terms; in Section 4, we identify σ in Jacobian terms.

We end this section with some remarks on notation. In the Introduction, we gave the elliptic extension of σ the name rn_2 for two reasons: on the one hand, rn_2 is a 'root' function from which the various functions in [7] may be derived; on the other hand, the proximity of rn to sn reflects the fact that rn_2 is a signature four replacement for the

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Jacobian modular sine. Nevertheless, for largely typographical reasons, we continue to use the notation σ for the elliptic extension of $\sin \frac{1}{2}\psi$.

3. Weierstrassian representation

Our identification of σ in Weierstrassian terms uses the result of [9, Example 2, page 454], which is a result attributed to Weierstrass himself. Explicitly, let f be the quartic polynomial given by

$$f(z) = a_0 z^4 + 4a_1 z^3 + 6a_2 z^2 + 4a_3 z + a_4$$

with quadrinvariant

$$g_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2$$

and cubinvariant

$$g_3 = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 - a_1^2 a_4$$

and assume that the zeros of f are simple. Then the initial value problem

$$(w^\circ)^2 = f(w), \quad w(0) = a$$

has solutions given by

$$w = a + \frac{Ap^\circ + \frac{1}{2} f^\circ(a)[p - \frac{1}{24} f^{\circ\circ}(a)] + \frac{1}{24} f^{\circ\circ\circ}(a)}{2[p - \frac{1}{24} f^{\circ\circ}(a)]^2 - \frac{1}{48} f(a) f^{\circ\circ\circ\circ}(a)},$$

where A is a square root of $f(a)$ and where $p = \wp(-; g_2, g_3)$ is the Weierstrass $\wp$-function with g_2 and g_3 as invariants.

THEOREM 3.1. *The function σ is given by*

$$\sigma = \frac{\frac{1}{4}\kappa p^\circ}{(\frac{1}{4}\kappa)^2 - (\frac{1}{12} + p)^2} = \frac{\frac{1}{4}\kappa p^\circ}{(\frac{1}{4}\kappa - \frac{1}{12} - p)(\frac{1}{4}\kappa + \frac{1}{12} + p)},$$

where $p = \wp(-; g_2, g_3)$ is the Weierstrass function with invariants

$$g_2 = \tfrac{1}{12}(1 + 3\kappa^2), \quad g_3 = \tfrac{1}{216}(1 - 9\kappa^2).$$

PROOF. Apply the result of Weierstrass to the quartic that appears on the right-hand side of the differential equation in Theorem 2.3. The square root $A = -\frac{1}{2}\kappa$ of $\frac{1}{4}\kappa^2$ is preferred because $\sigma(t) = \sin \frac{1}{2}\psi(t) > 0$ when $t > 0$ is small. □

The Weierstrass function p has a fundamental pair of periods $(2\omega, 2\omega')$ with $\omega > 0$ and $-i\omega' > 0$; introduce the third half-period ω'' by the symmetrical condition $\omega + \omega' + \omega'' = 0$, as is customary. We present explicit evaluations of these fundamental periods in Section 5; their precise values are not important at present. The differential equation $(p^\circ)^2 = 4p^3 - g_2 p - g_3$ satisfied by p factorises as

$$(p^\circ)^2 = 4(p - \tfrac{1}{6})(p + \tfrac{1}{12} - \tfrac{1}{4}\kappa)(p + \tfrac{1}{12} + \tfrac{1}{4}\kappa)$$

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so that p has (stationary) midpoint values given in decreasing order by

$$e := p(\omega) = \tfrac{1}{6}, \quad e'' := p(\omega'') = -\tfrac{1}{12} + \tfrac{1}{4}\kappa, \quad e' := p(\omega') = -\tfrac{1}{12} - \tfrac{1}{4}\kappa,$$

and, for σ, we have the equivalent expression

$$\sigma = -\tfrac{1}{4}\kappa \frac{p^\circ}{(p - e'')(p - e')}.$$

Such an expression for σ (namely, p° times a rational function of p) is to be expected of an odd elliptic function in terms of a $\wp$-function having the same periods. The following result enables us to confirm that σ and p are, in fact, coperiodic.

