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Periodic and solitary waves in a Korteweg–de Vries equation with delay

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For a perturbed generalized Korteweg–de Vries equation with a distributed delay, we prove the existence of both periodic and solitary waves by using the geometric singular perturbation theory and the Melnikov method. We further obtain monotonicity and boundedness of the speed of the periodic wave with respect to the total energy of the unperturbed system. Finally, we establish a relation between the wave speed and the wavelength.

Keywords: Korteweg–de Vries equation with delay; periodic wave; solitary wave; geometric singular perturbation theory; Abelian integral

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1. Introduction

The Korteweg–de Vries equation (KdV for short)

$$u_t + \alpha u u_x + u_{xxx} = 0 \tag{1.1}$$

was first derived by Korteweg and de Vries [22] in 1895 for modelling propagation of small amplitude long water waves in a uniform channel, where α is a constant coefficient, and u is a function in both the spatial variable x and the time t describing the waves. The KdV equation and its generalizations have modelled a large number of different physical phenomena [7].

As we know, a large class of generalizations of the KdV equation can be expressed as the generic functional form [30]

$$u_t + (F(u))_x + u_{xxx} = 0, \tag{1.2}$$

where $F(u)$ is a nonlinear function satisfying $F(0) = 0$. If $F \sim u^2$, equation (1.2) corresponds to the classic KdV equation (1.1) [3, 14, 20, 22]. If $F(u) \sim u^p$ with $p \in \mathbb{N} \setminus \{1\}$, equation (1.2) is called the generalized KdV equation [18, 26–28]. Especially, when $p = 3$, equation (1.2) corresponds to the modified KdV equation [17, 18], which possesses many polynomial conservation laws. While when $p > 3$, there remain only three polynomial conservation laws, see e.g. [28]. Of course, KdV equation has also many other generalizations, see e.g. Goubet [15], Chu *et al.* [6], Isaza and León [20] and the references therein.

On perturbation of the KdV equation, there are also many results with respect to the travelling waves. Derks and Gils [8] in 1993 studied uniqueness of the travelling wave solutions in the following equation

$$u_t + uu_x + u_{xxx} + \epsilon(u_{xx} + u_{xxxx}) = 0, \tag{1.3}$$

where $\epsilon > 0$ is a small parameter. One year later, Ogawa [29] investigated the existence of the solitary waves and periodic waves to system (1.3) by using geometric singular perturbation theory (GSPT for short). In the same paper, Ogawa also studied monotonicity properties for the speed of periodic waves with the total energy of the Hamiltonian via Abelian integrals, and provided a relationship between the wave speed and the wavelength. In 2014, Yan *et al.* [34], applying GSPT, researched the existence of the solitary waves and the periodic waves in a perturbed generalized KdV equation

$$u_t + u^n u_x + \beta u_{xxx} + \epsilon(u_{xx} + u_{xxxx}) = 0,$$

with $n > 0$ integer and $0 < \epsilon \ll 1$. Moreover, the authors in [34] discussed the limit speed of the wave and its upper and lower bounds via the Abelian integral theory. Applying GSPT, Mansour [25] studied the existence of the travelling wave solution in a generalized dissipative perturbed KdV equation

$$u_t + \alpha u^n u_x + \beta u_{xxx} + \epsilon(a u_{xx} + b(uu_x)_x + c u_{xxxx}) = 0, \tag{1.4}$$

where $0 < \epsilon \ll 1$, and the coefficients α, β, a, b, c are constants. Zhuang *et al.* [36] verified the existence of the solitary wave solutions for a perturbed generalized KdV equation

$$u_t + \alpha u^{n+1} u_x + \beta u_{xxx} + \epsilon(u_{xx} + u_{xxxx}) = 0,$$

where $0 < \epsilon \ll 1$, and α and β are positive parameters.

Recall that the geometric singular perturbation theory was initially developed by Fenichel [11–13], for studying the existence of locally invariant manifolds, of invariant stable (unstable) manifolds and of invariant foliations, which are perturbed from a so-called normally hyperbolic critical manifold and its associated centre-stable (centre-unstable) manifolds. This theory has been greatly developed in the past decades, see e.g. the literatures [1, 5, 23, 31] and the references therein. It also has diverse applications in many disciplines for verifying the existence of travelling and periodic waves, and local and global dynamics, for instance, the vector–disease models [24, 32], Belousov–Zhabotinskii system [10], Schrödinger equation [35], Burgers–KdV equation [33], KPP equation [2], Camassa–Holm [9] equation and so on.

On application of the GSPT to the systems with delay for finding travelling waves, there are also many results, see e.g. [9, 10, 24, 32, 35]. Du et al. [9] in 2018 considered the existence of the solitary wave solution for the perturbed delayed Camassa–Holm equation

$$u_t - u_{xxt} + 2ku_x + 3(f * u)u_x + \tau u_{xx} = 2u_x u_{xx} + uu_{xxx},$$

where $0 < \tau \ll 1$, and the convolution $f * u$ is defined as either the local distributed delay

$$(f * u)(x, t) = \int_{-\infty}^{t} f(t - s)u(x, s)\,\mathrm{d}s, \tag{1.5}$$

or the nonlocal distributed delay

$$(f * u)(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} f(x - y, t - s)u(y, s)\,\mathrm{d}y\,\mathrm{d}s.$$

In the expression (1.5), the kernels

$$f(t) = \frac{1}{\tau}\,\mathrm{e}^{-\frac{t}{\tau}} \text{ and } f(t) = \frac{t}{\tau^2}\,\mathrm{e}^{-\frac{t}{\tau}}$$

are frequently adopted in the literature for delay differential equations. The first of the two kernels is called the weak generic kernel, and the second is called the strong generic kernel. Moreover, the two kernels are all nonnegative and satisfy $\int_0^{+\infty} f(t)\,\mathrm{d}t = 1$.

In this paper, we study the existence and monotonicity of the periodic waves, and also the existence of the solitary wave of the following perturbed generalized KdV equation with delay

$$U_t + a_0((f * U)U^2)_x + a_1 U_{xxx} + \tau U_{xxxx} = 0, \tag{1.6}$$

where $a_0,\ a_1 > 0$ are constant coefficients, $0 < \tau \ll 1$ is a perturbation parameter and the convolution $f * U$ is that in (1.5) with the weak generic kernel $f(t) = \frac{1}{\tau}\,\mathrm{e}^{-\frac{t}{\tau}}$. Note that $((f * U)U^2)_x$ is the convection term with delay and U_{xxxx} is the dissipation term. Moreover, when $\tau \to 0$, the convolution has the limit $f * U \to U$. Thus, when $\tau \to 0$, equation (1.6) is reduced to the famous modified KdV equation

$$U_t + 3a_0 U^2 U_x + a_1 U_{xxx} = 0. \tag{1.7}$$

That is, equation (1.6) is a small perturbation of the modified KdV equation (1.7). The perturbation $f * U$ contains the local distributed delay, which can be seen as a continuous version of the discrete delay $U(t - \tau)$ with the small delay τ. Note that the normalized weak generic kernel ensures that the steady-states are not

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affected with variations in the delay. To our knowledge, the normalized distributed delay was introduced in many different models for further depicting their dynamics, for instance, the Camassa–Holm equation [9], the chemostat-type model [19] and the diffusive Nicholson's blowflies equation [16] and so on. Hakkaev *et al.* [17] studied the modified KdV equation (1.7) with $a_0 = 2$ and $a_1 = 1$, and obtained a family of specific type of periodic travelling wave solutions. They further proved the orbital stability of these periodic waves. As a bridge to prove our main results below for equation (1.6), we achieved for equation (1.7) some other family of periodic wave solutions than those in [17] and also a solitary wave, which was not studied in [17]. Besides the travelling wave solutions, there are also some researches in other directions on equation (1.7), e.g. Hayashi and Naumkin [18] studied the so-called final state problem for equation (1.7) with $a_1 = 1/3$, which is one of the special cases of their general equations.

