Curvature functionals on convex bodies

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Abstract. We investigate the *weighted L_p affine surface areas* which appear in the recently established L_p Steiner formula of the L_p Brunn–Minkowski theory. We show that they are valuations on the set of convex bodies and prove isoperimetric inequalities for them. We show that they are related to f divergences of the cone measures of the convex body and its polar, namely the Kullback–Leibler divergence and the Rényi divergence.

1 Introduction

In [34], an L_p Steiner formula was proved for the L_p affine surface area, namely, if a convex body K is C_+^2, then we have, for all suitable t and for all $p \in \mathbb{R}$, $p \neq -n$, that

$$(1.1) \qquad as_p(K + tB_2^n) = \sum_{k=0}^{\infty} \left[\sum_{m=0}^{k} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{W}_{m,k}^p(K) \right] t^k,$$

where

$$(1.2) \qquad as_p(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1}$$

is the L_p affine surface area of a convex body K, $N(x)$ is the outer normal to K in $x \in \partial K$, the boundary of K, $H_{n-1}(x)$ is the Gauss curvature in x, $\mathcal{H}^{n-1}$ is the usual surface area measure on ∂K, and $\binom{\alpha}{k}$ are binomial coefficients (see (2.3)). The Euclidean unit ball centered at 0 is denoted by B_2^n.

Identity (1.1) is the analog of the classical Steiner formula (e.g., [13, 29]) of the Brunn–Minkowski theory in the more recent L_p Brunn–Minkowski theory. This theory has as its starting point Lutwak's seminal paper [23] and it has been developed immensely (e.g., [3–6, 14, 17, 21, 24, 41–44]). In analogy to the classical theory, the coefficients $\mathcal{W}_{m,k}^p(K)$ are called L_p Steiner coefficients and they are defined in [34] for a (general) convex body K in $\mathbb{R}^n$, for all $k, m \in \mathbb{N} \cup \{0\}$ as

$$\mathcal{W}_{m,k}^p(K) =$$
$$\int_{\partial K} \langle x, N(x) \rangle^{m-k+\frac{n(1-p)}{n+p}} H_{n-1}^{\frac{p}{n+p}} \sum_{\substack{i_1,\ldots,i_{n-1} \geq 0 \\ i_1+2i_2+\cdots+(n-1)i_{n-1}=m}} c(n,p,m) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j} d\mathcal{H}^{n-1},$$

where the H_j are the jth normalized elementary symmetric functions of the principal curvatures. The $c(n, p, m)$ are certain binomial coefficients (see [34, 35] for the details). The L_p Steiner coefficients were studied in [35], where it was proved, among other results, that they are valuations on the set of convex bodies.

1.1 Main results

In this paper, we look at the L_p Steiner formula with a different focus. Expressions also appearing naturally in formula (1.1) are *weighted L_p affine surface areas*, $\mu_{\vec{i}} - as_p(K)$, which we define in Section 2.2.

    We investigate in detail those weighted L_p affine surface areas in Section 3. We show that they are homogeneous of a certain degree and are invariant under rotations and reflections. We show that they are valuations on the set of convex bodies. Valuations have become a vitally import subject of study in convexity and affine and differential geometry (e.g., [1, 10, 15, 16, 22, 30]). The weighted L_p affine surface areas satisfy isoperimetric inequalities which generalize the L_p affine isoperimetric inequalities of [23, 39]. This is shown in Theorem 3.2.

Theorem 3.2   *Let $s \neq -n, r \neq -n, t \neq -n$ be real numbers. Let K be a C_+^2 convex body in $\mathbb{R}^n$ with centroid at the origin.*

(i) If $\frac{(n+r)(t-s)}{(n+t)(r-s)} > 1$, then

$$\mu_{\vec{i}} - as_r(K) \leq \left(\mu_{\vec{i}} - as_t(K) \right)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \left(\mu_{\vec{i}} - as_s(K) \right)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}.$$

(ii) If $\frac{(n+r)t}{(n+t)r} > 1$, then

$$\frac{\mu_{\vec{i}} - as_r(K)}{\mu_{\vec{i}} - \text{vol}_n(K)} \leq n^{\frac{n(t-r)}{t(n+r)}} \left(\frac{\mu_{\vec{i}} - as_t(K)}{\mu_{\vec{i}} - \text{vol}_n(K)} \right)^{\frac{r(n+t)}{t(n+r)}}.$$

Equality holds in the above inequalities, if and only if K is an ellipsoid.

We show in Section 3.3, that the weighted L_p affine surface areas have natural geometric interpretations in terms of certain convex bodies associated with the given convex body K.

    We prove a monotonicity behavior in the parameter p for the weighted L_p affine surface areas which allows to establish asymptotics for the weighted L_p affine surface areas. These asymptotics connects them to entropy powers, namely to the Kullback–Leibler divergence D_{KL} of the cone measures of K and its polar K°, Q_K, and P_K. We quote the relevant Theorem 3.8 and refer to Section 3.4 for the details. We put

$$\omega_{m,k,\vec{i}}^p(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x) \rangle^{\frac{n(p-1)}{n+p}}} \langle x, N(x) \rangle^{m-k} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j}(x) \right\} d\mathcal{H}^{n-1}(x),$$

and then the following theorem holds.

Theorem 3.8 *Let K be a C_+^2 convex body in $\mathbb{R}^n$ with centroid at the origin. Then,*

(i)

$$\lim_{p \to \infty} \left(\frac{\omega_{m,k,\vec{i}}^p(K)}{\omega_{m,k,\vec{i}}^\infty(K)} \right)^{n+p} = \exp\left(-\frac{n\, D_{KL}(P_K \| Q_K)}{\mu_{\vec{i}} - \mathrm{vol}_n(K^\circ)} \right).$$

(ii)

$$\lim_{p \to 0} \left(\frac{\omega_{m,k,\vec{i}}^p(K^\circ)}{\omega_{m,k,\vec{i}}^0(K^\circ)} \right)^{\frac{n(n+p)}{p}} = \exp\left(-\frac{n\, D_{KL}(P_{K^\circ} \| Q_{K^\circ})}{\mu_{\vec{i}} - \mathrm{vol}_n(K^\circ)} \right).$$

This leads naturally to consider more general f-divergences than just the Kullbak–Leibler divergence. We treat that in Section 4, where we also observe that the weighted L_p affine surface areas themselves are special f-divergences.

Throughout the paper, we assume that the convex bodies K are C_+^2, i.e., K has twice continuously differentiable boundary with strictly positive Gauss curvature everywhere and such that 0 is the centroid of K, $0 = \frac{1}{\mathrm{vol}_n(K)} \int_K x\, dx$.

2 Weighted L_p-affine surface areas

2.1 Background from differential geometry

For more information and the details in this section, we refer to, e.g., [13, 29].

Let K be a convex body of class C^2. For a point x on the boundary ∂K of K, we denote by $N(x)$ the unique outward unit normal vector of K at x. The map $N_K : \partial K \to S^{n-1}$ is called the spherical image map or Gauss map of K and is of class C^1. Its differential is called the Weingarten map. The eigenvalues of the Weingarten map are the principal curvatures $k_i(x)$ of K at x.

