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THE J_0-RADICAL OF A MATRIX NEARRING CAN BE INTERMEDIATE

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ABSTRACT. An example is constructed to show that the J_0-radical of a matrix nearring can be an intermediate ideal. This solves a conjecture put forward in [1].

1. **Introduction.** Soon after the discovery of intermediate ideals in matrix nearrings (see [1] and [4]), several questions were raised in connection with these ideals. A fact which followed immediately was that the J_2-radical of a matrix nearring can never be intermediate—for any (zerosymmetric) nearring R we have $J_2\big(\mathbb{M}_n(R)\big) = \big(J_2(R)\big)^*$ (see [7, Theorem 4.4]). Because this relation does not hold for the J_0-radical in general (see [2]), the question was raised in [1] whether the J_0-radical of a matrix nearring can be intermediate. The object of this note is to provide an example of a finite zerosymmetric abelian nearring R for which $J_0\big(\mathbb{M}_n(R)\big)$ is an intermediate ideal.

2. **Preliminaries.** We will assume R to be a right zerosymmetric nearring with identity 1. For a natural number n we define R^n to be the direct sum of n copies of the (not necessarily abelian) group $(R,+)$. For $r \in R$ and $1 \leq i, j \leq n$ we define the function $f_{ij}^r\colon R^n \longrightarrow R^n$ by $f_{ij}^r\alpha = \iota_i\big(r\pi_j(\alpha)\big)$ for each $\alpha \in R^n$, where $\iota_i\colon R \longrightarrow R^n$ and $\pi_i\colon R^n \longrightarrow R$ are the i-th injection and projection functions respectively. The subnearring of $M(R^n)$ generated by the set $\{f_{ij}^r \mid r \in R, 1 \leq i,j \leq n\}$ is called the *$n \times n$ matrix nearring over R* and denoted $\mathbb{M}_n(R)$. It is easy to verify that $\mathbb{M}_n(R)$ is also a right zerosymmetric nearring with identity.

For an ideal $A \trianglelefteq R$ there are two ways to construct an ideal in $\mathbb{M}_n(R)$ which relates naturally to A (see [6]), namely

$$A^+ := \mathrm{id}\langle f_{ij}^a \mid a \in A, 1 \leq i,j \leq n\rangle$$

and

$$A^* := \{U \in \mathbb{M}_n(R) \mid U\alpha \in A^n \text{ for all } \alpha \in R^n\}.$$

It easily follows that $A^+ \subseteq A^*$, and several examples exist (see [6], [2], [1] and [4]) to show that $A^+ \underset{\neq}{\subset} A^*$ is possible. It is also possible that ideals of $\mathbb{M}_n(R)$ can be properly situated between A^+ and A^* (see [1] and [4]). Any ideal I of $\mathbb{M}_n(R)$ with the property

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that $A^+ \underset{\neq}{\subset} I \underset{\neq}{\subset} A^*$ for some ideal A of R is called an *intermediate ideal* of $\mathbb{M}_n(R)$. In [1] it is shown that an intermediate ideal can never be of the form B^+ or B^* for any ideal B of R. In the next section we construct a nearring R for which $J_0\big(\mathbb{M}_n(R)\big)$ is intermediate.

3. **An example.** We need the following result.

LEMMA 3.1. *Suppose R is a zerosymmetric nearring with an ideal A such that $A^2 = \{0\}$. Then $(A^+)^2 = \{0\}$ in $\mathbb{M}_n(R)$.*

PROOF. From [6, Proposition 7] it follows that $(A^+)^2 \subseteq A^+A^* \subseteq \{0\}^* = \{0\}$. ∎

The example: Consider the following abelian groups:

$$M := \mathbb{Z}_2 \oplus \mathbb{Z}_2$$
$$N := M \oplus \mathbb{Z}_2$$
$$G := N \oplus \mathbb{Z}_2.$$

Let M_1, M_2, M_3 denote the two-element subgroups of M with $m_i \in M_i$ the nonzero element for each $i = 1, 2, 3$. Similarly, let N_1, N_2, N_3, N_4 denote the two-element subgroups of N which are not subgroups of M, with $n_i \in N_i$ the nonzero element for each $i = 1, 2, 3, 4$. Finally, let $g_1, g_2, \ldots, g_8$ denote the elements of $G \setminus N$. We identify $M \oplus \{0\} \oplus \{0\}$ with M and $N \oplus \{0\}$ with N, and $\bar{0} := (0, 0, 0, 0)$ denotes the neutral element of G.