We note that equation (1.7) is also the unperturbed system with $n = 2$ of the generalized dissipation perturbed KdV equation (2) in [25], i.e. (1.4) above, which describes the waves in many applied disciplines, such as the plasma waves, the thermoconvective liquid layer and nonlinear electromagnetic waves and so on. Mansour [25] constructed travelling waves of the perturbed equation (1.4), which include the solitary waves and oscillatory kink or shock waves. Here our equation (1.6) is a perturbation of equation (1.7), where the convection term $((f * U)U^2)_x$, as $\tau \to 0$, has the limit $u^2 u_x$. We prove not only the existence of solitary waves, but also the existence of periodic waves for equation (1.6). Meanwhile, we establish also a relationship between the wave speed and wavelength of periodic waves.

The remaining part of this paper is organized as follows. Section 2 recalls the Fenichel first invariant manifold theorem, which will be used in the proof of our main results, theorems 2.1 and 2.2, which will be stated also in this section. Section 3 is a proof of theorem 2.1, which is separated in three subsections. The first one is on persistence of the periodic orbits of the unperturbed system (1.7) under the small perturbation via the Melnikov method. These periodic orbits provide periodic waves of equation (1.6). The second one is on properties of the limit wave speed of the periodic wave. The third one determines the existence of the solitary waves for $0 < \tau \ll 1$. Section 4 is a proof of theorem 2.2, where we also provide more information on monotonicity and boundedness of the periods of the period waves. The last section is a conclusion.

2. Preliminaries and the main results

In this section, we first recall some fundamental known results on the Fenichel invariant manifold theory. Then we present our main results.

2.1. Preliminaries

In this subsection, we recall the Fenichel invariant manifold theorem for slow–fast systems, or singularly perturbed systems. Consider the slow–fast systems

$$\begin{aligned} x'(t) &= f(x, y, \varepsilon), \\ y'(t) &= \varepsilon g(x, y, \varepsilon), \end{aligned} \qquad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \tag{2.1}$$

where $\varepsilon > 0$ is a small real parameter, and f and g are C^∞ functions in their variables. Here, the prime is the derivative in the time t. For $\varepsilon \neq 0$, after the time rescaling $\tau = \varepsilon t$, system (2.1) can be written as

$$\begin{aligned} \varepsilon \dot{x}(\tau) &= f(x, y, \varepsilon), \\ \dot{y}(\tau) &= g(x, y, \varepsilon), \end{aligned} \qquad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m. \tag{2.2}$$

Systems (2.1) and (2.2) are called, respectively, the fast and slow systems, and their limits when $\varepsilon \to 0$

$$\begin{aligned} x'(t) &= f(x, y, 0), \\ y'(t) &= 0, \end{aligned} \qquad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$$

and

$$\begin{aligned} 0 &= f(x, y, 0), \\ \dot{y}(\tau) &= g(x, y, 0), \end{aligned} \qquad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

are called the layer and reduced systems, respectively.

The set $\Omega := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m |\ f(x, y, 0) = 0\}$ is called a critical set, and it plays a key role in the study of dynamics of system (2.1). Choose a connected branch Ω_0 of Ω. If the Jacobian matrix of $f(x, y, 0)$ with respect to x restricted to Ω_0 has all eigenvalues with nonvanishing real parts, we call Ω_0 normally hyperbolic.

A subset M of $\mathbb{R}^n \times \mathbb{R}^m$ is locally invariant under the flow of system (2.1) if it has a neighbourhood U, as long as an orbit of the system leaves M, it will leave U.

We now state the Fenichel first invariant manifold theorem [**9**, **13**, **21**], which will be one of our main tools in the next proofs of theorems 2.1 and 2.2.

Fenichel first invariant manifold theorem. Assume that

- M_0 is a compact, normally hyperbolic critical manifold of system (2.1), and
- it has a coordinate expression $M_0 := \{x = \psi(y) |\ \ y \in K_0\}$ with $K_0 \subset \mathbb{R}^m$ compact and $\psi(y) \in C^\infty(K_0)$.

Then for $\varepsilon > 0$ sufficiently small, system (2.1) has a locally invariant C^r manifold M_ε, with any prescribed $r \in \mathbb{N}$, which has the expression $x = \psi_\varepsilon(y)$, $x \in K_0$, such that

$$\lim_{\varepsilon \to 0} \psi_\varepsilon(y) = \psi(y), \quad y \in K_0.$$

Restricted to M_ε, system (2.1) is reduced to

$$\dot{y}(\tau) = g(\psi_\varepsilon(y), y, \varepsilon), \tag{2.3}$$

which, as $\varepsilon \to 0$, has the limit $\dot{y}(\tau) = g(\psi(y), y, 0)$.

The Fenichel first invariant manifold theorem indicates that in the normally hyperbolic case, system (2.3) is a regular perturbation of the reduced system $\dot{y}(\tau) = g(\psi(y), y, 0)$. Consequently, all hyperbolic and structurally stable objectives of the reduced system will be preserved by system (2.3) for $0 < \varepsilon \ll 1$.

2.2. Statement of the main results

In this subsection, we present our main results. Since we want to find the travelling wave solution of equation (1.6), we set $U(x,t)=\phi(\xi)=\phi(x-ct)$. Then equation (1.6) becomes the next ordinary differential equation

$$-c\phi'+a_0(\eta_1\phi^2)'+a_1\phi'''+\tau\phi''''=0, \tag{2.4}$$

where $'=\frac{d}{d\xi}$, and

$$\eta_1=\int_0^{+\infty}\frac{1}{\tau}\,\mathrm{e}^{-\frac{s}{\tau}}\phi(\xi+cs)\,\mathrm{d}s.$$

Integrating equation (2.4) with respect to ξ and omitting the integral constant, we obtain the following equation

$$-c\phi+a_0\eta_1\phi^2+a_1\phi''+\tau\phi'''=0. \tag{2.5}$$

After the rescalings $\phi(\xi)=\sqrt{c}u(z)$ and $\xi=\frac{z}{\sqrt{c}}$, equation (2.5) can be further written in

$$-u+a_0\eta u^2+a_1\ddot{u}+\sqrt{c}\tau\dddot{u}=0. \tag{2.6}$$

Here the dot denotes the derivative in z, and

$$\eta=\int_0^{+\infty}\frac{1}{\tau}\,\mathrm{e}^{-\frac{s}{\tau}}u(z+c^{\frac{3}{2}}s)\,\mathrm{d}s. \tag{2.7}$$

Direct computations show that $\eta\to u$ as $\tau\to 0$. Thus, when $\tau\to 0$, equation (2.6) has the limit

$$-u+a_0u^3+a_1\ddot{u}=0, \tag{2.8}$$

which is called unperturbed equation of equation (2.6). Equation (2.8) can be written in an equivalent way as the following Hamiltonian system

$$\dot{u}=v,\quad \dot{v}=wu-bu^3,$$

with the Hamiltonian

$$H(u,v)=\frac{v^2}{2}-\frac{wu^2}{2}+\frac{bu^4}{4},$$

where $w=a_1^{-1}>0$ and $b=a_0a_1^{-1}>0$. The level curve $H(u,v)=h$ is

- a homoclinic orbit of the Hamiltonian system at the origin in the half plane $u>0$ when $h=0$,
- a periodic orbit for each $h\in(-1/(4a_0a_1),0)$, which is in the interior region enclosed by the homoclinic orbit,
- the centre when $h=-1/(4a_0a_1)$.

