Integral Functions with Gap Power Series

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1. Let

$$f(z) = \sum_{0}^{\infty} a_n z^{\lambda_n} \tag{1}$$

be an integral function, λ_n being a strictly increasing sequence of non-negative integers. We shall use the notations

$$M(r) = \max_{|z|=r} |f(z)|, \; m(r) = \min_{|z|=r} |f(z)|,$$

$$\mu(r) = \max_{n=0,1,2,\ldots} |a_n| r^n,$$

describing $M(r)$ as the maximum modulus, $m(r)$ as the minimum modulus and $\mu(r)$ as the maximum term of $f(z)$.

The present paper is a development of a remark by Pólya (*Math. Zeit.*, 29 (1929), 549-640, last sentence of the paper) that if

$$\underline{\lim} \frac{\log(\lambda_{n+1} - \lambda_n)}{\log \lambda_n} > \tfrac{1}{2} \tag{2}$$

then

$$\overline{\lim_{r\to\infty}} \frac{m(r)}{M(r)} = \overline{\lim_{r\to\infty}} \frac{\mu(r)}{M(r)} = 1. \tag{3}$$

Our first result is

THEOREM 1.

If

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+1} - \lambda_n} < \infty, \tag{4}$$

then (3) *holds.*

Theorem 1 is clearly a sharpened form of Pólya's result, for from (2) it evidently follows that for sufficiently large n

$$\lambda_{n+1} - \lambda_n > \lambda_n^{\frac{1}{2}+\epsilon} > n^{1+\delta} \text{ for some positive } \epsilon \text{ and } \delta.$$

Theorem 1 is best possible, as is shown by our next result.

THEOREM 2.

If

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+1} - \lambda_n} = \infty, \tag{4}$$

then there exists an integral function of the form (1) *such that*

$$\overline{\lim_{r\to\infty}} \frac{\mu(r)}{M(r)} \leqq \tfrac{1}{2}, \quad \overline{\lim_{r\to\infty}} \frac{m(r)}{M(r)} \leqq \tfrac{1}{2}. \tag{6}$$

We generalise these theorems in two ways. First, relaxing the gap hypothesis we have

THEOREM 3.

If for a positive integer h

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+h} - \lambda_n} < \infty \tag{7}$$

then

$$\overline{\lim_{r\to\infty}} \frac{\mu(r)}{M(r)} \geqq \frac{1}{2h-1}; \tag{8}$$

but if

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+h} - \lambda_n} = \infty \tag{9}$$

for every h, then there exists an integral function of the form (1) *such that*

$$\lim_{r\to\infty} \frac{\mu(r)}{M(r)} = \lim_{r\to\infty} \frac{m(r)}{M(r)} = 0. \tag{10}$$

The conjecture that under condition (7) we could derive

$$\overline{\lim_{r\to\infty}} \frac{m(r)}{M(r)} > 0 \tag{11}$$

is disproved trivially by the example

$$\sum_{0}^{\infty} z^{n^3}/(n^3)! + \sum_{0}^{\infty} z^{n^3+1}/(n^3+1)!\,.$$

Our second generalisation relaxes the gap condition of Theorem 1 in a different way, but imposes in addition a condition on the order of the function. We have

THEOREM 4.

If as n tends to infinity

$$\sum_{k=0}^{n} \frac{1}{\lambda_{k+1} - \lambda_k} = o(\log \lambda_n), \tag{12}$$

and the function $f(z)$ is of finite order, or if

$$\sum_{k=0}^{n} \frac{1}{\lambda_{k+1} - \lambda_k} = O(\log \lambda_n), \tag{13}$$

and $f(z)$ is of zero order, then (2) *holds.*

This theorem cannot be materially strengthened since the example constructed for Theorem 2 will be of finite order if

$$\lim_{n\to\infty} \frac{1}{\log \lambda_n} \sum_{k=0}^{n} \frac{1}{\lambda_{k+1}-\lambda_k} > 0$$

and of zero order if

$$\lim_{n\to\infty} \frac{1}{\log \lambda_n} \sum_{k=0}^{n} \frac{1}{\lambda_{k+1}-\lambda_k} = \infty .$$