The jth normalized elementary symmetric functions of the principal curvatures are denoted by H_j. They are defined as follows:

$$(2.1) \qquad H_j = \binom{n-1}{j}^{-1} \sum_{1 \le i_1 < \cdots < i_j \le n-1} k_{i_1} \cdots k_{i_j},$$

for $j = 1, \ldots, n-1$ and $H_0 = 1$. Note that

$$H_1 = \frac{1}{n-1} \sum_{1 \le i \le n-1} k_i$$

is the mean curvature, that is, the average of principal curvatures, and

$$H_{n-1} = \prod_{i=1}^{n-1} k_i$$

is the Gauss curvature.

We say that K is of class C_+^2 if K is of class C^2 and the Gauss map ν is a diffeomorphism. This means in particular that N_K has a smooth inverse. This assumption is stronger than just C^2, and is equivalent to the assumption that all principal curvatures

are strictly positive, or that the Gauss curvature $H_{n-1} \neq 0$. It also means that the differential of N_K, i.e., the Weingarten map, is of maximal rank everywhere.

Let K be of class C_+^2. For $u \in \mathbb{R}^n \setminus \{0\}$, let $\xi_K(u)$ be the unique point on the boundary of K at which u is an outward normal vector. The map ξ_K is defined on $\mathbb{R}^n \setminus \{0\}$. Its restriction to the sphere S^{n-1}, the map $\bar{\xi}_K : S^{n-1} \to \partial K$, is called the reverse spherical image map, or reverse Gauss map. The differential of $\bar{\xi}_K$ is called the reverse Weingarten map. The eigenvalues of the reverse Weingarten map are called the principal radii of curvature $r_1, \ldots, r_{n-1}$ of K at $u \in S^{n-1}$.

The jth normalized elementary symmetric functions of the principal radii of curvature are denoted by s_j. In particular, $s_0 = 1$, and for $1 \leq j \leq n-1$, they are defined by

$$(2.2) \qquad s_j = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} r_{i_1} \cdots r_{i_j}.$$

Note that the principal curvatures are functions on the boundary of K and the principal radii of curvature are functions on the sphere.

Now, we describe the connection between H_j and s_j. For a body K of class C_+^2, we have for $u \in S^{n-1}$ that $\bar{\xi}_K(u) = N_K^{-1}(u)$. In particular, the principal radii of curvature are reciprocals of the principal curvatures, that is,

$$r_i(u) = \frac{1}{k_i(\bar{\xi}_K(u))}.$$

This implies that for $x \in \partial K$ with $N_K(x) = u$,

$$s_j = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} \frac{1}{k_{i_1}(\bar{\xi}_K(u)) \cdots k_{i_j}(\bar{\xi}_K(u))} = \frac{H_{n-1-j}}{H_{n-1}}\Big(\bar{\xi}_K(u)\Big),$$

and

$$H_j = \frac{s_{n-1-j}}{s_{n-1}}\Big(N_K(x)\Big),$$

for $j = 1, \ldots, n-1$.

2.2 Definitions

For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, the generalized binomial coefficients are defined as

$$(2.3) \qquad \binom{\alpha}{k} = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k < 0 \text{ or } \alpha = 0, \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}, & \text{if } k > 0. \end{cases}$$

For fixed $k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and fixed sequence $\vec{i} = \{i_j\}_{j=0}^{n-1}$ such that $i_1 + 2i_2 + \cdots (n-1)i_{n-1} = m$ and all $p \in \mathbb{R}$, $p \neq -n$, we define

(2.4)
$$\omega^p_{m,k,\vec{i}}(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x)\rangle^{\frac{n(p-1)}{n+p}}} \langle x, N(x)\rangle^{m-k} \prod_{j=1}^{n-1}\left\{\binom{n-1}{j}^{i_j} H_j^{i_j}(x)\right\} d\mathcal{H}^{n-1}(x)$$

and

$$(2.5) \quad u^p_{m,k,\vec{i}}(K) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} h_K(u)^{m-k} \frac{\prod_{j=1}^{n-1}\binom{n-1}{j}^{i_j} s_{n-1-j}^{i_j}(u)}{s_{n-1}^{\sum_j i_j + \frac{p}{n+p}}(u)} d\mathcal{H}^{n-1}(u).$$

Remark 2.1 Note that the above quantities are vanishing for polytopes. Therefore, we will treat only C_+^2 convex bodies throughout the text and then

$$(2.6) \qquad \omega^p_{m,k,\vec{i}}(K) = u^p_{m,k,\vec{i}}(K).$$

Denote

$$c(n) = \prod_{j=1}^{n-1}\binom{n-1}{j}^{i_j}.$$

Let $\mu_{m,k,\vec{i}}$ be the measure on ∂K with density

$$d\mu_{m,k,\vec{i}}(x) = c(n)\langle x, N(x)\rangle^{m-k} \prod_{j=1}^{n-1} H_j^{i_j}(x)\, d\mathcal{H}^{n-1}(x),$$

with respect to the surface measure $\mathcal{H}^{n-1}$ on ∂K, and let $v_{m,k,\vec{i}}$ be the measure on S^{n-1} with density

$$dv_{m,k,\vec{i}}(u) = c(n) h_K^{m-k}(u) \frac{\prod_{j=1}^{n-1} s_{n-1-j}^{i_j}(u)}{s_{n-1}^{\sum_j i_j + \frac{p}{n+p}}(u)} d\mathcal{H}^{n-1}(u)$$

with respect to the surface measure $\mathcal{H}^{n-1}$ on S^{n-1}. To keep notations simple, we mostly write $\mu_{\vec{i}}$ and $v_{\vec{i}}$ instead of $\mu_{m,k,\vec{i}}$ and $v_{m,k,\vec{i}}$.

We then define the *weighted L_p-affine surface areas* by

$$(2.7) \qquad \mu_{\vec{i}} - as_p(K) = \omega^p_{m,k,\vec{i}}(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu_{\vec{i}}(x).$$

Definition (1.2) explains that those can be considered as L_p-affine surface area weighted by the measure $\mu_{\vec{i}}$. In particular,

$$\omega^0_{m,k,\vec{i}}(K) = \int_{\partial K} \langle x, N(x)\rangle d\mu_{\vec{i}}(x) = n(\mu_{\vec{i}} - \text{vol}_n(K))$$

is a weighted volume of K, weighted by the measure $\mu_{\vec{i}}$ and

$$\omega_{m,k,\vec{i}}^{\infty}(K) = \int_{\partial K} \frac{H_{n-1}}{\langle x, N(x)\rangle^n} d\mu_{\vec{i}}(x) = n(\mu_{\vec{i}} - \mathrm{vol}_n(K^{\circ}))$$

is a weighted volume of K°, weighted by the measure $\mu_{\vec{i}}$, where

$$K^{\circ} = \{y \in \mathbb{R}^n : \langle y, x\rangle \le 1, \text{ for all } x \in K\}$$

is the polar body of K.

We can take another point of view for the measure $d\mathcal{H}^{n-1}$ on S^{n-1} via the density $f_p(K, u) = \left(\frac{f_K(u)}{h_K^{p-1}(u)}\right)^{\frac{n}{n+p}}$. This density was introduced by Lutwak [23]. Then

$$u_{m,k,\vec{i}}^p(K) = c(n) \int_{S^{n-1}} f_p(K, u) h_K(u)^{m-k} \frac{\prod\limits_{j=1}^{n-1} s_{n-1-j}^{i_j}(u)}{s_{n-1}^{\sum_j i_j + \frac{p}{n+p}}(u)} d\mathcal{H}^{n-1}(u).$$

2.3   Special cases

Note that

$$\omega_{0,0,0}^p(K) = u_{0,0,0}^p(K) = as_p(K),$$

where we denote the sequence $\vec{i} = \{0, \dots, 0\}$ as 0.