We remark that for $h<-1/(4a_0a_1)$ or $h>0$, the level curve $H(u,v)=h$ is out of our interest.

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Now we can state our main results. The first one is on the existence of periodic waves and of a solitary wave, and their limiting properties.

Theorem 2.1. *For equation (1.6), there exists a sufficiently small $\tau^* > 0$, such that for $\tau \in (0, \tau^*)$, the following results hold.*

(a) *For $h \in (-1/(4a_0a_1), 0)$, equation (1.6) has a travelling wave solution*

$$U = \sqrt{c}u(\tau, h, c, z),$$

with the wave speed $c = c(\tau, h) > 0$, where $u(\tau, h, c, z)$ is a solution of equation (2.6) satisfying

$$\frac{\partial}{\partial z}u(\tau, h, c, 0) = 0, \quad \frac{\partial^2}{\partial z^2}u(\tau, h, c, 0) > 0 \quad \textit{for} \quad h < 0.$$

(b) *The limit $\lim_{\tau\to 0} u(\tau, h, c, z) = u_0(h, z)$ holds uniformly in z, where $u_0(h, z)$ is a solution of equation (2.8) on the level curve $H = h \in (-1/(4a_0a_1), 0]$.*

(c) *For $h \in (-1/(4a_0a_1), 0)$, the function $u(\tau, h, c, z) \geqslant 0$ represents a periodic wave of equation (1.6). Whereas, for $h = 0$, the function $u(\tau, h, c, z) \geqslant 0$ indicates a solitary wave of equation (1.6).*

(d) *The wave speed $c = c(\tau, h)$ is smooth in both τ and h, and satisfies $\lim_{\tau\to 0} c(\tau, h) = c_0(h)$, with $c_0(h) \in [7/(4a_1), 2/a_1)$ a decreasing smooth function on $(-1/(4a_0a_1), 0]$ and having the limits*

$$\lim_{h\to -\frac{1}{4a_0a_1}} c_0(h) = \frac{2}{a_1}, \lim_{h\to 0} c_0(h) = \frac{7}{4a_1}.$$

The second one provides a relation between the wave speed and the wavelength of the periodic wave of equation (1.6).

Theorem 2.2. *For $0 < \tau \ll 1$, the wave speed c_0 and the wavelength λ_0 of the periodic wave solution to equation (1.6) satisfy*

$$c_0 = \tilde{c}_0(\lambda_0), \quad \tilde{c}_0'(\lambda_0) < 0, \quad \lambda_0 \in (a_1\pi, +\infty).$$

In the next section, we will prove theorem 2.1. The proof of theorem 2.2 will be given in § 4. In both sections, we will provide more information on the properties of the periods and the wave speeds of the periodic waves, respectively.

3. The proof of theorem 2.1

This section is a proof of theorem 2.1. We separate it in three subsections, which are on the existence of periodic waves, properties of wave speeds and the solitary wave.

3.1. Existence of periodic waves

Write the third-order differential equation (2.6) in an equivalent way as a system of the first-order ordinary differential equations

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= p, \\ \sqrt{c}\tau\dot{p} &= u - a_0\eta u^2 - a_1 p. \end{aligned} \tag{3.1}$$

Differentiating expression (2.7) with respect to z yields $c^{\frac{3}{2}}\tau\dot{\eta} = \eta - u$. It together with (3.1) forms the following system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= p, \\ \sqrt{c}\tau\dot{p} &= u - a_0\eta u^2 - a_1 p, \\ c^{\frac{3}{2}}\tau\dot{\eta} &= \eta - u. \end{aligned} \tag{3.2}$$

For $\tau > 0$ sufficiently small, system (3.2) is a singularly perturbed system with two fast variables p and η, and two slow variables u and v.

According to the singular perturbation theory, we consider the critical set

$$M_0 = \left\{(u, v, p, \eta)|\ p = \ a_1^{-1}u - a_0 a_1^{-1}\eta u^2,\ \eta = u\right\},$$

which is a two-dimensional smooth manifold. Restricted to this critical manifold, the reduced system of system (3.2) is

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= wu - bu^3, \end{aligned} \tag{3.3}$$

where $w = a_1^{-1} > 0, b = a_0 a_1^{-1} > 0$. Obviously, the reduced system (3.3) has three equilibria: $(0, 0)$ a saddle, and $(\pm\sqrt{w/b}, 0)$ both centres. Note that system (3.3) is a Hamiltonian one, with the Hamiltonian

$$H(u, v) = \frac{v^2}{2} - \frac{wu^2}{2} + \frac{bu^4}{4},$$

which has two symmetric homoclinic orbits to the origin contained in the level set $H(u, v) = 0$, and a family of periodic orbits of the reduced system contained in the level set $H(u, v) = h$ for $h \in (-w^2/(4b), 0)$.

Now we are back to the full system (3.2). When $\tau \neq 0$, the slow system (3.2) can be written in an equivalent way, via the time rescaling $z = \tau s$, as the next fast

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system

$$\begin{aligned} \frac{du}{ds} &= \tau v, \\ \frac{dv}{ds} &= \tau p, \\ \sqrt{c}\frac{dp}{ds} &= u - a_0\eta u^2 - a_1 p, \\ c^{\frac{3}{2}}\frac{d\eta}{ds} &= \eta - u. \end{aligned} \tag{3.4}$$

Some easy calculations show that the linearized matrix of (3.4) restricted to M_0 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{c}}(1-2a_0\eta u) & 0 & -\frac{a_1}{\sqrt{c}} & -\frac{a_0}{\sqrt{c}}u^2 \\ -c^{-\dfrac{3}{2}} & 0 & 0 & c^{-\dfrac{3}{2}} \end{pmatrix},$$

which has four eigenvalues: $0,\ 0, -a_1c^{-\frac{1}{2}},\ c^{-\frac{3}{2}}$. This means that the critical manifold M_0 is normally hyperbolic. By the Fenichel first invariant manifold theorem, for $0 < \tau \ll 1$, system (3.2) has an invariant slow manifold M_τ, which can be expressed as

$$M_\tau = \left\{(u,v,p,\eta)\in\mathbb{R}^4 |\ p = a_1^{-1}u - a_0a_1^{-1}\eta u^2 + g(u,v,\tau),\ \eta = u + h(u,v,\tau)\right\},$$

where g and h depend smoothly on their variables, and $g(u,v,0) = h(u,v,0) = 0$. To compute the asymptotic expression of the functions g and h, we expand them in the parameter τ as

$$\begin{aligned} g(u,v,\tau) &= \tau g_1(u,v) + \tau^2 g_2(u,v) + \cdots, \\ h(u,v,\tau) &= \tau h_1(u,v) + \tau^2 h_2(u,v) + \cdots. \end{aligned} \tag{3.5}$$