2. *Proof of Theorem* 1. To prove the theorem we need an elementary inequality. If $\epsilon_0 + \epsilon_1 + \epsilon_2 + \ldots$ is a convergent series of non-negative numbers and if a sequence δ_n is defined by

$$\delta_n = \max_{i<n<j} \frac{1}{(j-i+1)^{\frac{3}{2}}} \sum_{v=i}^{j} \epsilon_v, \tag{14}$$

then

$$\sum_0^\infty \delta_n \leqq (1 + 2 \sum_{n=2}^{\infty} n^{-\frac{3}{2}}) \sum_0^\infty \epsilon_v. \tag{15}$$

We have

$$\sum_0^\infty \delta_n = \sum_0^\infty \sum_0^\infty A_{v,n} \epsilon_v,$$

where $A_{v,n} = (j_n - i_n + 1)^{-3/2}$ or zero, as v falls in $i_n \leqq v \leqq j_n$ or not, i_n, j_n being the values of i, j for which the maximum in (14) is attained. Since $i_n \leqq n \leqq j_n$ also it follows that $j_n - i_n \geqq | v-n |$. Consequently

$$\sum_0^\infty \delta_n \leqq \sum_0^\infty \sum_0^\infty \frac{\epsilon_v}{(|v-n|+1)^{3/2}}$$

$$\leqq (1 + 2\sum_0^\infty n^{-3.2}) \sum_0^\infty \epsilon^n.$$

We now assume (4) and set

$$\epsilon_n = 1/(\lambda_{n+1} - \lambda_n). \tag{16}$$

Defining δ_n as in (14), we have $\sum_0^\infty \delta_n < \infty$ by (15). Let c_n be a sequence of positive numbers tending to infinity so slowly that

$$\sum_0^\infty c_n \delta_n < \infty . \tag{17}$$

Now let $A_n \leqq | z | \leqq A_{n+1}$, $n = 0, 1, 2, \ldots$, be the sequence of intervals in which a single term $a_k z^{\lambda_k}$ remains the maximum term. k will depend on n and increases with n, but we need not express this dependence in our notation. From (17) we have $\prod_0^\infty (1 + 2c_k \delta_k)^2 < \infty$, and hence there exist arbitrarily large values of n such that

$$A_{n+1}/A_n > (1 + 2\, c_k \delta_k)^2. \tag{18}$$

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We understand by n such a value and by k the associated integer. Since $a_k z^{\lambda_k}$ is the maximum term for $A_n \leqq |z| \leqq A_{n+1}$, we have

$$|a_v| \leqq |a_k| A_n^{\lambda_k - \lambda_v} \qquad (v < k)$$
$$|a_v| \leqq |a_k| A_{n+1}^{-(\lambda_v - \lambda_k)} \qquad (v > k). \qquad (19)$$

Using these inequalities with $r = |z| = (A_n A_{n+1})^{\frac{1}{2}}$, we have

$$|a_v| r^{\lambda_v} \leqq |a_k| r^{\lambda_k} (A_n / A_{n+1})^{\frac{1}{2}(\lambda_k - \lambda_v)}$$
$$\leqq |a_k| r^{\lambda_k} (1 + 2 c_k \delta_k)^{-(\lambda_k - \lambda_v)} \qquad (v < k), \qquad (20)$$
$$|a_v| r^{\lambda_v} \leqq |a_k| r^{\lambda_k} (1 + 2 c_k \delta_k)^{-(\lambda_v - \lambda_k)} \qquad (v > k).$$

But by the definition of δ_n and the inequality of the harmonic and arithmetic means,

$$\delta_k \geqq \left(\frac{1}{\lambda_{v+1} - \lambda_v} + \frac{1}{\lambda_{v+2} - \lambda_{v+1}} + \ldots + \frac{1}{\lambda_k - \lambda_{k-1}}\right)(k - v)^{-\frac{3}{2}}$$
$$\geqq \frac{1}{(k-v)^{\frac{1}{2}}} \left(\frac{k - v}{\lambda_k - \lambda_v}\right) = \frac{(k-v)^{\frac{1}{2}}}{\lambda_k - \lambda_v} \qquad (v < k). \qquad (21)$$