1. When $k = m$, we get

$$\omega_{m,m,\vec{i}}^p(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x)\rangle^{\frac{n(p-1)}{n+p}}} \prod_{j=1}^{n-1}\left\{\binom{n-1}{j}^{i_j} H_j^{i_j}(x)\right\} d\mathcal{H}^{n-1}(x).$$

2. $m = 0$ implies that $\vec{i} = 0$ and (2.4) simplifies to

$$\omega_{0,k,0}^p(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x)\rangle^{\frac{n(p-1)}{n+p}+k}} d\mathcal{H}^{n-1}(x) = as_{p+\frac{k}{n}(n+p), -k}(K)$$

(see [34, equation (29)]).

3. For the Euclidean unit ball B_2^n, we get

(2.8) $$\omega_{m,k,\vec{i}}^p(B_2^n) = \mathrm{vol}_{n-1}(B_2^n) \prod_{j=1}^{n-1}\binom{n-1}{j}^{i_j} = c(n) \, \mathrm{vol}_{n-1}(\partial B_2^n),$$

which does not depend on k. Note that if $i_0 \ne 0$ and $i_1 = \cdots = i_{n-1} = 0$, that is, $\vec{i} = \{i_0, 0, \dots, 0\}$, then $\omega_{m,k,\vec{i}}^p(B_2^n) = \mathrm{vol}_{n-1}(\partial B_2^n)$.

4. If $p = 0$, we have

$$\omega_{m,k,\vec{i}}^0(K) = c(n) \int_{\partial K} \langle x, N(x)\rangle^{m-k+1} \prod_{j=1}^{n-1} H_j^{i_j}(x) \, d\mathcal{H}^{n-1}(x).$$

When in addition $k = m$, we get

$$\omega^0_{m,m,\vec{i}}(K) = c(n) \int_{\partial K} \langle x, N(x) \rangle \prod_{j=1}^{n-1} H_j^{i_j}(x) \, d\mathcal{H}^{n-1}(x).$$

5. If $p = \infty$, we get that

$$\omega^\infty_{m,m,\vec{i}}(K) = c(n) \int_{\partial K} \frac{H_{n-1}}{\langle x, N(x) \rangle^n} \prod_{j=1}^{n-1} H_j^{i_j}(x) \, d\mathcal{H}^{n-1}(x).$$

3 Properties of the weighted L_p-affine surface areas

3.1 Valuation, invariance, and homogeneity

Proposition 3.1   *Let K be a C_+^2 convex body in $\mathbb{R}^n$ with centroid at the origin. Let $p \in \mathbb{R}$, $p \neq -n$. Then $\omega^p_{m,k,\vec{i}}(K)$ is an $\left(n\frac{n-p}{n+p} - k \right)$-homogeneous valuation that is invariant under rotations and reflections.*

Proof   The proof follows immediately from results in [35]. We present an outline of the proof for completeness.

1. *Valuation.* As was shown in [35, Theorem 5.9], for all $1 \leq i_1, \ldots, i_{n-1} \leq n-1$, $1 \leq j \leq n-1$, and $\alpha_1, \ldots, \alpha_j \geq 0$,

$$\int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x) \rangle^{\frac{n(p-1)}{n+p}}} \langle x, N(x) \rangle^{k-m} \prod_{i=1}^{j} k_{i_j}^{\alpha_j} \, d\mathcal{H}^{n-1}(x)$$

is a valuation. It immediately follows that $\omega^p_{m,k,\vec{i}}(K)$ are valuations as the linear combination of valuations is again a valuation.

2. *Homogeneity.* Similarly to [35, Theorem 5.1] we can show that $\omega^p_{m,k,\vec{i}}(K)$ are homogeneous of order $n\frac{n-p}{n+p} - k$. Applying [35, Proposition 5.4] with $T = a \, Id$, we get that $\omega^p_{m,k,\vec{i}}(K) = a^{k-n\frac{n-p}{n+p}} \omega^p_{m,k,\vec{i}}(aK)$.

3. *Invariance.* If T is a rotation or a reflection, then $|\det T| = 1$, $\| T^{-1t}(N_K(T^{-1}(y))) \| = \| N_K(T^{-1}(y)) \| = 1$ and for all $1 \leq j \leq n-1$,

$$\{H_j(y) : y \in \partial T(K)\} = \{H_j(x) : x \in \partial K\}.$$

Thus, using these observations and [35, Proposition 5.4], we get

$$\omega^p_{m,k,\vec{i}}(K) =$$

$$\int_{\partial T(K)} \langle y, N_{T(K)}(y) \rangle^{m-k+\frac{n(1-p)}{n+p}} H_{n-1}^{\frac{p}{n+p}}(y) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j}(y) \, d\mathcal{H}^{n-1}(y) = \omega^p_{m,k,\vec{i}}(T(K)).$$

$\blacksquare$

3.2 Inequalities

Theorem 3.2   *Let $s \ne -n, r \ne -n, t \ne -n$ be real numbers. Let K be a C_+^2 convex body in $\mathbb{R}^n$ with centroid at the origin.*

(i)   *If $\frac{(n+r)(t-s)}{(n+t)(r-s)} > 1$, then*

$$\mu_{\vec{i}} - as_r(K) \le \left(\mu_{\vec{i}} - as_t(K)\right)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \left(\mu_{\vec{i}} - as_s(K)\right)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}.$$

(ii)   *If $\frac{(n+r)t}{(n+t)r} > 1$, then*

$$\frac{\mu_{\vec{i}} - as_r(K)}{\mu_{\vec{i}} - \mathrm{vol}_n(K)} \le n^{\frac{n(t-r)}{t(n+r)}} \left(\frac{\mu_{\vec{i}} - as_t(K)}{\mu_{\vec{i}} - \mathrm{vol}_n(K)}\right)^{\frac{r(n+t)}{t(n+r)}}.$$

Equality holds in the above inequalities, if and only if K is an ellipsoid.

