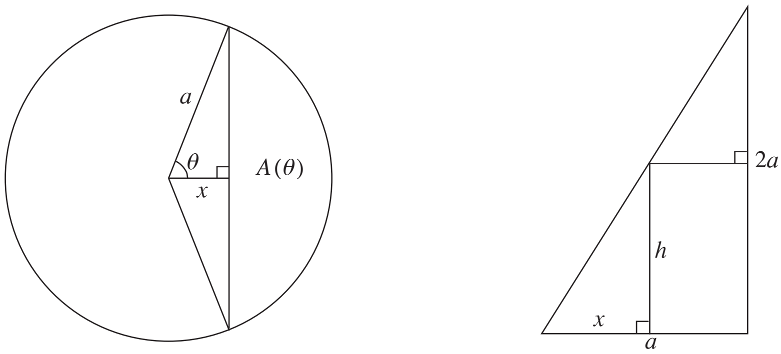
On 107.03: Nick Lord writes: I enjoyed this neat triple integral evaluation of the volume of the ungula. But it is worth noting that there is a single integral derivation which, although slightly longer, makes the calculation more accessible to Sixth Formers. The Figure shows the horizontal and vertical cross-sections of the ungula at height h, $0 \leqslant h \leqslant 2a$.



(a) (b)

FIGURE: Cross-sections of the ungula: (a) horizontal segment, (b) vertical triangle.

The volume, V, is given by $V = \int_0^{2a} A(\theta)\,dh$ where, from the Figure, $h = 2x = 2a\cos\theta$ and $A(\theta) = \frac{1}{2}a^2(2\theta - \sin 2\theta) = a^2(\theta - \sin\theta\cos\theta)$.

Thus

$$V = 2a^3 \int_0^{\pi/2} (\theta - \sin\theta\cos\theta)\sin\theta\,d\theta$$

$$= 2a^3\left[-\theta\cos\theta + \sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2} = \frac{4}{3}a^3.$$

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On 107.09: Nick Lord writes: There is a quick geometric proof of the theorem in this note, which gave the radius r of the circle which is tangent to side AC and passes through vertex B of triangle ABC as

$$r^2 = \frac{a^4b^2}{2(a^2b^2 + a^2c^2 + b^2c^2) - a^4 - b^4 - c^4}.$$

From the expanded form of Heron's formula for the area, Δ, of triangle ABC, this formula is equivalent to $r = \dfrac{a^2b}{4\Delta}$. In the Figure, the alternate segment theorem guarantees that a triangle similar to ABC may be inscribed in the circle with $r = \frac{a}{c}R$, where R is the circumradius of ABC. Since $\Delta = \frac{1}{2}bc\sin A = \dfrac{abc}{4\Delta}$, it follows that $r = \dfrac{a}{c}\cdot\dfrac{abc}{4\Delta} = \dfrac{a^2b}{4R}$.

It is also worth noting that we can alternatively merge the two proofs to give an unusual derivation of Heron's formula.