Plugging the expression of M_τ with (3.5) into the slow system (3.2), we obtain

$$\sqrt{c}\tau\left(\frac{v}{a_1} - \frac{a_0}{a_1}\dot{\eta}u^2 - \frac{2a_0}{a_1}\eta uv + O(\tau)\right) = -a_1g_1\tau - a_1g_1\tau^2 + o(\tau^2),$$

$$c^{\frac{3}{2}}\tau\left(v + \tau(\frac{\partial h_1}{\partial u}\dot{u} + \frac{\partial h_1}{\partial v}\dot{v}) + o(\tau)\right) = \tau h_1(u,v) + \tau^2 h_2(u,v) + o(\tau^2).$$

Equating the coefficients of τ^1 in the both sides of the above equations yields

$$\begin{aligned} g_1(u,v) &= -\frac{\sqrt{c}}{a_1}\left(\frac{v}{a_1} - \frac{3a_0}{a_1}u^2v\right), \\ h_1(u,v) &= c^{\frac{3}{2}}v. \end{aligned}$$

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Then the slow manifold M_τ has a more precise expression

$$\begin{aligned} p &= \frac{u}{a_1} - \frac{a_0}{a_1}u^3 - \frac{\sqrt{c}}{a_1}\left(a_0 c u^2 v + \frac{v}{a_1} - \frac{3a_0}{a_1}u^2 v\right)\tau + o(\tau), \\ \eta &= u + c^{\frac{3}{2}}\tau v + o(\tau), \end{aligned} \qquad (u,v) \in \mathbb{R}^2.$$

Restricted to the invariant slow manifold M_τ, the slow system (3.2) is reduced to the following two-dimensional system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= wu - bu^3 - \frac{\sqrt{c}}{a_1}\left(a_0 c u^2 v + \frac{v}{a_1} - \frac{3a_0}{a_1}u^2 v\right)\tau + o(\tau), \end{aligned} \tag{3.6}$$

with $w = a_1^{-1} > 0$ and $b = a_0 a_1^{-1} > 0$. By the Fenichel invariant manifold theorem, it follows that system (3.6) is a regular perturbation of system (3.3).

Now we investigate the existence of the periodic orbits of system (3.6), which are the perturbation of the periodic orbits inside the period annulus of the Hamiltonian system (3.3). We should mention that the periodic orbits of system (3.6) is also the periodic orbits of system (3.2), and so they provide the periodic waves of equation (1.6).

To prove the existence of the periodic orbits of system (3.6), we will use the Melnikov method. Fix an initial data $(\alpha, 0)$ with $0 < \alpha < \sqrt{w/b}$, and let $(u_\tau(z), v_\tau(z))$ be the solution of system (3.6) satisfying $(u_\tau, v_\tau)(0) = (\alpha, 0)$. Then there exist $z_1 > 0$ and $z_2 < 0$, so that

$$v_\tau(z) > 0 \text{ for } 0 < z < z_1, \quad v_\tau(z_1) = 0$$

and

$$v_\tau(z) < 0 \text{ for } z_2 < z < 0, \quad v_\tau(z_2) = 0.$$

Let $(\alpha, 0)$ be the point on the level curve $H(u_\tau, v_\tau) = h$ for $h \in (-w^2/(4b), 0)$. Then α is a monotonic function in h. For exhibiting dependence of the solution $(u_\tau(z), v_\tau(z))$ on all the variables and parameters that are involved, we also write

$$u_\tau(z) = u(\tau, h, c, z), \quad v_\tau(z) = v(\tau, h, c, z).$$

Direct calculation by (3.6) gives

$$\frac{\partial}{\partial z}u(\tau, h, c, 0) = 0, \quad \frac{\partial^2}{\partial z^2}u(\tau, h, c, 0) > 0.$$

According to Carr [**4**, Chapter 4], we define the function

$$\Phi(h, c, \tau) := \int_{z_2}^{z_1} \dot{H}(u_\tau, v_\tau)\,\mathrm{d}z, \tag{3.7}$$

where $\dot{H}$ is the derivative of the Hamiltonian H along the orbit of system (3.6). Some calculations show that

$$\dot{H}(u_\tau, v_\tau) = \frac{\tau\sqrt{c}}{a_1}\left(a_0 c u_\tau^2 v_\tau^2 + \frac{v_\tau^2}{a_1} - \frac{3a_0}{a_1}u_\tau^2 v_\tau^2 + O(\tau)\right). \tag{3.8}$$

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Note that $\Phi(h, c, \tau)$ is, in fact, the difference of the values of H at the two first intersecting points of the positive and negative orbits passing $(\alpha, 0)$ with the positive u-axis, i.e.

$$\Phi(h, c, \tau) = H(u_\tau(z_1), v_\tau(z_1)) - H(u_\tau(z_2), v_\tau(z_2)).$$

Hence, $\Phi(h, c, \tau) = 0$ if and only if $(u_\tau(z), v_\tau(z))$ is a periodic solution of system (3.6).

Our next goal is to find solutions $c = c(h, \tau)$ of $\Phi(h, c, \tau) = 0$. By (3.7) and (3.8), we have the expression

$$\Phi(h, c, \tau) = \tau\tilde{\Phi}(h, c, \tau),$$

and the limit

$$\lim_{\tau\to 0} \tilde{\Phi}(h, c, \tau) = \int_{z_{20}}^{z_{10}} \frac{\sqrt{c}}{a_1}(a_0 c {u_0}^2 {v_0}^2 + \frac{{v_0}^2}{a_1} - \frac{3a_0}{a_1}{u_0}^2{v_0}^2)\,\mathrm{d}z =: \sqrt{c}a_1^{-2}M(h, c),$$

which follows from the classical qualitative theory, where (u_0, v_0) is the solution of the unperturbed system (3.3) starting from the initial point $(\alpha, 0)$, and z_{10} and z_{20} are respectively the positive and negative times of this orbit arriving again at the u-axis. After these manipulations, we can write $\Phi(h, c, \tau)$ in (3.7) as

$$\Phi(h, c, \tau) = \sqrt{c}a_1^{-2}M(h, c)\tau + O(\tau^2).$$

The function $M(h, c)$ is called the Melnikov function, or the Pontryagin–Melnikov function.

Note that

$$\frac{\partial M}{\partial c}(h, c) = a_0 a_1 \int_{z_{20}}^{z_{10}} {u_0}^2{v_0}^2\,\mathrm{d}z > 0.$$

So for each $h = H(\alpha, 0) \in (-w^2/(4b), 0)$, the functional equation $M(h, c) = 0$ has a unique simple solution $c = c_0(h)$, whose precise expression will be obtained from (3.10) below. Hence, it follows from the implicit function theorem that for each $h = H(\alpha, 0) \in (-w^2/(4b), 0)$, the functional equation $\Phi(h, c, \tau) = 0$ has a unique simple solution $c = c(h, \tau)$ defined in a small neighbourhood of $(h, 0)$. This proves that for any $h = H(\alpha, 0) \in (-w^2/(4b), 0)$, and $0 < \tau \ll 1$, if $c = c(h, \tau)$ system (3.6) has an isolated periodic orbit passing $(\alpha, 0)$ in the (u, v) plane. Consequently, system (3.2) has an isolated periodic orbit when $c = c(h, \tau)$. It provides a periodic wave of equation (1.6).

Up to now, we have proved statements (a), (b) and (c) of theorem 2.1 on the periodic waves.

Next we will study the properties of the wave speed $c(h, \tau)$ associated to the periodic wave of equation (1.6).