THEOREM 3.2. *The function σ satisfies the identity $\sigma(z + \omega) = -\sigma(z)$.*

PROOF. The Weierstrass function p satisfies the familiar identity

$$(p(z + \omega) - e)(p(z) - e) = (e'' - e)(e' - e) =: E$$

from which

$$p(z + \omega) = e + \frac{E}{p(z) - e}$$

and therefore

$$p^\circ(z + \omega) = -\frac{E\, p^\circ(z)}{(p(z) - e)^2}.$$

Introduce the function s by $\sigma = -\frac{1}{4}\kappa s$ so that

$$s = \frac{p^\circ}{(p - e')(p - e'')}$$

and it follows, by direct calculation, that

$$s(z + \omega) = -\frac{E\, p^\circ(z)}{(E + (e - e')(p(z) - e))(E + (e - e'')(p(z) - e))};$$

here,

$$E + (e' - e)(e) = e''(e' - e) \quad \text{and} \quad E + (e'' - e)(e) = e'(e'' - e)$$

so that, after cancellation of E, it follows that $s(z + \omega) = -s(z)$. It is perhaps of interest to note that s has this property when p is any Weierstrass function. All that remains of the proof is to reinstate the multiplier $-\frac{1}{4}\kappa$. □

Here, we may note that the identity $\sigma(u + 2L) = -\sigma(u)$ in the proof of Theorem 2.2 extends to the complex plane by analytic continuation.

THEOREM 3.3. *The functions σ and p have the same periods.*

PROOF. The reformulated expression for σ that came before Theorem 3.2 makes it plain that each period of p is a period of σ. When α is a period of σ, two possibilities stem from $\sigma(\alpha) = \sigma(0) = 0$. One possibility is that p has a pole at α,

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where the expression for σ has a triple pole in its numerator but a quadruple pole in its denominator; such a point is a period of p. The other possibility is that α is a zero of p° but not of $(p - e'')(p - e')$; such a point is congruent to ω and could not have been a period of σ on account of Theorem 3.2. □

We are also able to identify the zeros and poles of σ.

THEOREM 3.4. *The function σ has simple zeros at* 0 *and ω and simple poles at ω' and ω''.*

PROOF. Inspect the expression for σ before Theorem 3.2: at each of ω'' and ω' the numerator has a simple zero but the denominator has a double zero; at ω only the numerator has a zero, while at 0 the numerator has a triple pole but the denominator has a quadruple pole. □

In Theorem 3.2, we show that an ω shift reverses σ. The effects on σ of half-period shifts by ω' and ω'' are as follows.

THEOREM 3.5.

$$\sigma(z + \omega') = \frac{\frac{1}{2}p^\circ(z)}{\frac{1}{6} - p(z)} \quad \text{and} \quad \sigma(z + \omega'') = \frac{\frac{1}{2}p^\circ(z)}{p(z) - \frac{1}{6}}.$$

PROOF. As σ is reversed by an ω shift, each of these two formulas follows from the other. Either may be verified by calculations along the lines of those in the proof of Theorem 3.2; we leave these calculations as straightforward exercises. □

In connection with the effect of an ω'' shift, the devotee of [9] will be pleased to consult Miscellaneous Example 12 on page 457 for the case when $c = -1/6$ and $e = \kappa/2$.

The following result makes more transparent the effect on σ of an ω' shift.

THEOREM 3.6. $\sigma(z + \omega')\sigma(z) = \frac{1}{2}\kappa$.

PROOF. This follows easily by applying Theorem 3.5 to the expression for σ and recalling the differential equation that p satisfies. Instead, we may play a familiar elliptic game: Theorem 3.4 tells us that the ω' shift interchanges the (simple) zeros and (simple) poles of σ; the product of σ and its ω' shift is thus an elliptic function without poles and so has a constant value, the calculation of which is left as another exercise. □