Consequently

$$(1 + 2c_k \gamma_k)^{-(\lambda_k - \lambda_v)} \leqq e^{-c_k (k-v)^{\frac{1}{2}}} \qquad (v < k). \qquad (22)$$

From this and a similar inequality when $v > k$, it follows from (20) that as $n \to \infty$ (and so $k \to \infty$, $r \to \infty$, $c_n \to \infty$)

$$\sum_0^{k-1} |a_v| r^{\lambda_v} + \sum_{k+1}^{\infty} |a_v| r^{\lambda_v} = o(|a_k| r^{\lambda_k}). \qquad (23)$$

From this follow first the second and then evidently the first statement of (3).

3. *Proof of Theorem* 2. Now suppose that $\sum_0^{\infty} 1/(\lambda_{n+1} - \lambda_n)$ diverges. We choose the coefficients a_n by the following rules.

$$a_0 = 1, \qquad a_n = a_{n+1} A_n^{-(\lambda_n - \lambda_{n-1})}, \qquad (24)$$

where

$$A_n = \prod_{v=0}^{n-1} \left(1 + \frac{\epsilon_v}{\lambda_v - \lambda_{v-1}}\right), \qquad A_0 = 1, \qquad A_1 = \left(1 + \frac{1}{\lambda_0 + 1}\right) \qquad (25)$$

and ϵ_n is a a sequence of positive numbers tending to zero and such that $\Sigma\, \epsilon_n/(\lambda_{n+1} - \lambda_n)$ diverges.

Evidently $A_n \to \infty$ and $f(z) = \sum_0^{\infty} a_n z^{\lambda_n}$ is an integral function.

Since

$$\frac{a_{n+1} r^{\lambda_{n+1}}}{a_n r^{\lambda_n}} = \frac{r^{\lambda_{n+1} - \lambda_n}}{A_{n+1}^{\lambda_{n+1} - \lambda_n}}, \qquad (26)$$

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the maximum term $\mu(r)$ is $a_n r^{\lambda_n}$ for

$$A_n \leqq r \leqq A_{n+1}. \tag{27}$$

Clearly

$$M(r) = \sum_0^\infty a_n r^{\lambda_n} > a_n r^{\lambda_n} + a_{n+1} r^{\lambda_{n+1}}. \tag{28}$$

Now for $A_n \leqq r \leqq A_{n+1}$ we have

$$\frac{a_{n+1} r^{\lambda_{n+1}}}{a_n r^{\lambda_n}} = \left(\frac{r}{A_{n+1}}\right)^{\lambda_{n+1}-\lambda_n} \geqq \left(\frac{A_n}{A_{n+1}}\right)^{\lambda_{n+1}-\lambda_n} \tag{29}$$

$$= \left(1 + \frac{\epsilon_n}{\lambda_{n+1}-\lambda_n}\right)^{-(\lambda_{n+1}-\lambda_n)} > e^{-\epsilon_n},$$

and it follows that $M(r) > (2-\epsilon)\mu(r)$ for all sufficiently large r.

This proves the first inequality of (6). To establish the second we argue as follows. With $A_n \leqq r \leqq A_{n+1}$ and $z = re^{\pi i/(\lambda_{n+1}-\lambda_n)}$ we have, for n sufficiently large,

$$|f(z)| \leqq M(r) - a_n r^{\lambda_n} - a_{n+1} r^{\lambda_{n+1}} + (a_n r^{\lambda_n} - a_{n+1} r^{\lambda_{n+1}}) \tag{30}$$

$$= M(r) - 2a_{n+1} r^{\lambda_{n+1}} \leqq M(r) - (2-\epsilon)\mu(r).$$

If $\mu(r) \geqq \frac{1}{4} M(r)$, it follows that $m(r) \leqq (\frac{1}{2}+\epsilon) M(r)$.