Proof

(i) By Hölder's inequality—which enforces the condition $\frac{(n+r)(s-t)}{(n+t)(s-r)} > 1$, we then get

(3.1)

$$
\begin{aligned}
\mu_{\vec{i}} - as_r(K) &= \omega_{m,k,\vec{i}}^r(K) \\
&= \int_{\partial K} \frac{H_{n-1}(x)^{\frac{r}{n+r}}}{\langle x, N(x)\rangle^{\frac{n(r-1)}{n+r}}} \langle x, N(x)\rangle^{m-k} \prod_{j=1}^{n-1}\left\{\binom{n-1}{j}^{i_j} H_j^{i_j}(x)\right\} d\mathcal{H}^{n-1}(x) \\
&= \int_{\partial K} \left(\frac{H_{n-1}(x)^{\frac{t}{n+t}}}{\langle x, N(x)\rangle^{\frac{n(t-1)}{n+t}}}\right)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \left(\frac{H_{n-1}(x)^{\frac{s}{n+s}}}{\langle x, N(x)\rangle^{\frac{n(s-1)}{n+s}}}\right)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}} d\mu_{\vec{i}}(x) \\
&\le \left(\omega_{m,k,\vec{i}}^t(K)\right)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \left(\omega_{m,k,\vec{i}}^s(K)\right)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}} \\
&= \left(\mu_{\vec{i}} - as_t(K)\right)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \left(\mu_{\vec{i}} - as_s(K)\right)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}.
\end{aligned}
$$

(ii) Similarly, again using Hölder's inequality—which now enforces the condition $\frac{(n+r)t}{(n+t)r} > 1$,

$$
\begin{aligned}
\mu_{\vec{i}} - as_r(K) &= \omega_{m,k,\vec{i}}^r(K) \\
&= \int_{\partial K} \frac{H_{n-1}(x)^{\frac{r}{n+r}}}{\langle x, N(x)\rangle^{\frac{n(r-1)}{n+r}}} d\mu_{\vec{i}}(x) = \int_{\partial K} \left(\frac{H_{n-1}(x)^{\frac{t}{n+t}}}{\langle x, N(x)\rangle^{\frac{n(t-1)}{n+t}}}\right)^{\frac{r(n+t)}{t(n+r)}} \frac{d\mu_{\vec{i}}(x)}{\langle x, N(x)\rangle^{\frac{(r-t)n}{(n+r)t}}} \\
&\le \left(\omega_{m,k,\vec{i}}^t(K)\right)^{\frac{r(n+t)}{t(n+r)}} \left(\omega_{m,k,\vec{i}}^0(K)\right)^{\frac{(t-r)n}{(n+r)t}} \\
&= \left(\mu_{\vec{i}} - as_t(K)\right)^{\frac{r(n+t)}{t(n+r)}} \left(n(\mu_{\vec{i}} - \mathrm{vol}_n(K))\right)^{\frac{n(t-r)}{t(n+r)}}.
\end{aligned}
$$

The equality characterizations follow from the equality characterization of Hölder's inequality.

Equality holds in (i) and (ii) if and only if equality holds in Hölder's inequality which happens if and only if

$$\frac{H_{n-1}}{\langle x, N(x) \rangle^{n+1}} = \text{constant}$$

$\mu_{\vec{i}}$-almost everywhere on ∂K. As ∂K is C_+^2, $\{x \in \partial K : \mu_{\vec{i}}(x) = 0\} = \varnothing$ and therefore

$$\frac{H_{n-1}}{\langle x, N(x) \rangle^{n+1}} = \text{constant}$$

holds for all $x \in \partial K$. Thus, we can use the following theorem by Petty [27], which then finishes the proof. ∎

Theorem 3.3 [27] *Let K be a convex body in $\mathbb{R}^n$ that is C_+^2. K is an ellipsoid if and only if, for all x in ∂K,*

$$\frac{H_{n-1}}{\langle x, N(x) \rangle^{n+1}} = c,$$

where $c > 0$ is a constant.

Theorem 3.4 *Let K be a C_+^2 convex body in $\mathbb{R}^n$ with centroid at the origin. If $r < s < k$, then*

$$\omega_{m,s,\vec{i}}^p(K) \le \left(\omega_{m,k,\vec{i}}^p(K) \right)^{\frac{k-s}{k-r}} \left(\omega_{m,r,\vec{i}}^p(K) \right)^{\frac{s-r}{k-r}}.$$

Equality holds if and only if K is a ball.

Proof

$$\omega_{m,s,\vec{i}}^p(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x) \rangle^{\frac{n(p-1)}{n+p}}} \langle x, N(x) \rangle^{m-s} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j}(x) \right\} d\mathcal{H}^{n-1}(x)$$

(3.2)

$$= \int_{\partial K} \left(\frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x) \rangle^{\frac{n(p-1)}{n+p}}} \langle x, N(x) \rangle^{m-k} c(n) \prod_{j=1}^{n-1} H_j^{i_j} \right)^{\frac{s-r}{k-r}}$$

$$\cdot \left(\frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x) \rangle^{\frac{n(p-1)}{n+p}}} \langle x, N(x) \rangle^{m-r} c(n) \prod_{j=1}^{n-1} H_j^{i_j} \right)^{\frac{k-s}{k-r}} d\mathcal{H}^{n-1}(x)$$

$$\le \left(\omega_{m,k,\vec{i}}^p(K) \right)^{\frac{s-r}{k-r}} \left(\omega_{m,r,\vec{i}}^p(K) \right)^{\frac{k-s}{k-r}}.$$

Equality holds if and only if equality holds in Hölder's inequality which happens if and only if $\langle x, N(x) \rangle = \text{constant}$ if and only if K is a ball. ∎

Remark 3.5 When $\vec{i} = 0$ and $k = 0$, we recover L_p affine isoperimetric inequalities of [39].

We also obtain monotonicity behaviors.

Corollary 3.6   *Let K be a C_+^2 convex body in $\mathbb{R}^n$ with centroid at the origin. Let $p \neq -n$ be a real number.*

   (i) *The function $p \to \left(\frac{\omega_{m,k,\bar{i}}^p(K)}{\omega_{m,k,\bar{i}}^0(K)} \right)^{\frac{n+p}{p}}$ is increasing in $p \in (-n, \infty)$ and $p \in (-\infty, -n)$.*

   (ii) *The function $p \to \left(\frac{\omega_{m,k,\bar{i}}^p(K)}{\omega_{m,k,\bar{i}}^0(K)} \right)^{n+p}$ is decreasing in $p \in (-n, \infty)$ and $p \in (-\infty, -n)$.*

   (iii) *The inequalities are strict unless K is an ellipsoid.*

Proof   The statement (i) follows immediately from Theorem 3.2(ii). The statement (ii) follows from Theorem 3.2(i), by letting $s \to \infty$. The statement (iii) follows from the equality characterizations.   ∎

3.3   Geometric interpretations

We recall several constructions of convex bodies associated with a given convex body K. Namely:

   1. *Weighted floating bodies* [36]

   Let K be a convex body in $\mathbb{R}^n$ and denote by λ the Lebesgue measure on $\mathbb{R}^n$. Let $s \geq 0$, and let $f : K \to \mathbb{R}$ be an integrable function such that $f > 0$ λ-a.e. The weighted floating body $F(K, f, s)$ was defined in [36] (see also [3–4]) as the intersection of all closed half-spaces H^+ whose defining hyperplanes H cut off a set of $(f\lambda)$-measure less than or equal to s from K,

$$F(K, f, s) = \bigcap_{\int_{K \cap H^-} f \, d\lambda \leq s} H^+.$$

It was shown in [36] that

$$2 \left(\frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{n+1} \right)^{\frac{2}{n+1}} \lim_{s \to 0^+} \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(F(K, f, s))}{s^{\frac{2}{n+1}}} = \int_{\partial K} \frac{H_{n-1}^{\frac{1}{n+1}}}{f^{\frac{2}{n+1}}} \, d\mathcal{H}^{n-1}.$$

   2. *Surface bodies* [33]