4. Jacobian representation

The elliptic function σ can also be given a manifestly Jacobian cast. Return to the differential equation

$$(\sigma^\circ)^2 = \tfrac{1}{4}\kappa^2 - \sigma^2 + \sigma^4$$

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of Theorem 2.3 and compare it with the differential equation

$$(\mathrm{sn}^\circ)^2 = (1 - \mathrm{sn}^2)(1 - k^2\mathrm{sn}^2)$$

that is satisfied by the Jacobian modular sine function $\mathrm{sn} = \mathrm{sn}(-, k)$ of modulus k. Alongside the modulus $\kappa \in (0, 1)$ we set the complementary modulus $\lambda = (1 - \kappa^2)^{1/2} \in (0, 1)$ and introduce the abbreviations

$$\mu_\pm = \left(\frac{1 \pm \lambda}{2}\right)^{1/2} \in (0, 1)$$

so that

$$\mu_+\mu_- = \left(\frac{1 - \lambda^2}{4}\right)^{1/2} = \frac{1}{2}\kappa$$

and

$$\mu_+^2 + \mu_-^2 = 1.$$

THEOREM 4.1. *The function σ is given by $\sigma(z) = \mu_- \,\mathrm{sn}(\mu_+ z, k)$, where*

$$k = \frac{\mu_-}{\mu_+} = \left(\frac{1 - \lambda}{1 + \lambda}\right)^{1/2}.$$

PROOF. With $s(z) := \mu_- \,\mathrm{sn}(\mu_+ z, k)$, we have

$$s^\circ(z) = \mu_-\mu_+ \,\mathrm{sn}^\circ(\mu_+ z, k) = \tfrac{1}{2}\kappa \,\mathrm{sn}^\circ(\mu_+ z, k)$$

from which the differential equation satisfied by $\mathrm{sn}(-, k)$ yields

$$s^\circ(z)^2 = \frac{1}{4}\kappa^2 \left(1 - \frac{s(z)^2}{\mu_-^2}\right)\left(1 - \left(\frac{\mu_-}{\mu_+}\right)^2 \frac{s(z)^2}{\mu_-^2}\right)$$

and mild simplification reveals that s satisfies the differential equation

$$(s^\circ)^2 = \tfrac{1}{4}\kappa^2 - s^2 + s^4;$$

of course, $s(0) = 0$ and $s(t) > 0$ when $t > 0$ is small. Thus, s is none other than σ. □

Note that Theorem 3.2 embodies the property $\mathrm{sn}(u + 2K, k) = -\,\mathrm{sn}(u, k)$, while Theorem 3.6 reflects the property $\mathrm{sn}(u + iK', k) = k/\mathrm{sn}(u, k)$. As a curiosity, note further that the real Jacobi transformation

$$k \,\mathrm{sn}(u, k) = \mathrm{sn}(ku, 1/k)$$

assumes here the symmetrical form

$$\mu_- \,\mathrm{sn}\left(\mu_+ \, z, \, \frac{\mu_-}{\mu_+}\right) = \mu_+ \,\mathrm{sn}\left(\mu_- \, z, \, \frac{\mu_+}{\mu_-}\right).$$

Along with the modular sine sn, we have the modular cosine cn and the delta amplitude dn, all to the same modulus k. Accordingly, the elliptic function σ belongs to a triple (σ, γ, δ) of 'Jacobian' elliptic functions defined by

$$\sigma(z) = \mu_- \,\mathrm{sn}(\mu_+ z, k), \quad \gamma(z) = \mu_- \,\mathrm{cn}(\mu_+ z, k), \quad \delta(z) = \mu_+ \,\mathrm{dn}(\mu_+ z, k).$$

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Here, the choice of μ_+ as the multiplier in δ leads to more elegant relationships, as follows.

THEOREM 4.2. *The functions σ, γ and δ satisfy the algebraic equations*

$$\gamma^2 + \sigma^2 = \mu_-^2 \quad \text{and} \quad \delta^2 + \sigma^2 = \mu_+^2.$$

PROOF. These equations follow directly from those for the Jacobian functions sn, cn and dn: the first from $\mathrm{cn}^2 + \mathrm{sn}^2 = 1$ and the second from $\mathrm{dn}^2 + k^2\mathrm{sn}^2 = 1$. □

By subtraction, we also have $\delta^2 - \gamma^2 = \lambda$.