Define the nearring R as follows:

$$\begin{aligned} R := \{f \in M_0(G) \mid\ & f(M_i) \subseteq M_i, 1 \leq i \leq 3;\ f(N_j) \subseteq N_j, 1 \leq j \leq 4; \\ & g, g' \in G \text{ and } g - g' \in M \Rightarrow f(g) - f(g') \in M; \\ & g, g' \in G \text{ and } g - g' \in N \Rightarrow f(g) - f(g') \in N\}. \end{aligned}$$

Then R is a right, zerosymmetric abelian nearring with identity 1. Moreover, R is finite with $|R| = 2^{23}$.

Define the R-subgroups K and L of R as follows:

$$K := \{f \in R \mid f(g_i) \in M,\ 1 \leq i \leq 8;\ \bar{0} \text{ otherwise}\}$$
$$L := \{f \in R \mid f(g_i) \in N,\ 1 \leq i \leq 8;\ \bar{0} \text{ otherwise}\}.$$

We may now draw several conclusions.

I. *K and L are R-ideals of R.*

PROOF. This follows because M and N are R-ideals of ${}_RG$. ∎

Observation I enables us to consider the R-modules ${}_RK$, ${}_R(L/K)$ and ${}_R(R/L)$.

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II. $J_0(R) = \mathrm{Ann}_R N \cap \mathrm{Ann}_R(G/N)$.

PROOF. Each of $M_i(1 \leq i \leq 3)$ and $N_j(1 \leq j \leq 4)$ and G/N, is an R-module of type 0 (they are all of order 2 and nontrivial). We also have that

$$\mathrm{Ann}_R N = \Big[\bigcap_{i=1}^{3} \mathrm{Ann}_R M_i\Big] \cap \Big[\bigcap_{j=1}^{4} \mathrm{Ann}_R N_j\Big].$$

It follows that

$$J_0(R) \subseteq \mathrm{Ann}_R N \cap \mathrm{Ann}_R(G/N).$$

But since $\mathrm{Ann}_R N \cap \mathrm{Ann}_R(G/N)$ is a nilpotent ideal of R, the reverse inclusion also follows. ∎

Note that the R-modules $M_i(1 \leq i \leq 3)$, $N_j(1 \leq j \leq 4)$ and G/N are in fact of type 2, implying that $J_0(R) = J_2(R)$.

III. $\big(J_0(R)\big)^+$ *is a nilpotent ideal of nilpotency degree* 2 *in* $\mathbb{M}_2(R)$.

PROOF. By II we have that $\big(J_0(R)\big)^2 = \{0\}$. The result now follows directly from Lemma 3.1. ∎

IV. $J_0\big(\mathbb{M}_2(R)\big)$ *contains a nilpotent element of nilpotency degree* 3.

PROOF. By I we can consider K^2, L^2/K^2 and R^2/L^2 as $\mathbb{M}_2(R)$-modules (see also [3, Proposition 4.1]). Define

$$A := \mathrm{Ann}_{\mathbb{M}_2(R)} K^2 \cap \mathrm{Ann}_{\mathbb{M}_2(R)}(L^2/K^2) \cap \mathrm{Ann}_{\mathbb{M}_2(R)}(R^2/L^2).$$

Then A is a nilpotent ideal of $\mathbb{M}_2(R)$ ($A^3 = \{0\}$) yielding

$$A \subseteq J_0\big(\mathbb{M}_2(R)\big).$$

Consider the elements $g_1, n_1, n_2, m_3 \in G$, where

$$\begin{aligned} g_1 &:= (0,0,0,1) \\ n_1 &:= (0,1,1,0) \in N \\ n_2 &:= (1,0,1,0) \in N \\ m_3 &:= (1,1,0,0) \in M; \end{aligned}$$