3.2. Properties of the wave speed of the periodic wave

This subsection is to investigate the properties of the limit wave speed $c_0(h)$ of $c(\tau, h)$ as $\tau \to 0$, where $c(\tau, h)$ is the wave speed of the periodic wave to equation (1.6), which is determined by the solutions $(u_\tau(z), v_\tau(z)) =$

$(u(\tau, h, c(h,\tau), z), v(\tau, h, c(h,\tau), z))$ of system (3.2) with $(u_\tau(0), v_\tau(0)) = (\alpha, 0)$. Recall that the relation between α and h is given by $H(\alpha, 0) = h \in (-w^2/(4b), 0)$,

First, for $h = H(\alpha, 0) \in (-w^2/(4b), 0)$, one has

$$\begin{aligned}\int_{z_{20}}^{z_{10}} {u_0}^2{v_0}^2\,\mathrm{d}z &= \int_{z_{20}}^{z_{10}} ({u_0}^2 - \frac{w}{b})\dot{u}_0\,\mathrm{d}u_0 + \frac{w}{b}\int_{z_{20}}^{z_{10}} {v_0}^2\,\mathrm{d}z \\ &= -2\int_{z_{20}}^{z_{10}} {u_0}^2{v_0}^2\,\mathrm{d}z + \frac{1}{b}\int_{z_{20}}^{z_{10}} \ddot{u}_0^2\,\mathrm{d}z + \frac{w}{b}\int_{z_{20}}^{z_{10}} {v_0}^2\,\mathrm{d}z.\end{aligned}$$

It follows that

$$\int_{z_{20}}^{z_{10}} {u_0}^2{v_0}^2\,\mathrm{d}z = \frac{1}{3b}\int_{z_{20}}^{z_{10}} \ddot{u}_0^2\,\mathrm{d}z + \frac{w}{3b}\int_{z_{20}}^{z_{10}} {v_0}^2\,\mathrm{d}z.$$

Then the equation $M(h, c) = 0$ is equivalent to

$$(a_0a_1c - 3a_0)\int_{z_{20}}^{z_{10}} {u_0}^2{v_0}^2\,\mathrm{d}z + \int_{z_{20}}^{z_{10}} {v_0}^2\,\mathrm{d}z = 0, \tag{3.9}$$

and is further equivalent to

$$\frac{1}{a_0a_1c - 3a_0} = -\frac{1}{3b}\frac{\int_{z_{20}}^{z_{10}} \ddot{u}_0^2\,\mathrm{d}z}{\int_{z_{20}}^{z_{10}} \dot{u}_0^2\,\mathrm{d}z} - \frac{w}{3b}, \tag{3.10}$$

with $0 < c < 3a_1^{-1} = 3w$.

Next, for simplification to notations, we replace $(u_0(z), v_0(z))$ by $(u(z), v(z))$. Set $k = 4h \in (-1/(a_0a_1), 0)$, and let $\alpha(k)$ and $\beta(k)$ be the two positive real roots of the equation

$$2wu^2 - bu^4 + k = 0, \tag{3.11}$$

with $0 < \alpha(k) < \beta(k)$. Recall that the functional equation (3.11) is, in fact, $H(u, 0) = k/4$. Let

$$v := E(u) = \sqrt{2wu^2 - bu^4 + k}, \quad u \in [\alpha(k), \beta(k)],$$

be the closed segment γ_0^+ of the periodic orbit $\Gamma_0(k)$ over the u-axis, which is contained in $H(u, v) = k/4$. Let

$$P(k) := \frac{1}{2}\int_{z_{20}}^{z_{10}} \ddot{u}^2\,\mathrm{d}z, \quad Q(k) := \frac{1}{2}\int_{z_{20}}^{z_{10}} \dot{u}^2\,\mathrm{d}z.$$

Then by (3.3) one has

$$\begin{aligned}P(k) &= \frac{1}{2}\int_{z_{20}}^{z_{10}} (wu - bu^3)^2\,\mathrm{d}z = \sqrt{2}\int_{\alpha}^{\beta} \frac{w^2u^2 + b^2u^6 - 2wbu^4}{E(u)}\,\mathrm{d}u, \\ Q(k) &= \frac{1}{2}\int_{z_{20}}^{z_{10}} \dot{u}^2\,\mathrm{d}z = \frac{\sqrt{2}}{2}\int_{\alpha}^{\beta} E(u)\,\mathrm{d}u.\end{aligned} \tag{3.12}$$

For $k \in (-1/(a_0a_1), 0)$, we denote by $c(k)$ the solution of equation (3.9), instead of the previous $c_0(h)$. The next result characterizes the properties of the limit $c(k)$ of the wave speed of the periodic wave.

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PROPOSITION 3.1. *The function $c(k)$ has the next properties*:

$$c'(k) < 0, \quad k \in \left(\frac{-1}{a_0a_1},\ 0\right),$$

$$\lim_{k\to -1/(a_0a_1)} c = \frac{2}{a_1}, \quad \lim_{k\to 0} c = \frac{7}{4a_1}.$$

To prove this proposition, according to (3.9), (3.10) and (3.12), we define the functions

$$J_n(k) = \int_\alpha^\beta u^n E(u)\,\mathrm{d}u, \quad n = 0, 1, 2, \cdots .$$

Then

$$J_n'(k) = \frac{1}{2}\int_\alpha^\beta \frac{u^n}{E(u)}\,\mathrm{d}u,$$

which follows from $E(u) = \sqrt{2wu^2 - bu^4 + k}$ and $\frac{\mathrm{d}E}{\mathrm{d}k} = \frac{1}{2E}$. Under these notations, P and Q can be written as

$$\begin{aligned} P &= 2\sqrt{2}w^2 J_2'(k) + 2\sqrt{2}b^2 J_6'(k) - 4\sqrt{2}wbJ_4'(k), \\ Q &= \frac{\sqrt{2}}{2}J_0(k). \end{aligned} \tag{3.13}$$

Since $E^2 = 2wu^2 - bu^4 + k$, we have

$$E\frac{\mathrm{d}E}{\mathrm{d}u} = 2wu - 2bu^3.$$

LEMMA 3.2.

$$\begin{pmatrix} J_0 \\ J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} \dfrac{4k}{3} & 0 & \dfrac{4w}{3} \\ 0 & \dfrac{w^2}{b} + k & 0 \\ \dfrac{4wk}{15b} & 0 & \dfrac{16w^2 + 12kb}{15b} \end{pmatrix} \begin{pmatrix} J_0' \\ J_1' \\ J_2' \end{pmatrix}$$

Proof. Hereafter, for simplicity, we omit the upper and lower limits of the integrals. Direct calculations show that

$$\begin{aligned} J_0 &= \int (2wu^2 - bu^4 + k)\frac{\mathrm{d}u}{E} = \int \left(2wu^2 - u(wu - \frac{E\mathrm{d}E}{2\mathrm{d}u}) + k\right)\frac{\mathrm{d}u}{E} \\ &= w\int \frac{u^2}{E}\,\mathrm{d}u + \frac{1}{2}\int u\,\mathrm{d}E + k\int \frac{\mathrm{d}u}{E} = 2wJ_2' - \frac{1}{2}J_0 + 2kJ_0'. \end{aligned}$$

It induces

$$J_0 = \frac{4k}{3}J_0' + \frac{4w}{3}J_2'.$$

Similar manipulation gives the expressions of J_1 and J_2 in J_0', J_1' and J_2'. □

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Next result exhibits the expressions of J_3, J_4, J_5 and J_6 in J_0, J_1 and J_2.