THEOREM 4.3. *The functions σ, γ and δ satisfy the differential equations*

$$\sigma^\circ = \gamma\,\delta, \quad \gamma^\circ = -\sigma\,\delta, \quad \delta^\circ = -\sigma\,\gamma.$$

PROOF. These follow at once from the familiar Jacobian identities

$$\mathrm{sn}^\circ = \mathrm{cn}\cdot\mathrm{dn}, \quad \mathrm{cn}^\circ = -\,\mathrm{sn}\cdot\mathrm{dn}, \quad \mathrm{dn}^\circ = -\,k^2\,\mathrm{sn}\cdot\mathrm{cn}.$$ □

5. Elliptic periods

Here, we derive explicit expressions for the fundamental periods 2ω and $2\omega'$ of the elliptic function σ; equivalently, of its coperiodic Weierstrass function p. We offer two such expressions: one in terms of the hypergeometric function $F(\frac{1}{2}, \frac{1}{2}; 1; -)$ that is appropriate to the classical theory of elliptic functions and one in terms of the signature four hypergeometric function $F(\frac{1}{4}, \frac{3}{4}; 1; -)$. Before proceeding, it is helpful to dress the notation with the modulus on which our construction is based: thus, $2\omega_\kappa$ and $2\omega'_\kappa$ will be the fundamental periods of σ_κ and p_κ.

Expressions in terms of the classical hypergeometric function $F(\frac{1}{2}, \frac{1}{2}; 1; -)$ are provided at once by Theorem 4.1. Here, recall that $\lambda = (1-\kappa^2)^{1/2}$ is the modulus complementary to κ and that $\mu_+ = (\frac{1}{2}(1+\lambda))^{1/2}$.

THEOREM 5.1. *The fundamental half-periods ω_κ and ω'_κ of σ_κ are given by*

$$\mu_+\,\omega_\kappa = \pi\,F\Big(\frac{1}{2}, \frac{1}{2}; 1; \frac{1-\lambda}{1+\lambda}\Big) \quad \textit{and} \quad \mu_+\,\omega'_\kappa = \frac{1}{2}\pi\,i\,F\Big(\frac{1}{2}, \frac{1}{2}; 1; \frac{2\lambda}{1+\lambda}\Big).$$

PROOF. Recall (say, from [9, Ch. XXII]) that $\mathrm{sn}(-, k)$ has fundamental periods $4K$ and $2iK'$, where

$$K = \tfrac{1}{2}\pi\,F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) \quad \text{and} \quad K' = \tfrac{1}{2}\pi\,F(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-k^2).$$

It follows immediately from Theorem 4.1 that σ has $4K/\mu_+$ as real period and $2iK'/\mu_+$ as imaginary period. Finally, recall from Theorem 4.1 the expression for k in terms of λ. □

An $F(\frac{1}{4}, \frac{3}{4}; 1; -)$ expression for the real period $2\omega_\kappa$ is already in hand: reference to Theorem 2.2 procures the expression $\omega_\kappa = \pi\,F(\frac{1}{4}, \frac{3}{4}; 1; \kappa^2)$ for the real half-period. Access to the imaginary half-period ω'_κ will be facilitated by means of an auxiliary Weierstrass function.

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Explicitly, alongside the Weierstrass function

$$p_\kappa = \wp(-;\omega_\kappa,\omega'_\kappa) = \wp(-;g_2,g_3)$$

that is coperiodic with σ_κ, we shall consider the Weierstrass function

$$q_\kappa = \wp(-;\omega_\kappa/2,\omega'_\kappa) = \wp(-;h_2,h_3)$$

that comes by halving the real (half-)period.

THEOREM 5.2. *The invariants h_2 and h_3 of q_κ are given by*

$$h_2 = \tfrac{4}{3} - \kappa^2 \quad \text{and} \quad h_3 = \tfrac{8}{27} - \tfrac{1}{3}\kappa^2.$$

PROOF. Reference to [4, Section 9.8] provides us with expressions for the invariants of q_κ in terms of those for p_κ. Quite generally, such period dimidiation has the following effect:

$$h_2 = 60p(\omega)^2 - 4g_2, \quad h_3 = 56p(\omega)^3 + 8g_3.$$

With $p_\kappa(\omega) = 1/6$ and with the invariants of p_κ provided by Theorem 3.1, we calculate the invariants of q_κ to be as advertised. □

The Weierstrass functions q_κ and p_λ are, therefore, related as follows.

THEOREM 5.3. $q_\kappa(z) = -2p_\lambda(i\sqrt{2}\,z)$.