and the elements $a, b, c, d \in R$, where

$$\begin{aligned} a(g_i) &:= n_1, 1 \leq i \leq 8;\ \bar{0} \text{ otherwise} \\ b(g_i) &:= n_2, 1 \leq i \leq 8;\ \bar{0} \text{ otherwise} \\ c(m_3) &:= m_3;\ \bar{0} \text{ otherwise} \\ d(n_j) &:= n_j, 1 \leq j \leq 4;\ \bar{0} \text{ otherwise;} \end{aligned}$$

and finally here, the matrix $V \in \mathbb{M}_2(R)$, defined by

$$V := f_{11}^a + f_{21}^b + f_{11}^c(f_{11}^d + f_{12}^d).$$

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We show that $V \in \mathcal{A}$. Take any $\langle k, k' \rangle \in K^2$. Then $V\langle k, k' \rangle = \langle ak + c(dk + dk'), bk \rangle$. Now, for any $i, 1 \leq i \leq 8$, we have $k(g_i), k'(g_i) \in M$ while $a(M) = b(M) = d(M) = \{\bar{0}\}$. Since $k(N) = k'(N) = \{\bar{0}\}$, it follows that $ak + c(dk + dk') = bk = 0$, implying that

$$V \in \operatorname{Ann}_{\mathbb{M}_2(R)} K^2. \tag{1}$$

Now take any $\langle l, l' \rangle \in L^2$. Then $V\langle l, l' \rangle = \langle al + c(dl + dl'), bl \rangle$. Since $l(G), l'(G) \subseteq N$ and $a(N) = b(N) = \{\bar{0}\}$, while $c(G) \subseteq M$, it follows that $[al + c(dl + dl')](g_i), bl(g_i) \in M$ for all $i, 1 \leq i \leq 8$. Also, $l(N) = l'(N) = \{\bar{0}\}$, and we deduce that

$$V \in \operatorname{Ann}_{\mathbb{M}_2(R)}(L^2/K^2). \tag{2}$$

Finally, take any $\langle r, r' \rangle \in R^2$. Then $V\langle r, r' \rangle = \langle ar + c(dr + dr'), br \rangle$. Since $a(G), b(G), c(G) \subseteq N$, it follows that $[ar + c(dr + dr')](g_i), br(g_i) \in N$ for all $i, 1 \leq i \leq 8$. Consider $n_j \in N_j$ $(1 \leq j \leq 4)$. Then

$$[ar + c(dr + dr')](n_j) \in a(N_j) + c\big(d(N_j) + d(N_j)\big) \subseteq c(N_j) = \{\bar{0}\}.$$

Also, $br(n_j) = \bar{0}$. Now consider $m_i \in M_i$ $(1 \leq i \leq 3)$. Then

$$[ar + c(dr + dr')](m_i) \in a(M_i) + c\big(d(M_i) + d(M_i)\big) \subseteq c(M_i) = \{\bar{0}\},$$

while $br(m_i) = \bar{0}$. It follows that $\langle ar + c(dr + dr'), br \rangle \in L^2$, *i.e.*,

$$V \in \operatorname{Ann}_{\mathbb{M}_2(R)}(R^2/L^2). \tag{3}$$

Our claim that $V \in \mathcal{A}$ is now established by virtue of (1), (2) and (3). Now consider

$$\begin{aligned} V^2\langle 1, 0 \rangle &= V\langle a + cd, b \rangle \\ &= V\langle a, b \rangle \text{ since } cd = 0 \\ &= \langle a^2 + c(da + db), ba \rangle \\ &= \langle c(da + db), 0 \rangle \text{ since } a^2 = ba = 0. \end{aligned}$$

This, together with

$$\begin{aligned} c(da + db)(g_1) &= c\big(d(n_1) + d(n_2)\big) \\ &= c(n_1 + n_2) \\ &= c(m_3), \text{ since } n_1 + n_2 = m_3 \\ &= m_3 \neq \bar{0}, \end{aligned}$$

shows that $V^2 \neq 0$. Since $\mathcal{A} \subseteq J_0\big(\mathbb{M}_2(R)\big)$, IV is proved. ∎

V. $\big(J_0(R)\big)^+ \underset{\neq}{\subset} J_0\big(\mathbb{M}_2(R)\big)$.

PROOF. This follows directly from III and IV. ∎

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VI. $J_0\big(\mathbb{M}_2(R)\big) \subsetneq \big(J_0(R)\big)^*$.