Let K be a convex body, and let $f : \partial K \to \mathbb{R}$ be a nonnegative, integrable function with $\int_{\partial K} f \, \mathcal{H}^{n-1} = 1$. The probability measure $\mathbb{P}_f$ is the measure on ∂K with density f. Let $s \geq 0$. The surface body $S(K, f, s)$ was defined in [33] as the intersection of all the closed half-spaces H^+ whose defining hyperplanes H cut off a set of $\mathbb{P}_f$-measure less than or equal to s from ∂K,

$$S(K, f, s) = \bigcap_{\mathbb{P}_f(\partial K \cap H^-) \leq s} H^+.$$

It was shown in [33] that

$$2 \left(\mathrm{vol}_{n-1}(B_2^{n-1}) \right)^{\frac{2}{n-1}} \lim_{s \to 0^+} \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(S(K, f, s))}{s^{\frac{2}{n-1}}} = \int_{\partial K} \frac{H_{n-1}^{\frac{1}{n-1}}}{f^{\frac{2}{n-1}}} \, d\mathcal{H}^{n-1}.$$

3. *Random polytopes* [32]

A random polytope is the convex hull of finitely many points that are chosen with respect to a probability measure. In [32], random polytopes are considered where the points are chosen from ∂K with respect to $\mathbb{P}_f$, where $f : \partial K \to \mathbb{R}$ is an integrable, nonnegative function with $\int_{\partial K} f \, d\mathcal{H}^{n-1} = 1$ and $d\mathbb{P}_f = f \, d\mathcal{H}^{n-1}$. Then the expected volume of such a random polytope is

$$\mathbb{E}(f, N) = \mathbb{E}(\mathbb{P}_f, N) = \int_{\partial K} \cdots \int_{\partial K} \text{vol}_n([x_1, \ldots, x_N]) d\mathbb{P}_f(x_1) \ldots d\mathbb{P}_f(x_N),$$

where $[x_1, \ldots, x_N]$ is the convex hull of the points $x_1, \ldots, x_N$. It was shown in [32] that under mild smoothness assumptions on K,

$$\lim_{N \to \infty} \frac{\text{vol}_n(K) - \mathbb{E}(f, N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}} = c_n \int_{\partial K} \frac{H_{n-1}^{\frac{1}{n-1}}}{f^{\frac{2}{n-1}}} \, d\mathcal{H}^{n-1},$$

where $c_n = \dfrac{(n-1)^{\frac{n+1}{n-1}} \Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!(\text{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}}$.

4. *Ulam floating bodies* [19]

Given a Borel measure μ on $\mathbb{R}^n$, the metronoid associated with μ was introduced by Huang and Slomka [18] and is the convex set defined by

$$\mathcal{M}(\mu) = \bigcup_{\substack{0 \le f \le 1, \\ \int_{\mathbb{R}^n} f d\mu = 1}} \left\{ \int_{\mathbb{R}^n} y f(y) \, d\mu(y) \right\},$$

where the union is taken over all functions $0 \le f \le 1$ for which $\int_{\mathbb{R}^n} f d\mu = 1$ and $\int_{\mathbb{R}^n} y f(y) \, d\mu(y)$ exists. The metronoid $\mathcal{M}_\delta(K)$ is generated by the uniform measure on K with total mass $\delta^{-1}|K|$. Namely, let $\mathbb{1}_K$ be the characteristic function of K, and let μ be the measure whose density with respect to Lebesgue measure is $\delta^{-1}\mathbb{1}_K$. Then $\mathcal{M}_\delta(K) := \mathcal{M}(\mu)$. In [19] weighted variations of $\mathcal{M}_\delta(K)$ were defined where the weight is given by a positive continuous function $f : K \to \mathbb{R}$,

$$\mathcal{M}_\delta(K, f) := \mathcal{M}\left(\frac{f(x)}{\delta} \mathbb{1}_K(x) \, dx\right).$$

It was shown in [19] that

$$\lim_{\delta \to 0^+} \frac{\text{vol}_n(K) - \text{vol}_n(\mathcal{M}_\delta(K, f))}{\delta^{\frac{2}{n+1}}} = 2 \frac{n+1}{n+3} \left(\frac{\text{vol}_{n-1}(B_2^{n-1})}{n+1}\right)^{\frac{2}{n+1}} \int_{\partial K} \frac{H_{n-1}^{\frac{1}{n+1}}}{f^{\frac{2}{n+1}}} \, d\mathcal{H}^{n-1}.$$

This leads to geometric interpretations of the weighted L_p-affine surface areas in terms of these associated bodies. That is, if we let

$$f(x) = \frac{H_{n-1}(x)^{\frac{n(1-p)}{2(n+p)}}}{\left[c(n) H_j^{i_j}(x)\right]^{\frac{n+1}{2}}} \langle x, N(x) \rangle^{\left(\frac{n(p-1)}{n+p} + k - m\right)\frac{n+1}{2}}$$

in the case of weighted floating bodies and Ulam floating bodies, and

$$f(x) = \frac{H_{n-1}(x)^{\frac{2p+n(1-p)}{2(n+p)}}}{\left[c(n) H_j^{i_j}(x)\right]^{\frac{n-1}{2}}} \langle x, N(x) \rangle^{\left(\frac{n(p-1)}{n+p} + k - m\right)\frac{n-1}{2}}$$

in the case of surface bodies and random polytopes, respectively, and denote by $K_{f,s}$ the corresponding associated body, i.e.,

$$K_{f,s} = F(K, f, s), \quad \text{or } K_{f,s} = S(K, f, s), \quad \text{or } K_{f,s} = \mathcal{M}_s(K, f), \quad \text{or } \mathrm{vol}_n(K_{f,s}) = \mathbb{E}(f, N),$$

then, with the properly adjusted constant c_n, we get the following proposition.

Proposition 3.7 *Let K be a C_+^2 convex body in $\mathbb{R}^n$ with centroid at the origin. Let $p \neq -n$ be a real number.*

$$c_n \lim_{s \to 0} \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_{f,s})}{s^{\frac{2}{n-1}}} = \omega_{m,k,\vec{i}}^p(K).$$

3.4 Asymptotics for the weighted L_p-affine surface areas

Let (X, μ) be a measure space, and let $dP = p d\mu$ and $Q = q d\mu$ be measures in X that are absolutely continuous with respect to the measure μ. The *Kullback–Leibler divergence* or *relative entropy* from P to Q is defined as in [11]

$$(3.3) \qquad\qquad D_{KL}(P\|Q) = \int\limits_X p \log \frac{p}{q} d\mu.$$

The *information inequality* (also called *Gibb's inequality*) [11] holds for the Kullback–Leibler divergence. Let P and Q be as above, then

$$(3.4) \qquad\qquad D_{KL}(P\|Q) \geq 0$$

with equality if and only if $P = Q$.