PROOF. Quite generally, a Weierstrass function $\wp(-;G_2,G_3)$ satisfies the homogeneity relation: $\wp(z;\mu^4G_2,\mu^6G_3) = \mu^2\wp(\mu z;G_2,G_3)$. Here, take $G_2 = \frac{1}{12}(1+3\lambda^2)$ and $G_3 = \frac{1}{216}(1-9\lambda^2)$; with $\mu = i\sqrt{2}$,

$$\mu^4G_2 = \tfrac{1}{3}(1+3\lambda^2) = \tfrac{4}{3} - \kappa^2 \quad \text{and} \quad \mu^6G_3 = \tfrac{1}{27}(9\lambda^2 - 1) = \tfrac{8}{27} - \tfrac{1}{3}\kappa^2.$$

Finally, invoke Theorem 3.1 for the modulus κ and for the complementary modulus λ. □

We may now make the imaginary period of σ_κ explicit in $F(\frac{1}{4},\frac{3}{4};1;-)$ terms; for reference, we record also the real period, as previously addressed.

THEOREM 5.4. *The fundamental half-periods ω_κ and ω'_κ of σ_κ are given by*

$$\omega_\kappa = \pi F(\tfrac{1}{4},\tfrac{3}{4};1;\kappa^2) \quad \text{and} \quad \omega'_\kappa = i\pi F(\tfrac{1}{4},\tfrac{3}{4};1-\kappa^2)/\sqrt{2}.$$

PROOF. Compare expressions for the real half-period of q_κ: on the one hand, it is $\omega_\kappa/2$ by definition; on the other hand, it is $-i\omega'_\lambda/\sqrt{2}$ on account of Theorem 5.3. Switching moduli, we deduce that $\omega'_\kappa = i\omega_\lambda/\sqrt{2}$; recalling our identification of the real half-period, we conclude that

$$\omega'_\kappa = i\pi F(\tfrac{1}{4},\tfrac{3}{4};1;\lambda^2)/\sqrt{2} = i\pi F(\tfrac{1}{4},\tfrac{3}{4};1;1-\kappa^2)/\sqrt{2}.$$ □

These concrete expressions for the fundamental periods of σ_κ at once furnish transformation laws relating the signature four hypergeometric function $F(\frac{1}{4},\frac{3}{4};1;-)$ to the 'classical' hypergeometric function $F(\frac{1}{2},\frac{1}{2};1;-)$.

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THEOREM 5.5. *If* $0 < \lambda < 1$, *then*

$$F\Big(\frac{1}{4},\frac{3}{4};1;1-\lambda^2\Big)=\sqrt{\frac{2}{1+\lambda}}F\Big(\frac{1}{2},\frac{1}{2};1;\frac{1-\lambda}{1+\lambda}\Big)\quad\text{and}\quad F\Big(\frac{1}{4},\frac{3}{4};1;\lambda^2\Big)=\sqrt{\frac{1}{1+\lambda}}F\Big(\frac{1}{2},\frac{1}{2};1;\frac{2\lambda}{1+\lambda}\Big).$$

PROOF. This follows from direct comparison of the expressions for ω_κ and ω'_κ in Theorem 5.1 and Theorem 5.4. □

These identities also appear in [2]: the first appears as Theorem 9.2 and is derived by manipulation of an entry in the second notebook of Ramanujan; the second appears as Theorem 9.1 and is precisely an entry in the notebook. In our approach, these identities fall directly from the periods of the elliptic function σ_κ and thereby cement the place of this function in the signature four elliptic theory.

As in [2], these transformation laws engender a relationship between the signature four base

$$q_4(x) = \exp\Big[-\sqrt{2}\,\pi\frac{F(\frac{1}{4},\frac{3}{4};1;1-x)}{F(\frac{1}{4},\frac{3}{4};1;x)}\Big]$$

and the classical base

$$q(y) = \exp\Big[-\pi\frac{F(\frac{1}{2},\frac{1}{2};1;1-y)}{F(\frac{1}{2},\frac{1}{2};1;y)}\Big];$$

in fact, with $x = \lambda^2$ and $y = 2\lambda/(1+\lambda)$, we see from Theorem 5.5 that

$$q_4(\lambda^2) = q\Big(\frac{2\lambda}{1+\lambda}\Big)^2.$$

Again as in [2], this relationship between bases supports a transfer principle whereby classical elliptic results yield counterparts in signature four. Section 9 of [2] contains several such instances of passage from classical results to signature four counterparts. In our next section, we demonstrate how some such signature four identities may be derived directly from the elliptic function σ_κ.