PROOF. Consider the R-subgroup K_0 of K generated by the elements $k_1, k_2 \in K$ where

$$k_1(g_1) := m_1;\ \bar{0} \text{ otherwise,}$$
$$k_2(g_1) := m_2;\ \bar{0} \text{ otherwise.}$$

In other words,

$$K_0 = \{f \in R \mid f(g_1) \in M;\ \bar{0} \text{ otherwise}\}.$$

Now suppose $\{0\} \subsetneq K_0' \subsetneq K_0$ is a proper nontrivial R-subgroup of K_0. Then there exists $m \in M$ such that $m \notin K_0'(g_1)$. Choose $k_m \in K_0$ such that $k_m(g_1) = m$ and $r \in R$ such that $r(m) = m$ and $r(g) = \bar{0}$ if $g \neq m$. Now let $k \in K_0'$, $k \neq 0$. Then $[r(k+k_m) - rk_m](g_1) = -m \notin K_0'(g_1)$, which implies that $r(k+k_m) - rk_m \notin K_0'$, *i.e.*, K_0' is not an R-ideal of K_0. Consequently, K_0 is a simple R-module R-generated by two elements $k_1, k_2 \in K$. This means that $K_0^2 = K_0 \oplus K_0$ is a simple $\mathbb{M}_2(R)$-module generated by the single element $\langle k_1, k_2\rangle$, *i.e.*, K_0^2 is an $\mathbb{M}_2(R)$-module of type 0 (see [5, Lemma 3.1(b)]). This implies that

$$J_0\big(\mathbb{M}_2(R)\big) \subseteq \operatorname{Ann}_{\mathbb{M}_2(R)} K_0^2. \tag{4}$$

Now consider the matrix

$$W := f_{11}^s(f_{11}^t + f_{12}^t),$$

where $t(m_i) := m_i$, $i = 1, 2$; $\bar{0}$ otherwise, and $s(m_3) := m_3$; $\bar{0}$ otherwise. Take any $\langle r, r'\rangle \in R^2$. Then $W\langle r, r'\rangle = \langle s(tr + tr'), 0\rangle$. For m_i, $i = 1, 2$, we have

$$s(tr + tr')(m_i) \in s\big(t(M_i) + t(M_i)\big) \subseteq s(M_i) = \{\bar{0}\}.$$

Also, for m_3 we see that

$$(tr + tr')(m_3) \in s\big(t(M_3) + t(M_3)\big) \subseteq s(\{\bar{0}\}) = \{\bar{0}\}.$$

For n_j, $1 \le j \le 4$, we obtain $s(tr + tr')(n_j) = \{\bar{0}\}$, since $s(n_j) = \bar{0}$, $1 \le j \le 4$. Hence, we see that $s(tr+tr') \in \operatorname{Ann}_R N$. Also, since $s(G) \subseteq N$, it follows that $s(tr+tr') \in \operatorname{Ann}_R(G/N)$, whence $W \in \big(J_0(R)\big)^*$. Now consider $\langle k_1, k_2\rangle \in K_0^2$. Then $W\langle k_1, k_2\rangle = \langle s(tk_1 + tk_2), 0\rangle$ and $s(tk_1 + tk_2)(g_1) = s\big(t(m_1) + t(m_2)\big) = s(m_1 + m_2) = s(m_3) = m_3 \neq \bar{0}$. Consequently, $W \notin \operatorname{Ann}_{\mathbb{M}_2(R)} K_0^2$. By (4), $W \notin J_0\big(\mathbb{M}_2(R)\big)$ and so $J_0\big(\mathbb{M}_2(R)\big) \subsetneq \big(J_0(R)\big)^*$. ■

VII. $J_0\big(\mathbb{M}_2(R)\big)$ *is an intermediate ideal.*

PROOF. This follows by V and VI. ■

One of the key properties of this example is that $\big(J_0(R)\big)^+ \subseteq J_0\big(\mathbb{M}_2(R)\big)$. It is not known whether this is true in general, although it seems to be a plausible conjecture, which we formalize as follows:

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CONJECTURE 2.2. *If R is a zerosymmetric nearring with identity, then* $\big(J_0(R)\big)^+ \subseteq J_0\big(\mathbb{M}_n(R)\big)$. ∎

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