We will apply this when $(X, \mu) = (\partial K, \mu_{\vec{i}})$, where K is C_+^2 and densities p and q with respect to $\mu_{\vec{i}}$ given by

$$(3.5) \qquad\qquad p_K(x) = \frac{H_{n-1}}{\langle x, N(x) \rangle^n}, \qquad q_K(x) = \langle x, N(x) \rangle.$$

We let

$$(3.6) \qquad\qquad P_K = \frac{H_{n-1}}{\langle x, N(x) \rangle^n} \mu_{\vec{i}}, \qquad Q_K = \langle x, N(x) \rangle \mu_{\vec{i}}.$$

Recall that classical cone measure cm_K on ∂K is defined as follows: for every measurable set $A \subseteq \partial K$,

$$cm_K(A) = \mathrm{vol}_n(\{ta : a \in A, t \in [0,1]\}).$$

It is wellknown (see, e.g., [26] for a proof) that the measures $\frac{H_{n-1}}{\langle x, N(x) \rangle^n} \mathcal{H}^{n-1}$ and $\langle x, N(x) \rangle \mathcal{H}^{n-1}$ are the cone measures of K and K°,

$$cm_K(A) = \frac{1}{n} \int_A \langle x, N(x) \rangle \mathcal{H}^{n-1}(x), \qquad cm_{K^\circ}(A) = \frac{1}{n} \int_A \frac{H_{n-1}}{\langle x, N(x) \rangle^n} \mathcal{H}^{n-1}(x).$$

The interpretation of cm_{K° as the "cone measure of K°" is via the Gauss map on K°, $N_{K^\circ} : \partial K^\circ \to S^{n-1}$, $x \longmapsto N_{K^\circ}(x)$ and the inverse of the Gauss map on $K N_K : \partial K \to S^{n-1}$, $x \longmapsto N_K(x)$,

$$N_K^{-1} N_{K^\circ} cm_{K^\circ},$$

where we use $N_K(x)$ to emphasize that it is the normal vector of a body K at $x \in \partial K$.

We now define the *weighted cone measures* $\mu_{\vec{i}} - cm_K$ of K and $\mu_{\vec{i}} - cm_{K^\circ}$ of K° by

$$(3.7) \qquad \mu_{\vec{i}} - cm_K = \frac{1}{n} \int_A \langle x, N(x) \rangle \mu_{\vec{i}}(x),$$

$$(3.8) \qquad \mu_{\vec{i}} - cm_{K^\circ}(A) = \frac{1}{n} \int_A \frac{H_{n-1}}{\langle x, N(x) \rangle^n} \mu_{\vec{i}}(x).$$

We show that the limits of the weighted L_p affine surface areas are entropy powers.

Theorem 3.8 *Let K be a C_+^2 convex body in $\mathbb{R}^n$ with centroid at the origin.*
(i)

$$\lim_{p \to \infty} \left(\frac{\omega_{m,k,\vec{i}}^p(K)}{\omega_{m,k,\vec{i}}^\infty(K)} \right)^{n+p} = \exp\left(-\frac{n \, D_{KL}(P_K \| Q_K)}{\mu_{\vec{i}} - \mathrm{vol}_n(K^\circ)} \right).$$

(ii)

$$\lim_{p \to 0} \left(\frac{\omega_{m,k,\vec{i}}^p(K^\circ)}{\omega_{m,k,\vec{i}}^0(K^\circ)} \right)^{\frac{n(n+p)}{p}} = \exp\left(-\frac{n \, D_{KL}(P_{K^\circ} \| Q_{K^\circ})}{\mu_{\vec{i}} - \mathrm{vol}_n(K^\circ)} \right).$$

Proof (i) We use L'Hospital's rule

$$\lim_{p \to \infty} \ln\left(\left(\frac{\omega_{m,k,\vec{i}}^p(K)}{\omega_{m,k,\vec{i}}^\infty(K)} \right)^{n+p} \right) = \lim_{p \to \infty} \frac{\ln\left(\frac{\omega_{m,k,\vec{i}}^p(K)}{\omega_{m,k,\vec{i}}^\infty(K)} \right)}{(n+p)^{-1}} = -\lim_{p \to \infty} (n+p)^2 \frac{\frac{d}{dp}\left(\omega_{m,k,\vec{i}}^p(K) \right)}{\omega_{m,k,\vec{i}}^p(K)}$$

$$= -\lim_{p \to \infty} \frac{(n+p)^2}{\omega_{m,k,\vec{i}}^p(K)} \int_{\partial K} \frac{d}{dp}\left(\exp\left(\ln\left(\frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N_K(x) \rangle^{\frac{n(p-1)}{n+p}}} \right) \right) \right) d\mu_{\vec{i}}(x)$$

$$= -\lim_{p \to \infty} \frac{n}{\omega_{m,k,\vec{i}}^p(K)} \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x) \rangle^{\frac{n(p-1)}{n+p}}} \ln\left(\frac{H_{n-1}(x)}{\langle x, N_K(x) \rangle^{n+1}} \right) d\mu_{\vec{i}}(x)$$

$$= -\frac{n}{\omega_{m,k,\vec{i}}^{\infty}(K)} \int_{\partial K} \frac{H_{n-1}(x)}{\langle x, N_K(x)\rangle^n} \ln\left(\frac{H_{n-1}(x)}{\langle x, N_K(x)\rangle^{n+1}}\right) d\mu_{\vec{i}}(x)$$

$$= -\frac{n}{\omega_{m,k,\vec{i}}^{\infty}(K)} D_{KL}(P_K \| Q_K).$$

(ii) We use L'Hospital's rule

$$\lim_{p\to 0} \ln\left(\left(\frac{\omega_{m,k,\vec{i}}^{p}(K^{\circ})}{\omega_{m,k,\vec{i}}^{0}(K^{\circ})}\right)^{\frac{n(n+p)}{p}}\right) = \lim_{p\to 0} n \frac{\ln\left(\frac{\omega_{m,k,\vec{i}}^{p}(K^{\circ})}{\omega_{m,k,\vec{i}}^{0}(K^{\circ})}\right)}{p(n+p)^{-1}} = \lim_{p\to 0} (n+p)^2 \frac{\frac{d}{dp}\omega_{m,k,\vec{i}}^{p}(K^{\circ})}{\omega_{m,k,\vec{i}}^{p}(K^{\circ})}$$

$$= \lim_{p\to 0} \frac{(n+p)^2}{\omega_{m,k,\vec{i}}^{p}(K^{\circ})} \int_{\partial K^{\circ}} \frac{d}{dp}\left(\exp\left(\ln\left(\frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N_{K^{\circ}}(x)\rangle^{\frac{n(p-1)}{n+p}}}\right)\right)\right) d\mu_{\vec{i}}(x)$$

$$= \lim_{p\to 0} \frac{n}{\omega_{m,k,\vec{i}}^{p}(K^{\circ})} \int_{\partial K^{\circ}} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N_{K^{\circ}}(x)\rangle^{\frac{n(p-1)}{n+p}}} \ln\left(\frac{H_{n-1}(x)}{\langle x, N_{K^{\circ}}(x)\rangle^{n+1}}\right) d\mu_{\vec{i}}(x)$$

$$= \frac{n}{\omega_{m,k,\vec{i}}^{0}(K^{\circ})} \int_{\partial K^{\circ}} \langle x, N_{K^{\circ}}(x)\rangle \ln\left(\frac{H_{n-1}(x)}{\langle x, N_{K^{\circ}}(x)\rangle^{n+1}}\right) d\mu_{\vec{i}}(x)$$

$$= -\frac{n}{\omega_{m,k,\vec{i}}^{0}(K^{\circ})} D_{KL}(Q_{K^{\circ}} \| P_{K^{\circ}}). \qquad \blacksquare$$

4 f-divergence

The results in the previous section lead naturally to consider more general f-divergences than just the Kullback–Leibler divergence.