We close this section by pointing out that the identities in Theorem 5.5 may, of course, be derived otherwise; as an illustration, the real quarter-period

$$L = \tfrac{1}{2}\pi F(\tfrac{1}{4},\tfrac{3}{4};1;\kappa^2)$$

may be cast in $F(\frac{1}{2},\frac{1}{2};1;-)$ terms by integration, as follows. Let $\theta = \arcsin\kappa$ be the modular angle, so that $\cos\theta = \lambda$ and $\sigma(L) = \sin\frac{1}{2}\psi(L) = \sin\frac{1}{2}\theta$. From

$$\sin^2\tfrac{1}{2}\theta = \tfrac{1}{2}(1-\cos\theta) = \tfrac{1}{2}(1-\lambda),$$

it follows that $\sin\frac{1}{2}\theta = \mu_-$ and, similarly, $\cos\frac{1}{2}\theta = \mu_+$ in the notation set up ahead of Theorem 4.1. From the differential equation in Theorem 2.3, we now deduce that

$$L = \int_0^{\mu_-}\frac{d\sigma}{\sqrt{\sigma^4-\sigma^2+\frac{1}{4}\kappa^2}},$$

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where $\kappa^2 = 4\sin^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta = 4\mu_-^2\mu_+^2$ and, therefore,

$$L = \int_0^{\mu_-}\frac{d\sigma}{\sqrt{(\mu_-^2-\sigma^2)(\mu_+^2-\sigma^2)}}.$$

Put $\sigma = \mu_- \sin t$ for $0 \leqslant t \leqslant \frac{1}{2}\pi$ to see that

$$L = \int_0^{\frac{1}{2}\pi}\frac{dt}{\sqrt{\mu_+^2 - \mu_-^2\sin^2 t}}$$

and conclude that

$$\mu_+ L = \int_0^{\frac{1}{2}\pi}\frac{dt}{\sqrt{1-k^2\sin^2 t}} = \frac{1}{2}\pi\, F\Big(\frac{1}{2},\frac{1}{2};1;k^2\Big)$$

by the familiar evaluation of a complete elliptic integral of the first kind.

6. Eisenstein series

We open this section by recalling the familiar connection between Weierstrass functions and Eisenstein series. For additional details, see [6, Ch. VII] or [1, Ch. 7].

Let $p = \wp(-;\omega,\omega') = \wp(-;g_2,g_3)$ be a quite arbitrary Weierstrass function, with 2ω and $2\omega'$ as fundamental periods and with g_2 and g_3 as invariants; take the period ratio $\tau := \omega'/\omega$ to have positive imaginary part as usual. The quadrinvariant g_2 is given by

$$g_2 = \frac{60}{(2\omega)^4}\sum{}'\frac{1}{(m+n\tau)^4}$$

and the Eisenstein series $E_4(\tau)$ by

$$E_4(\tau) = \frac{45}{\pi^4}\sum{}'\frac{1}{(m+n\tau)^4},$$

where, in each case, the prime $'$ indicates that summation takes place over all pairs of integers (m,n) other than $(0,0)$; thus,

$$E_4(\tau) = \frac{3}{4}\Big(\frac{2\omega}{\pi}\Big)^4 g_2.$$

Similarly, the cubinvariant g_3 is given by

$$g_3 = \frac{140}{(2\omega)^6}\sum{}'\frac{1}{(m+n\tau)^6}$$

and the Eisenstein series $E_6(\tau)$ by

$$E_6(\tau) = \frac{945}{2\cdot\pi^6}\sum{}'\frac{1}{(m+n\tau)^6}$$

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so that

$$E_6(\tau) = \frac{27}{8}\left(\frac{2\omega}{\pi}\right)^6 g_3.$$

It will now be convenient to employ the notational abbreviation $F_4 = F(\frac{1}{4}, \frac{3}{4}; 1; -)$: thus, $F_4(x) := F(\frac{1}{4}, \frac{3}{4}; 1; x)$. In [2], this is written $z(4; x)$ or just $z(4)$ when x is understood.

THEOREM 6.1. *Let* $0 < x < 1$. *If*

$$\tau = \frac{i}{\sqrt{2}}\,\frac{F_4(1-x)}{F_4(x)},$$

then $E_4(\tau) = F_4(x)^4\,(1+3x)$ *and* $E_6(\tau) = F_4(x)^6\,(1-9x)$.