Lemma 3.3.

$$
\begin{aligned}
J_3 &= \frac{w}{b} J_1, \\
J_4 &= \frac{8w}{7b} J_2 + \frac{k}{7b} J_0, \\
J_5 &= \frac{5w^2 + kb}{4b^2} J_1, \\
J_6 &= \frac{4wk}{21b^2} J_0 + \frac{2}{9} \frac{96w^2 + 21kb}{14b^2} J_2.
\end{aligned}
$$

Proof. The proof follows from the similar calculations as those to lemma 3.2. □

Furthermore, we need to compute the expressions of the second-order derivatives of J_0 and J_2.

Lemma 3.4.

$$
\begin{pmatrix} J_0'' \\ J_2'' \end{pmatrix} = \Delta(k) \begin{pmatrix} -k & -w - \dfrac{wk}{b} & k \end{pmatrix} \begin{pmatrix} J_0' \\ J_2' \end{pmatrix}, \quad \Delta(k) = \frac{b}{4k(w^2 + kb)}.
$$

Proof. According to lemma 3.2, we have

$$
J = A(k) J', \tag{3.14}
$$

where $J = (J_0, J_2)^{\mathrm{T}}$ and

$$
A(k) = \begin{pmatrix} \dfrac{4k}{3} & \dfrac{4w}{3} \\ \dfrac{4wk}{15b} & \dfrac{16w^2 + 12kb}{15b} \end{pmatrix}.
$$

Then the lemma follows from

$$
J' = A'J' + AJ'',
$$

and

$$
A^{-1}(E - A') = \Delta(k) \begin{pmatrix} -k & -w - wkb^{-1} & k \end{pmatrix}.
$$

Here

$$
A^{-1} = \frac{1}{4k(w^2 + kb)} \begin{pmatrix} 4w^2 + 3kb & -5bw & -wk & 5bk \end{pmatrix}. \tag{3.15}
$$

□

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Combining (3.13) and lemmas 3.2 and 3.3 together with some direct calculations, one gets

$$P(k) = -\frac{\sqrt{2}w}{2}J_0 + \frac{3\sqrt{2}b}{2}J_2.$$

Let

$$X(k) = \frac{P(k)}{Q(k)} = -w + 3b\frac{J_2}{J_0}.$$

By (3.10) one has

$$c(k) = \frac{3}{a_1}\left(1 - \frac{1}{a_1X(k)+1}\right). \tag{3.16}$$

Observe that the monotonicity of $c(k)$ and $X(k)$ is coincident. We turn to study monotonicity of $X(k)$. It is determined clearly by the properties of J_0 and J_2. Set

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x, \quad p>0,\ q>0,$$

which is the beta function.

Lemma 3.5.

$$J_0(0) = \frac{w\sqrt{2w}}{b}B\left(1,\frac{3}{2}\right), \quad J_2(0) = \frac{2w^2\sqrt{2w}}{b^2}B\left(2,\frac{3}{2}\right), \quad \frac{J_2(0)}{J_0(0)} = \frac{4w}{5b}.$$

Proof. By definition, one has clearly $\alpha(0) = 0, \beta(0) = \sqrt{2wb^{-1}}$, and

$$J_0(0) = \int_0^{\sqrt{\frac{2w}{b}}} \sqrt{2wu^2 - bu^4}\,\mathrm{d}u.$$

Simple calculations show that

$$J_0(0) = \frac{1}{2}\int_0^{\frac{2w}{b}} \sqrt{2w - bt}\,\mathrm{d}t = \frac{\sqrt{2w}}{2}\int_0^{\frac{2w}{b}} \sqrt{1 - \frac{bt}{2w}}\,\mathrm{d}t = \frac{w\sqrt{2w}}{b}B\left(1,\frac{3}{2}\right).$$

The expression $J_2(0)$ follows from the similar calculations. Applying the relation between the beta and gamma functions

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad \Gamma(s+1) = s\Gamma(s),$$

yields

$$\frac{J_2(0)}{J_0(0)} = \frac{4w}{5b}.$$

Recall that $\Gamma(s) = \int_0^{+\infty} x^{s-1}\,\mathrm{e}^{-x}\,\mathrm{d}x$. □

Lemma 3.6.

$$\lim_{k\to -\frac{w^2}{b}} \frac{J_2(k)}{J_0(k)} = \frac{w}{b}.$$

Proof. Since $0 < \alpha(k) \leqslant u \leqslant \beta(k)$ along the periodic orbit, one has

$$\int \alpha^2 E(u)\,\mathrm{d}u \leqslant \int u^2 E(u)\,\mathrm{d}u \leqslant \int \beta^2 E(u)\,\mathrm{d}u.$$

It follows

$$\alpha^2 \leqslant \frac{J_2(k)}{J_0(k)} \leqslant \beta^2.$$

Since $\alpha^2 \to \dfrac{w}{b}$ and $\beta^2 \to \dfrac{w}{b}$ as $k \to -\dfrac{w^2}{b}$, we arrive the conclusion. □

Lemma 3.7. *If* $X'(k_0) = 0$ *for some* $-\dfrac{w^2}{b} < k_0 < 0$, *then* $X''(k_0) < 0$.

Proof. The assumption $X'(k_0) = 0$ is equivalent to $J_2'(k_0)J_0(k_0) = J_2(k_0)J_0'(k_0)$, i.e.

$$\frac{J_2(k_0)}{J_0(k_0)} = \frac{J_2'(k_0)}{J_0'(k_0)}, \tag{3.17}$$

because $J_0(k_0), J_0'(k_0) > 0$. Let

$$\tilde{X}(k) = \frac{J_2(k)}{J_0(k)}, \quad \hat{X}(k) = \frac{J_2'(k)}{J_0'(k)}.$$

Then

$$J_2 = J_0\tilde{X}, \quad J_2' = J_0'\tilde{X} + J_0\tilde{X}', \quad J_2'' = J_0''\tilde{X} + 2J_0'\tilde{X}' + J_0\tilde{X}'',$$

and

$$J_2' = J_0'\hat{X}, \quad J_2'' = J_0''\hat{X} + J_0'\hat{X}'.$$

Note that the sign of $X''(k_0)$ coincides with that of $\tilde{X}''(k_0)$, and that $\tilde{X}'(k_0) = 0$ and $\tilde{X}(k_0) = \hat{X}(k_0)$ by (3.17). Then

$$\tilde{X}''(k_0) = \left.\frac{J_2'' - J_0''\tilde{X} - 2J_0'\tilde{X}'}{J_0}\right|_{k=k_0} = \left.\frac{J_0''(\hat{X} - \tilde{X}) + J_0'\hat{X}'}{J_0}\right|_{k=k_0} = \frac{J_0'(k_0)}{J_0(k_0)}\hat{X}'(k_0),$$

and

$$\begin{aligned}\hat{X}'(k_0) &= \frac{J_2''(k_0)J_0'(k_0) - J_2'(k_0)J_0''(k_0)}{(J_0'(k_0))^2}\\ &\overset{\text{lemma 3.4}}{=} \Delta(k_0)\left(2k_0\hat{X}(k_0) + w\hat{X}^2(k_0) - \frac{k_0 w}{b}\right).\end{aligned}$$

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Obviously, $2k_0\hat{X}(k_0) + w\hat{X}^2(k_0) - k_0 w/b$ is always positive, and $\Delta(k_0)$ is negative. It follows that $\hat{X}'(k_0) < 0$. Consequently

$$\tilde{X}''(k_0) = \frac{J_0'(k_0)}{J_0(k_0)}\hat{X}'(k_0) < 0. \qquad \square$$

LEMMA 3.8. *If* $X'(k_0) = 0$ *for* $-\dfrac{w^2}{b} < k_0 < 0$, *then* $X(k_0) \in (\dfrac{7w}{5}, 2w)$.