4.1 Background on f-divergence.

In information theory, probability theory and statistics, an f-divergence is a functional that measures the difference between two (probability) distributions. This notion was introduced by Csiszár [12], and independently Morimoto [25] and Ali and Silvery [2].

Let (X, μ) be a measure space, and let $P = p\mu$ and $Q = q\mu$ be (probability) measures on X that are absolutely continuous with respect to the measure μ. Let $f : (0, \infty) \to \mathbb{R}$ be a convex or a concave function. The $*$-adjoint function $f^* : (0, \infty) \to \mathbb{R}$ of f is defined by

(4.1) $$f^*(t) = t f(1/t), \quad t \in (0, \infty).$$

It is obvious that $(f^*)^* = f$ and that f^* is again convex, if f is convex, respectively, concave, if f is concave. Then the f-divergence $D_f(P, Q)$ of the measures P and Q is defined by

(4.2) $$D_f(P, Q) = \int_{\{pq>0\}} f\left(\frac{p}{q}\right) q d\mu + f(0) \, Q\left(\{x \in X : p(x) = 0\}\right) \\ + f^*(0) \, P\left(\{x \in X : q(x) = 0\}\right),$$

provided the expressions exist. Here,

$$(4.3) \qquad f(0) = \lim_{t \downarrow 0} f(t) \quad \text{and} \quad f^*(0) = \lim_{t \downarrow 0} f^*(t).$$

We make the convention that $0 \cdot \infty = 0$.

Please note that

$$(4.4) \qquad D_f(P, Q) = D_{f^*}(Q, P).$$

With (4.3) and as

$$f^*(0) \, P(\{x \in X : q(x) = 0\}) = \int_{\{q=0\}} f^*\left(\frac{q}{p}\right) p \, d\mu = \int_{\{q=0\}} f\left(\frac{p}{q}\right) q \, d\mu,$$

we can write in short

$$(4.5) \qquad D_f(P, Q) = \int_X f\left(\frac{p}{q}\right) q \, d\mu.$$

Examples of f-divergences are as follows:

1. For $f(t) = t \ln t$ (with $*$-adjoint function $f^*(t) = -\ln t$), the f-divergence is *Kullback–Leibler divergence* or *relative entropy* from P to Q (see [11])

$$(4.6) \qquad D_{KL}(P\|Q) = \int_X p \ln \frac{p}{q} d\mu.$$

2. For the convex or concave functions $f(t) = t^\alpha$, we obtain the *Hellinger integrals* (e.g., [20])

$$(4.7) \qquad H_\alpha(P, Q) = \int_X p^\alpha q^{1-\alpha} d\mu.$$

Those are related to the Rényi divergence of order α, $\alpha \neq 1$, introduced by Rényi [28] (for $\alpha > 0$) as

$$(4.8) \qquad D_\alpha(P\|Q) = \frac{1}{\alpha - 1} \ln \left(\int_X p^\alpha q^{1-\alpha} d\mu \right) = \frac{1}{\alpha - 1} \ln \left(H_\alpha(P, Q) \right).$$

The case $\alpha = 1$ is the relative entropy $D_{KL}(P\|Q)$.

More on f-divergence can be found in, e.g., [7–9, 20, 37, 38, 40].

4.2 *f*-divergence for the $\mu_{\vec{i}}$ -measure

In [38], f-divergence with respect to the surface area measure μ_K was introduced for a convex body K in $\mathbb{R}^n$ with 0 in its interior. We now introduce similarly f-divergence with respect to the measure $\mu_{\vec{i}}$.

Definition 4.1   Let $f : (0, \infty) \to \mathbb{R}$ be a convex or concave function. Let p_K and q_K be as in (3.5). Then the f-divergence $D_f(P_K, Q_K)$ of a convex body K in $\mathbb{R}^n$ with respect to the $\mu_{\vec{i}}$-measure is

$$D_f(P_K, Q_K) = \int_{\partial K} f\left(\frac{p_K}{q_K}\right) q_K d\mu_{\vec{i}}$$

$$= \int_{\partial K} f\left(\frac{H_{n-1}}{\langle x, N(x)\rangle^{n+1}}\right) \langle x, N(x)\rangle d\mu_{\vec{i}}.$$

In particular, for $f(t) = t\log t$, we recover the above Kullback–Leibler divergences of Section 3.4 and for $f(t) = t^{\frac{p}{n+p}}$, we obtain the weighted L_p-affine surface areas.

Remarks (i) By (4.4),

$$D_f(Q_K, P_K) = \int_{\partial K} f\left(\frac{q_K}{p_K}\right) p_K d\mu_{\vec{i}} = D_{f^*}(P_K, Q_K)$$

(4.9)
$$= \int_{\partial K} f^*\left(\frac{p_K}{q_K}\right) q_K d\mu_{\vec{i}}$$

$$= \int_{\partial K} f\left(\frac{\mathrm{vol}_n(K^\circ)\langle x, N_K(x)\rangle^{n+1}}{\mathrm{vol}_n(K) H_{n-1}(x)}\right) \frac{H_{n-1}(x)\, d\mathcal{H}^{n-1}}{n\,\mathrm{vol}_n(K^\circ)\langle x, N_K(x)\rangle^n}.$$

(ii) f-divergences can also be expressed as integrals over S^{n-1},

(4.10)
$$D_f(P_K, Q_K) = \int_{S^{n-1}} f\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(K^\circ) f_K(u) h_K(u)^{n+1}}\right) \frac{h_K(u) f_K(u)}{n\,\mathrm{vol}_n(K)}\, d\sigma$$

and

(4.11)
$$D_f(Q_K, P_K) = \int_{S^{n-1}} f\left(\frac{\mathrm{vol}_n(K^\circ) f_K(u) h_K(u)^{n+1}}{\mathrm{vol}_n(K)}\right) \frac{d\sigma_K}{n\,\mathrm{vol}_n(K^\circ) h_K(u)^n}.$$

(iii) Similar to (4.10) and (4.11), one can define mixed f-divergences for n convex bodies in $\mathbb{R}^n$. We will not treat those here but will concentrate on f-divergence for one convex body. We also refer to [41], where they have been investigated for functions in $Conv(0, \infty)$.

Proposition 4.1 *Let $f : (0, \infty) \to \mathbb{R}$ be a concave function. Then*

$$D_f(P_K, Q_K) \le f\left(\frac{\mu_{\vec{i}} - \mathrm{vol}_n(K^\circ)}{\mu_{\vec{i}} - \mathrm{vol}_n(K)}\right) \mu_{\vec{i}} - \mathrm{vol}_n(K).$$

If f is convex, the inequality is reversed. If f is linear, equality holds in Proposition 4.1. If f is not linear, equality holds iff K is an ellipsoid.