If $\mu(r) < \frac{1}{4} M(r)$ we argue differently. We use the relations

$$\{m(r)\}^2 \leqq \{M_2(r)\}^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_0^\infty a_n^2 r^{2\lambda_n}, \tag{31}$$

which lead to

$$\{M(r)\}^2 \geqq \sum_0^\infty a_v^2 r^{2\lambda_v} + \sum_0^\infty a_v r^{\lambda_v} \{f(r) - a_v r^{\lambda_v}\}$$

$$\geqq \{M_2(r)\}^2 + \sum_0^\infty a_v r^{\lambda_v} \{f(r) - \tfrac{1}{4} f(r)\} \tag{32}$$

and

$$\{m(r)\}^2 \leqq \{M_2(r)\}^2 \leqq \tfrac{1}{4}\{M(r)\}^2. \tag{33}$$

4. *Proof of Theorem* 3.

Suppose now that

$$\sum_{n=0}^\infty \frac{1}{\lambda_{n+h} - \lambda_n} < \infty, \tag{34}$$

where h is a positive integer greater than unity.

Defining δ_n as in (14) with $\epsilon_n = (\lambda_{n+h} - \lambda_n)^{-1}$ and choosing $c_n > 0$ so that $c_n \to +\infty$ and $\Sigma c_n \delta_n < \infty$, and again taking $A_n \leqq |z| < A_{n+1}$

to be the sequence of intervals in which a single term, say $a_k z^{\lambda_k}$, is the maximum term, we must have arbitrarily large values of n such that $A_{n+1}/A_n > (1 + 2c_k \delta_k)^2$, that is condition (18). With such values of n and associated k we still have (19) and (20), but we can no longer expect such a good result as (21) or its consequences (22) and (23). For $r = (A_n A_{n+1})^{\frac{1}{2}}$ and v "near" to k we can only say

$$| a_v | r^{\lambda_v} \leqq | a_k | r^{\lambda_k} \qquad (k - h < v < k + h). \tag{35}$$

For values of v which are not "too near" k we can give an analogue of (21) valid for $k - ph < v \leqq k - (p-1)h, \qquad p = 2, 3, \ldots,$ in

$$\delta_k \geqq \left(\frac{1}{\lambda_{k-(p-2)h} - \lambda_{k-(p-1)h}} + \ldots + \frac{1}{\lambda_{k-h} - \lambda_{k-2h}} + \frac{1}{\lambda_k - \lambda_{k-h}}\right)\frac{1}{(ph)^{\frac{3}{2}}}$$
$$\geqq \frac{(p-1)^2}{\lambda_k - \lambda_{k-(p-1)h}}\frac{1}{(ph)^{\frac{3}{2}}} \geqq \frac{p^{\frac{1}{2}}}{4h^{\frac{3}{2}}(\lambda_k - \lambda_v)}$$
$$\geqq \frac{(k-v)^{\frac{1}{2}}}{4h^2(\lambda_k - \lambda_v)}.$$

Consequently

$$(1 + 2c_k \delta_k)^{-(\lambda_k - \lambda_v)} \leqq e^{-c_k (k-v)^{\frac{1}{2}}/4h^2}.$$

From this and the similar inequalities with $v > k + h$ we have, as $n \to \infty$, the result

$$\sum_0^{k-h} | a_v | r^{\lambda_v} + \sum_{k+h}^{\infty} | a_v | r^{\lambda_v} = o(| a_k | r^{\lambda_k}), \tag{36}$$

and consequently with (35) we deduce

$$\underline{\lim}\, M(r)/\mu(r) \leqq (2h - 1)$$

or

$$\overline{\lim}\, \mu(r)/M(r) \geqq 1/(2h - 1),$$

which constitutes the first part of Theorem 3.

Now suppose that for some integer $h > 1$

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+h} - \lambda_n} = \infty.$$

Then evidently one of the series

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{nh+h+k} - \lambda_{nh+k}} \qquad (k = 0, 1, \ldots, h-1) \tag{37}$$

must diverge. There will be no loss of generality in supposing that the series with $k = 0$ diverges. We now, as in the proof of Theorem 2, define the series

$$f^*(z) = \sum_0^{\infty} a_n z^{\lambda^*_n}, \qquad \lambda^*_n = \lambda_{nh} \tag{38}$$