Proof. According to lemma 3.2, we have

$$3J_0 = 4kJ_0' + 4wJ_2', \tag{3.18}$$

and

$$15bJ_2 = 4wkJ_0' + (16w^2 + 12kb)J_2'. \tag{3.19}$$

Subtracting the multiplication of (3.18) by $4w$ with (3.19) gives

$$12wJ_0 - 15bJ_2 = 12wkJ_0' - 12kbJ_2'.$$

It can be written, via lemma 3.2, as

$$4w - 5b\frac{J_2}{J_0} = 4k\left(w\frac{J_0'}{J_0} - b\frac{J_2'}{J_0}\right) = 4k\left(w - b\frac{J_2'}{J_0'}\right)\frac{J_0'}{J_0}.$$

Since $\dfrac{J_0'(k_0)}{J_0(k_0)} > 0$ and $\tilde{X}(k_0) = \hat{X}(k_0)$, we get

$$4k_0(4w - 5b\tilde{X}(k_0))(w - b\tilde{X}(k_0)) > 0.$$

This shows

$$\frac{4w}{5b} < \tilde{X}(k_0) < \frac{w}{b}.$$

Moreover, by lemmas 3.5 and 3.6, we get

$$\lim_{k\to -\frac{w^2}{b}} X(k) = \lim_{k\to -\frac{w^2}{b}}\left(-w + 3b\frac{J_2(k)}{J_0(k)}\right) = 2w,$$

$$\lim_{k\to 0} X(k) = \lim_{k\to 0}\left(-w + 3b\frac{J_2(k)}{J_0(k)}\right) = \frac{7w}{5}.$$

This proves the lemma. $\square$

Combining lemmas 3.7 and 3.8 we achieve the next result.

LEMMA 3.9. *For* $-\dfrac{w^2}{b} < k < 0$, *we have* $X'(k) < 0$ *and* $X(k) \in \left(\frac{7w}{5}, 2w\right)$.

Lemmas 3.7, 3.8 and 3.9 together with equation (3.16) verify the two limits of $c(k)$ in proposition 3.1. This completes the proof of the proposition.

By proposition 3.1 and its proof, we have proved statement (d) of theorem 2.1.

In the next subsection, we will prove statements (a), (b) and (c) of theorem 2.1 on the solitary wave.

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3.3. Existence of the solitary wave

Subsection 3.1 provides a proof on the existence of the periodic waves of equation (1.6), which is a result from the perturbation of a period annulus of a Hamiltonian system with the total energy belonging to $(-w^2/(4b), 0)$. This subsection is to prove the existence of the solitary wave, which bifurcates from the homoclinic orbit of system (3.3) in the $u > 0$ half plane. To prove the existence of the solitary wave of equation (1.6), we only need to prove the existence of the homoclinic orbit of system (3.6) at the origin. Let $\gamma_1 := (u_1(z), v_1(z))$ be the 1-dimensional unstable manifold of system (3.6) at the origin with $v_1(z) > 0$, and let $\gamma_2 := (u_2(z), v_2(z))$ be the 1-dimensional stable manifold of system (3.6) at the origin with $v_2(z) < 0$. Then for $0 < \tau \ll 1$, the two invariant manifolds will intersect the positive u-axis. We denote these two intersecting points as their initial points of the unstable and stable manifolds. Then we have $v_1(0) = 0$ and $v_1(z) > 0$ for $-\infty < z < 0$, and $v_2(0) = 0$ and $v(z) < 0$ for $0 < z < +\infty$.

To prove the existence of a homoclinic orbit at the origin, we need to prove that the stable and unstable manifolds at the origin coincide on the positive u-axis. This is the same as $u_1(0) = u_2(0)$, which is equivalent to $H(u_1(0), 0) = H(u_2(0), 0)$. Recall that $H(u, v) = 0$ contains the homoclinic orbit γ_0 at the origin of the unperturbed system (3.3), and that when $\tau \to 0$, $\gamma_1 \cup \gamma_2$ has γ_0 as their limit.

To find the conditions ensuring $H(u_1(0), 0) = H(u_2(0), 0)$, we define

$$\Psi(c,\tau) := \int_{-\infty}^{0} \dot{H}(u_1, v_1)\,\mathrm{d}z - \int_{0}^{+\infty} \dot{H}(u_2, v_2)\,\mathrm{d}z.$$

By the standard Melnikov method together with some calculations, one gets from (3.6) that

$$\Psi(c,\tau) = \tau M(c) + O(\tau^2),$$

where

$$\begin{aligned}
M(c) &= \int_{-\infty}^{+\infty} \begin{pmatrix} v \\ wu - bu^3 \end{pmatrix} \wedge \begin{pmatrix} 0 & -\dfrac{\sqrt{c}}{a_1}(a_0 c u^2 v + \dfrac{v}{a_1} - \dfrac{3a_0}{a_1} u^2 v) \end{pmatrix} \mathrm{d}z \\
&= -\int_{-\infty}^{+\infty} \frac{\sqrt{c}}{a_1}(a_0 c u^2 v^2 + \frac{v^2}{a_1} - \frac{3a_0}{a_1} u^2 v^2)\,\mathrm{d}z \\
&= -\frac{\sqrt{c}}{a_1^2}\left((a_0 a_1 c - 3a_0)\int_{-\infty}^{+\infty} u^2 v^2\,\mathrm{d}z + \int_{-\infty}^{+\infty} v^2\,\mathrm{d}z\right),
\end{aligned}$$

where $(u(z), v(z))$ is the homoclinic orbit of system (3.3), the unperturbed system of system (3.6). Recall that the wedge operator $\wedge$ is defined as $f \wedge g = f_1 g_2 - f_2 g_1$ for $f = (f_1, f_2)$ and $g = (g_1, g_2)$.

Obviously, the Melnikov function $M(c)$ has a unique positive zero, which is simple, and is denoted by $c = c_0$. The implicit function theorem shows that for $0 < \tau \ll 1$, $\Psi(c, \tau) = 0$ has a unique positive solution $c = c(\tau)$, which satisfies $\lim_{\tau \to 0} c(\tau) = c_0$. This proves the existence of the homoclinic orbit at the origin for system (3.6) when $c = c(\tau)$ for $0 < \tau \ll 1$. Consequently, equation (1.6) has a solitary wave solution with the wave speed $c = c(\tau)$, whose leading term in the expansion in τ is

$$c_0 = \lim_{k\to 0} \frac{3}{a_1}\left(1 - \frac{1}{a_1 X(k) + 1}\right) = \frac{7}{4a_1}.$$

Here k and $X(k)$ have the same definitions as those in subsection 3.2, with difference only in replacing the limits α and β of the integrals by 0 and $u(0)$, respectively.

Up to now, we have completed the proof of statements (a), (b) and (c) of theorem 2.1 on the solitary wave, and consequently of the theorem.