PROOF. Let $x = \kappa^2$ and apply the foregoing recollections to the Weierstrass function p_κ that is coperiodic with σ_κ. From Theorem 5.4,

$$\frac{2\omega_k}{\pi} = 2F_4(\kappa^2),$$

and from Theorem 3.1, $g_2 = \frac{1}{12}(1+3\kappa^2)$, from which it follows at once that

$$E_4(\tau) = \tfrac{3}{4}\cdot(2\,F_4(\kappa^2))^4\cdot\tfrac{1}{12}(1+3\kappa^2) = F_4(\kappa^2)^4\,(1+3\kappa^2),$$

where

$$\tau = \frac{\omega'_\kappa}{\omega_\kappa} = \frac{i}{\sqrt{2}}\,\frac{F_4(1-\kappa^2)}{F_4(\kappa^2)},$$

again by Theorem 5.4. This handles E_4; E_6 is handled similarly. □

These formulas appear as Theorems 9.5 and 9.6 in [2] as applications of the signature four transfer principle. They also appear as (6.3) and (6.4) in [7], where they are approached quite differently, via theta functions and the Landen transformation.

THEOREM 6.2. *Let* $0 < x < 1$. *If*

$$\tau = i\sqrt{2}\,\frac{F_4(1-x)}{F_4(x)},$$

then $E_4(\tau) = F_4(x)^4\,(1-\frac{3}{4}x)$ *and* $E_6(\tau) = F_4(x)^6\,(1-\frac{9}{8}x)$.

PROOF. Let $x = \kappa^2$ and apply the recollections above to the Weierstrass function q_κ that was obtained from p_κ by halving the real period. For E_6,

$$E_6(\tau) = \tfrac{27}{8}\cdot F_4(\kappa^2)^6\cdot(\tfrac{8}{27}-\tfrac{1}{3}\kappa^2) = F_4(\kappa^2)^6(1-\tfrac{9}{8}\kappa^2)$$

using Theorems 5.4 and 5.2; for E_4 the calculations are similar. □

Again, these formulas are derived using the signature four transfer principle in [2], where they appear as Theorems 9.7 and 9.8. They also appear in [7], where in

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Corollary 3.4 they are derived substantially as in the present paper. Theorems 6.1 and 6.2 serve to cement still further the place of σ in signature four.

7. Shen functions

The elliptic function σ originated in a study of the elliptic function dn_2 that was introduced in [7] by Shen. Here, we set out the relationship of [7] to the present paper.

With minor notational differences, the construction in [7] is based on the same bijection $\phi : \mathbb{R} \to \mathbb{R}$, according to which

$$u = \int_0^{\phi(u)} F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt,$$

and on the same auxiliary function

$$\psi := \arcsin(\kappa \sin \phi)$$

with which we began Section 2; the elliptic function dn_2 is defined in [7] as the elliptic extension of the real function $\cos\psi$. The motivation for this construction comes from a classical approach to the Jacobian elliptic functions, in which the hypergeometric integrand $F(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; -)$ is simply replaced by $F(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; -)$; this motivation suggested the name dn_2 because use of the classical hypergeometric integrand, instead, leads directly to the Jacobian delta amplitude dn.

More explicitly, in [7], it is shown that the function $y = \cos\psi$ satisfies the differential equation

$$(y^\circ)^2 = 2(1 - y)(y^2 - \lambda^2),$$

while, of course, $y(0) = 1$. The solution to this initial value problem is a little easier to identify than was the solution in Theorem 3.1, because the right-hand side of the differential equation for y has the initial value as a zero, so that the simpler arguments on [9, page 453] apply. The result is that

$$y = 1 - \frac{\frac{1}{2}\kappa}{\frac{1}{3} + \wp(-; h_2, h_3)},$$

where the indicated Weierstrass function has invariants

$$h_2 = \tfrac{4}{3} - \kappa^2 \quad \text{and} \quad h_3 = \tfrac{8}{27} - \tfrac{1}{3}\kappa^2.$$

Notice that this Weierstrass function is precisely the function q_κ that we introduced immediately before Theorem 5.2. The elliptic extension dn_2 of $\cos\psi$ is, thus, given by

$$\mathrm{dn}_2 = 1 - \frac{\frac{1}{2}\kappa}{\frac{1}{3} + q_\kappa}$$

and q_κ is plainly its coperiodic Weierstrass function. Incidentally, the proof of Theorem 6.2 that appears in [7] as Corollary 3.4 was based on the periods of dn_2. The absence of σ from [7] necessitated use of the Landen transformation to deduce the

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equivalents (6.3) and (6.4) of our Theorem 6.1. That the Landen transformation should enter is, of course, not surprising, in view of the relationship between the Weierstrass functions p_κ and q_κ.