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with the properties that

$$\text{(i)}\quad \mu^*(r) = a_n^* r^{\lambda^*_n} \qquad \text{(ii)}\quad a^*_{n+1} r^{\lambda^*_{n+1}} \geqq (1-\epsilon) a^*_n r^{\lambda^*_n}$$

$$\text{for } A^*_n \leqq r \leqq A^*_{n+1}, \qquad n > n(\epsilon), \tag{39}$$

where $\mu^*(r)$ is the maximum term of $f^*(z)$ and A^*_n is defined from the sequence λ^*_n as A_n is defined from λ_n in (25). Let us now define $f(z) = \sum_0^\infty a_n z^{\lambda_n}$ by the conditions

$$a_{nh} = a^*_n,\ a_{nh+k} = a^*_n A_{n+h}^{-(\lambda_{nh+h} - \lambda_{nh})} \qquad (k = 1, 2, \ldots, h-1). \tag{40}$$

Then evidently for $A^*_n \leqq r \leqq A^*_{n+1}$ we shall have

$$a_{nh} r^{\lambda_{nh}} \geqq a_{nh+1} r^{\lambda_{nh+1}} \geqq \ldots \geqq a_{nh+h} r^{\lambda_{nh+h}}, \tag{41}$$

and $\mu(r)$ for the function $f(z)$ will be $a_{nh} r^{\lambda_{nh}}$, so that

$$M(r) = f(r) > (h + 1 - \epsilon) \mu(r) \qquad [r > r(\epsilon)]. \tag{42}$$

We approximate $m(r)$ by using

$$\{m(r)\}^2 \leqq \{M_2(r)\}^2 = \sum_0^\infty a_\nu^2 r^{2\lambda_\nu}. \tag{43}$$

Clearly

$$\{M(r)\}^2 = \sum_0^\infty a_\nu^2 r^{2\lambda_\nu} + \sum_0^\infty a_\nu r^{\lambda_\nu} \{M(r) - a_\nu r^{\lambda_\nu}\}$$

$$\geqq \{M_2(r)\}^2 + \{M(r)\}^2 - (h+1-\epsilon)^{-1} \{M(r)\}^2, \tag{44}$$

from which

$$m(r) \leqq M_2(r) \leqq (h+1-\epsilon)^{-\frac{1}{2}} M(r) \tag{45}$$

follows.

This does not quite complete the proof of Theorem 3 since $(h+1-\epsilon)^{-1}$ and $(h+1-\epsilon)^{-\frac{1}{2}}$, although arbitrarily small, are not zero. However we should only have to choose λ^*_n to be a subsequence of λ_n such that the interval $\lambda^*_n \leqq \lambda \leqq \lambda^*_{n+1}$ contains a number of λ_n increasing with λ^*_n but that $\sum (\lambda^*_{n+1} - \lambda^*_n)^{-1}$ diverges. It does not seem necessary to enumerate the details.

5. *Proof of Theorem* 4.

Given an increasing sequence of integers λ_n, let us first try to construct an integral function $\sum_0^\infty c_n x^{\lambda_n}$ with positive coefficients such that each term is in turn the maximum term and greatly exceeds in

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value the rest of the series. More precisely let $\delta > 0$ be a small prescribed number and let us choose the c_n in such a way that for a certain increasing sequence A_n of positive numbers the following conditions hold for all N. For $x = A_N$ we require that

$$c_{N+1}\, x^{\lambda_{N+1}} = \delta c_N\, x^{\lambda_N}$$
$$c_{N-1}\, x^{\lambda_{N-1}} = \delta c_N\, x^{\lambda_N}. \tag{46}$$

In this case we shall have, for $n > N$ and $x = A_N$,

$$c_{n+1}\, x^{\lambda_{n+1}} = \delta c_n\, x^{\lambda_n} \tag{47}$$

and consequently, for $x = A_N < A_n$,

$$c_{n+1}\, x^{\lambda_{n+1}} \leqq \delta c_n x^{\lambda_n}. \tag{48}$$

So for $x = A_N$, $p > 0$,

$$c_{N+p}\, x^{\lambda_{N+p}} \leqq \delta^p\, c_N\, x^{\lambda_N}$$
$$\sum_{N+1}^{\infty} c_n\, x^{\lambda_n} \leqq \frac{\delta}{1-\delta}\, c_N\, x^{\lambda_N}. \tag{49}$$

Similarly, for $x = A_N$,

$$\sum_{0}^{N+1} c_n\, x^{\lambda_n} \leqq \frac{\delta}{1-\delta}\, c_N\, x^{\lambda_N}. \tag{50}$$

We must now consider whether our conditions are possible.