4. The wavelength and wave speed

In this last section, we further study the properties of the wavelength and the wave speed. Let $T_\tau(k)$ be the period of the periodic orbit for system (2.6), and let $T(k) := \lim_{\tau\to 0} T_\tau(k)$. Then $T(k)$ is the period of the periodic solution $u(z) := u_k(z)$ of equation (2.8). Recall that $(u_k(z), v_k(z))$ is the periodic orbit of system (3.3), which is contained in the level curve $H = k/4$. Note that system (3.3) is symmetric with respect to $(v, z) \to (-v, -z)$, it follows that

$$\int_\alpha^\beta \frac{\mathrm{d}u}{v} = \int_0^{\frac{T}{2}} \mathrm{d}z = \frac{T}{2},$$

where $\alpha = \alpha(k)$ and $\beta = \beta(k)$ are the u coordinates of the intersection points of the periodic orbit with u-axis. Then

$$T = 2\int_\alpha^\beta \frac{\mathrm{d}u}{v} = 2\sqrt{2}\int_\alpha^\beta \frac{\mathrm{d}u}{E(u)} = 4\sqrt{2}J_0'(k).$$

Moreover, via simple calculation, one has

$$T'(k) = 4\sqrt{2}J_0''(k) = \frac{\sqrt{2}b}{k(w^2+kb)}(-kJ_0' - wJ_2') = -3\sqrt{2}J_0\triangle > 0.$$

This verifies the next result.

Lemma 4.1. *$T'(k) > 0$ for $k \in (-1/(a_0a_1), 0)$.*

Finally, we consider the limits of $T(k)$ at the endpoints of its domain. For obtaining the limit at $-w^2/b = -1/(a_0a_1)$, we first prove the next result.

Lemma 4.2.

$$\lim_{k\to -\frac{w^2}{b}} \frac{J_0}{w^2+kb} = \frac{\pi}{4b\sqrt{w}}.$$

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Proof. By definition of $\alpha(k)$ and $\beta(k)$, one gets

$$E^2(u) = b(u-\alpha)(\beta-u)(\alpha+u)(\beta+u) = -bu^4 + b(\alpha^2+\beta^2)u^2 - b\alpha^2\beta^2.$$

Obviously, $b(\alpha^2+\beta^2) = 2w$ and $b\alpha^2\beta^2 = -k$. It follows that $-\alpha^2(2w - b\alpha^2) = k$, and

$$(\alpha^2-\beta^2)^2 = (\alpha^2+\beta^2)^2 - 4\alpha^2\beta^2 = \frac{4w^2 + 4\alpha^2 b(\alpha^2 b - 2w)}{b^2}.$$

Consequently,

$$\lim_{k\to -\frac{w^2}{b}} \frac{J_0}{w^2+kb} = \lim_{k\to -\frac{w^2}{b}} \frac{\int_\alpha^\beta \sqrt{b(u^2-\alpha^2)(\beta^2-u^2)}\mathrm{d}u}{w^2+\alpha^2 b(b\alpha^2-2w)}.$$

Obviously,

$$\int_\alpha^\beta \sqrt{b(u^2-\alpha^2)(\beta^2-u^2)}\,\mathrm{d}u = \frac{\sqrt{b}}{2}\int_{\alpha^2}^{\beta^2} \frac{\sqrt{(m-\alpha^2)(\beta^2-m)}}{\sqrt{m}}\,\mathrm{d}m,$$

$$\begin{aligned}\frac{\sqrt{b}}{2\beta}\int_{\alpha^2}^{\beta^2} \sqrt{(m-\alpha^2)(\beta^2-m)}\,\mathrm{d}m &\leqslant \frac{\sqrt{b}}{2}\int_{\alpha^2}^{\beta^2} \frac{\sqrt{(m-\alpha^2)(\beta^2-m)}}{\sqrt{m}}\,\mathrm{d}m \\ &\leqslant \frac{\sqrt{b}}{2\alpha}\int_{\alpha^2}^{\beta^2} \sqrt{(m-\alpha^2)(\beta^2-m)}\,\mathrm{d}m,\end{aligned}$$

and

$$\int_{\alpha^2}^{\beta^2} \sqrt{(m-\alpha^2)(\beta^2-m)}\,\mathrm{d}m = \frac{\pi}{8}(\beta^2-\alpha^2)^2 = \frac{\pi}{2}\frac{w^2+\alpha^2 b(\alpha^2 b-2w)}{b^2}.$$

In addition,

$$\lim_{k\to -\frac{w^2}{b}} \alpha = \lim_{k\to -\frac{w^2}{b}} \beta = \lim_{k\to -\frac{w^2}{b}} u = \sqrt{\frac{w}{b}}.$$

One gets

$$\lim_{k\to -\frac{w^2}{b}} \frac{\frac{\sqrt{b}}{2\beta}\int_{\alpha^2}^{\beta^2} \sqrt{(m-\alpha^2)(\beta^2-m)}\,\mathrm{d}m}{w^2+\alpha^2 b(b\alpha^2-2w)} = \frac{\pi}{4b\sqrt{w}},$$

$$\lim_{k\to -\frac{w^2}{b}} \frac{\frac{\sqrt{b}}{2\alpha}\int_{\alpha^2}^{\beta^2} \sqrt{(m-\alpha^2)(\beta^2-m)}\,\mathrm{d}m}{w^2+\alpha^2 b(b\alpha^2-2w)} = \frac{\pi}{4b\sqrt{w}}.$$

Then the lemma follows from these last facts. □

Having the result in lemma 4.2, we can obtain the limit of the limit period.

LEMMA 4.3.

$$\lim_{k\to 0} T = +\infty, \qquad \lim_{k\to -\frac{w^2}{b}} T = \lim_{k\to -\frac{w^2}{b}} 4\sqrt{2}J_0' = \sqrt{2a_1}\pi.$$

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Proof. The first limit follows easily from the fact that the periodic orbit approaches the saddle during the limit process. The second limit follows from

$$\lim_{k\to -\frac{w^2}{b}} J_0' \overset{(3.14),(3.15)}{=} \lim_{k\to -\frac{w^2}{b}} \frac{J_0}{4k(w^2+kb)}\left(4w^2+3kb-5bw\frac{J_2}{J_0}\right)$$
$$\overset{\text{lemma } 3.5}{=} \lim_{k\to -\frac{w^2}{b}} \frac{bJ_0}{w^2+kb} \overset{\text{lemma } 4.2}{=} \frac{\pi}{4\sqrt{w}}.$$

□

Recall from § 3.2 that $\xi = z/\sqrt{c}$ for getting equation (2.6). Then the wavelength λ_0 of the periodic wave of equation (1.6) is $\lambda_0 = T/\sqrt{c_0}$ with c_0 satisfying (3.10). According to proposition 3.1, lemmas 4.1 and 4.3, we have the next result.

LEMMA 4.4. *For $k \in (-1/(a_0a_1), 0)$, it holds $\lambda_0'(k) > 0$, and*

$$\lim_{k\to 0} \lambda_0 = +\infty, \qquad \lim_{k\to -1/(a_0a_1)} \lambda_0 = a_1\pi.$$

Proof of theorem 2.2. It follows from proposition 3.1, lemma 4.4 and the properties of the derivatives of composite functions. □

5. Conclusion

In this paper, we have considered only the situation $u > 0$. In fact, the techniques can also be applied to the situation $u < 0$.

In addition, we have discussed here only the modified KdV equation with the weak generic delay kernel. If the kernel is replaced by the strong generic delay kernel, we can obtain similar results as those in theorems 2.1 and 2.2. Due to this reason, we omit the analysis on this case. For more general kernels, we do not know if this method is applicable or not. From our point of view, if the kernel is too general, it is hopeless. For some other special kernels, we need to explore new techniques to handle them.

The determination on the stability of those travelling waves obtained in theorem 2.1 for system (1.6) will be a good work. But up to now, we know very little about their further properties, so we have no idea to deal with their stability at the moment. On the solitary waves founded in theorem 2.1, another question is to know whether the solitary waves behave like solitons. We think these two questions are interesting and meaningful, and deserve to be further studied.

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