The origin of dn_2 as a signature four version of the Jacobian function dn invites consideration of the functions $\cos\phi$ and $\sin\phi$, as these yield cn and sn in the classical theory. It is shown in [7] that, in signature four, $\cos\phi$ does extend as an elliptic function cn_2 on the plane; in contrast, $\sin\phi$ does not even extend to a meromorphic function on the plane, as we show shortly.

In one significant respect, the elliptic function dn_2 is imperfect as a signature four counterpart to the classical Jacobian function dn: the classical function dn has two simple poles in each of its period parallelograms and is, therefore, of generally Jacobian type; but the signature four function dn_2 is of Weierstrassian type, as it has a double pole in each period parallelogram. A similar comment applies to the elliptic function cn_2; and sn_2 simply fails to exist as an elliptic function. These are the circumstances that prompted our search for a set of truly Jacobian functions in signature four; particularly, a signature four counterpart to the odd function sn.

The precise relationships between our functions σ, γ and δ and the functions in [7] are readily determined. First, the trigonometric duplication identity

$$\cos\psi = 1 - 2\sin^2 \tfrac{1}{2}\psi$$

on $\mathbb{R}$ extends at once by analytic continuation to the identifications

$$\mathrm{dn}_2 = 1 - 2\sigma^2 = 2\gamma^2 + \lambda = 2\delta^2 - \lambda$$

with the aid of Theorem 4.2. We extract from the proof of Theorem 2.3 the identification

$$\tfrac{1}{2}\kappa\,\mathrm{cn}_2 = \sigma^\circ$$

to which Theorem 4.3 at once contributes the identification

$$\tfrac{1}{2}\kappa\,\mathrm{cn}_2 = \gamma\,\delta.$$

As noted above, the case of sn_2 is different: from

$$\kappa^2\sin^2\phi = \sin^2\psi = 4\sin^2\tfrac{1}{2}\psi\,\cos^2\tfrac{1}{2}\psi,$$

it follows that

$$\tfrac{1}{4}\kappa^2\sin^2\phi = \sigma^2(1-\sigma^2)$$

on $\mathbb{R}$. The elliptic function $\sigma^2(1-\sigma^2)$ has simple zeros where $\sigma = \pm 1$ because $(\sigma^\circ)^2 = \frac{1}{4}\kappa^2 \neq 0$ there; this prevents $\sigma^2(1-\sigma^2)$ from having meromorphic square roots and so prevents $\sin\phi$ from having meromorphic extensions.

The preceding formulas for dn_2 and cn_2 have versions in terms of the classical Jacobian functions to modulus $k = \sqrt{(1-\lambda)/(1+\lambda)}$: explicitly, Theorem 4.1 at once gives

$$\mathrm{dn}_2(z) = 1 - (1-\lambda)\,\mathrm{sn}^2(\mu_+ z, k)$$

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and by differentiation gives

$$\mathrm{cn}_2(z) = \mathrm{cn}(\mu_+ z, k)\,\mathrm{dn}(\mu_+ z, k).$$

Finally, in [8], Shen presented elliptic or hyperelliptic functions of interest in each of the three standard alternative signatures. In each case, the proposed function satisfies a differential equation of the form

$$(y^\circ)^2 = T_n(y) - (1 - 2\kappa^2),$$

where T_n is the degree n Chebyshev polynomial of the first kind. The case appropriate to signature four has $n = 4$: the corresponding function y_4 satisfies

$$(y^\circ)^2 = 8y^4 - 8y^2 + 2\kappa^2$$

and is required to have as its initial value one of the four zeros of the quartic on the right; in fact, it is required that $y_4(0) = \cos \frac{1}{2}\theta$, where θ is the modular angle, as above. This function y_4 is plainly related to σ: indeed, $y_4(z) = \sigma(2\sqrt{2}z + c)$ for a suitable choice of the shift c.

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