(46) requires that

$$c_{N+1} = \delta c_N / A_N^{\lambda_{N+1} - \lambda_N}$$
$$c_N = \delta c_{N+1}\, A_{N+1}^{\lambda_{N+1} - \lambda_N}. \tag{51}$$

Eliminating c_N and c_{N+1}, we see that

$$A_{N+1}/A_N = \delta^{-2/(\lambda_{N+1} - \lambda_N)} = K^{1/(\lambda_{N+1} - \lambda_N)} \qquad (K > 1). \tag{52}$$

This defines the sequence A_n if we take $A_0 = 1$, and shows that it is increasing. With $c_1 = 1$ the sequence c_n is also defined, for the two conditions of (46) are now equivalent. The function $\sum_1^{\infty} c_n\, x^{\lambda_n}$ will be an integral function if A_n tends to infinity. Since

$$\log A_n = \log K \left\{\frac{1}{\lambda_1 - \lambda_0} + \frac{1}{\lambda_2 - \lambda_1} + \ldots + \frac{1}{\lambda_n - \lambda_{n-1}}\right\}, \tag{53}$$

this condition requires the divergence of $\sum_1^{\infty} 1/(\lambda_{n+1} - \lambda_n)$.

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The property of domination by single terms expressed by (49) and (50) will be carried over to the integral function $\sum_0^\infty a_n z^{\lambda_n}$ if we can assert that

$$\sum_0^\infty a_n z^{\lambda_n}/c_n \tag{54}$$

is an integral function. If we make the hypothesis that $\sum_0^\infty a_n z^{\lambda_n}$ is of finite order then $| a_n | < \lambda_n^{-a\lambda_n}$ for sufficiently large n and some positive a. To ensure that (54) does define an integral function we shall require to prove that for arbitrary $\epsilon > 0$ and sufficiently large n.

$$c_n > \lambda_n^{-\epsilon \lambda_n}. \tag{55}$$

This is equivalent to $\log c_n > -\epsilon \lambda_n \log \lambda_n$
and since

$$\log c_n = n \log \delta - \sum_{\nu=0}^{n-1} (\lambda_{\nu+1} - \lambda_\nu) \log A_\nu \tag{56}$$

this will follow from

$$\log A_n = o (\log \lambda_n) \tag{57}$$

or

$$\sum_1^n \frac{1}{\lambda_\nu - \lambda_{\nu-1}} = o (\log \lambda_n). \tag{58}$$

Now if we assume that $\sum_0^\infty a_n z^{\lambda_n}/c_n$ is an integral function it will follow that for sufficiently large values of z, say $z = R$, the maximum term of this function will occur with $n = N$ arbitrarily large. We shall have

$$| a_n | R^{\lambda_n}/c_n \leqq | a_N | R^{\lambda_N}/c_N.$$

$$\frac{| a_n | R^{\lambda_n}}{| a_N | R^{\lambda_N}} \leqq \frac{c_n}{c_N}$$

$$\frac{| a_n | (RA_N)^{\lambda_n}}{| a_N | (RA_N)^{\lambda_N}} \leqq \frac{c_n (A_N)^{\lambda_n}}{c_N (A_N)^{\lambda_N}}.$$

Thus the dominance expressed by (49) and (50) of a single term for $\Sigma c_n z^{\lambda_n}$ holds also for the function $\Sigma a_n z^\lambda$ with $| z | = RA_N$. Since δ may be chosen arbitrarily small Theorem 4 is proved for functions of finite order. If $\Sigma a_n x^{\lambda_n}$ is assumed to be of zero order we only require that $c_n > \lambda_n^{-h\lambda_n}$ for some positive h, and this clearly follows from (13).

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