Proof Let $f : (0, \infty) \to \mathbb{R}$ be a concave function. By Jensen's inequality,

$$D_f(P_K, Q_K) = \left(\int_{\partial K} f\left(\frac{H_{n-1}}{\langle x, N(x)\rangle^{n+1}}\right) \langle x, N(x)\rangle \frac{d\mu_{\vec{i}}}{\mu_{\vec{i}} - \mathrm{vol}_n(K)}\right) \mu_{\vec{i}} - \mathrm{vol}_n(K)$$

$$\le f\left(\int_{\partial K} \frac{H_{n-1}}{\langle x, N(x)\rangle^{n+1}} \langle x, N(x)\rangle \frac{d\mu_{\vec{i}}}{\mu_{\vec{i}} - \mathrm{vol}_n(K)}\right) \mu_{\vec{i}} - \mathrm{vol}_n(K)$$

$$= f\left(\frac{\mu_{\vec{i}} - \mathrm{vol}_n(K^\circ)}{\mu_{\vec{i}} - \mathrm{vol}_n(K)}\right) \mu_{\vec{i}} - \mathrm{vol}_n(K).$$

Equality holds in Jensen's inequality iff either f is linear or $\frac{p_K}{q_K}$ is constant. Indeed, if $f(t) = at + b$, then

$$D_f(P_K, Q_K) = \int_{\partial K} \left(a \frac{p_K}{q_K} + b \right) q_K \, d\mu_{\vec{i}} = a \int_{\partial K} p_K \, d\mu_{\vec{i}} + b \int_{\partial K} q_K \, d\mu_{\vec{i}}$$
$$= a \, \mu_{\vec{i}} - \mathrm{vol}_n(K^\circ) + b \, \mu_{\vec{i}} - \mathrm{vol}_n(K).$$

If f is not linear, equality holds iff $\frac{p_K}{q_K} = c$, where c is a constant. By the above quoted Theorem 3.3 of Petty, this holds iff K is an ellipsoid. Note that the constant c is different from 0, as we assume that K is C_+^2. ∎

Now we show that $D_f(P_K, Q_K)$ are valuations, i.e., for convex bodies K and L such that $K \cup L$ is convex,

$$(4.12) \qquad D_f(P_{K \cup L}, Q_{K \cup L}) + D_f(P_{K \cap L}, Q_{K \cap L}) = D_f(P_K, Q_K) + D_f(P_L, Q_L).$$

Proposition 4.2 *Let K be a convex body in $\mathbb{R}^n$ with the origin in its interior, and let $f : (0, \infty) \to \mathbb{R}$ be a convex function. Then $D_f(P_K, Q_K)$ are valuations.*

Proof To prove (4.12), we proceed as in Schütt [31]. For completeness, we include the argument. We decompose

$$\partial(K \cup L) = (\partial K \cap \partial L) \cup (\partial K \cap L^c) \cup (K^c \cap \partial L),$$
$$\partial(K \cap L) = (\partial K \cap \partial L) \cup (\partial K \cap \mathrm{int}L) \cup (\mathrm{int}K \cap \partial L),$$
$$\partial K = (\partial K \cap \partial L) \cup (\partial K \cap L^c) \cup (\partial K \cap \mathrm{int}L),$$
$$\partial L = (\partial K \cap \partial L) \cup (\partial K^c \cap \partial L) \cup (\mathrm{int}K \cap \partial L),$$

where all unions on the right-hand side are disjoint. Note that for x such that the curvatures $\kappa_K(x)$, $\kappa_L(x)$, $\kappa_{K \cup L}(x)$, and $\kappa_{K \cap L}(x)$ exist,

$$(4.13) \qquad \langle x, N_K(x) \rangle = \langle x, N_L(x) \rangle = \langle x, N_{K \cap L}(x) \rangle = \langle x, N_{K \cup L}(x) \rangle$$

and

$$(4.14) \qquad \kappa_{K \cup L}(x) = \min\{\kappa_K(x), \kappa_L(x)\}, \quad \kappa_{K \cap L}(x) = \max\{\kappa_K(x), \kappa_L(x)\}.$$

To prove (4.12), we split the involved integral using the above decompositions (4.13) and (4.14). ∎

References

[1] S. Alesker, *Continuous rotation invariant valuations on convex sets*. Ann. Math. **2**(1999), no. 149, 977–1005.

[2] M. S. Ali and D. Silvey, *A general class of coefficients of divergence of one distribution from another*. J. R. Stat. Soc. Ser. B. **28**(1966), 131–142.

[3] F. Besau, M. Ludwig, and E. M. Werner, *Weighted floating bodies and polytopal approximation*. Trans. Amer. Math. Soc. **370**(2018), 7129–7148.

[4] F. Besau and E. M. Werner, *The spherical convex floating body*. Adv. Math. **301**(2016), 867–901.

[5] F. Besau and E. M. Werner, *The floating body in real space forms*. J. Diff. Geometry **110**(2018), no. 2, 187–220.

[6] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, *The logarithmic Minkowski problem*. J. Amer. Math. Soc. **26**(2013), 831–852.

[7]   U. Caglar, A. V. Kolesnikov, and E. M. Werner, *Pinsker inequalities and related Monge–Ampère equations for log-concave functions*. Indiana Univ. Math. J. **71**(2022), no. 6, 2309–2333.

[8]   U. Caglar and E. M. Werner, *Divergence for s-concave and log concave functions*. Adv. Math. **257**(2014), 219–247.

[9]   U. Caglar and E. M. Werner, *Mixed f-divergence and inequalities for log concave functions*. Proc. Lond. Math. Soc. (**3**)(2015), no. 110, 271–290.

[10] A. Colesanti, M. Ludwig, and F. Mussnig, *Hessian valuations*. Indiana Univ. Math. J. **69**(2020), 1275–1315.

[11] T. Cover and J. Thomas, *Elements of information theory*. 2 nd ed., Wiley-Interscience (John Wiley and Sons), Hoboken, NJ, 2006.

[12] I. Csiszár, *Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten*. Publ. Math. Inst. Hungar. Acad. Sci. Ser. A **8**(1963), 84–108.

[13] R. J. Gardner, *Geometric tomography*. 2 nd ed., Cambridge University Press, New York, 2006.

[14] R. J. Gardner, D. Hug, W. Weil, and D. Ye, *The dual Orlicz–Brunn–Minkowski theory*. J. Math. Anal. Appl. **430**(2015), 810–829.

[15] C. Haberl, *Minkowski valuations intertwining with the special linear group*. J. Eur. Math. Soc. **14**(2012), 1565–1597.

[16] C. Haberl and L. Parapatits, *Moments and valuations*. Amer. J. Math. **138**(2017), 1575–1603.

[17] Y. Huang, E. Lutwak, D. Yang, and G. Zhang, *Geometric measures in the dual Brunn–Minkowski theory and their associated Minkowski problems*. Acta Math. **216**(2016), 325–388.

[18] H. Huang and B. A. Slomka, *Approximations of convex bodies by measure-generated sets*. Geom. Dedicata **200**(2019), 173–196.

[19] H. Huang, B. A. Slomka, and E. M. Werner, *Ulam floating bodies*. J. Lond. Math. Soc. **2**(2019), 425–446.

[20] F. Liese and I. Vajda, *On divergences and information in statistics and information theory*. IEEE Trans. Inform. Theory **52**(2006), 4394–4412.

[21] M. Ludwig and M. Reitzner, *A classification of SL(n) invariant valuations*. Ann. of Math. (2) **172**(2010), 1219–1267.

[22] M. Ludwig, *Valuations on Sobolev spaces*. Amer. J. Math. **134**(2012), 827–842.

[23] E. Lutwak, *The Brunn–Minkowski–Firey theory. II. Affine and geominimal surface areas*. Adv. Math. **118**(1996), 244–294.

[24] M. Meyer and E. Werner, *On the p-affine surface area*. Adv. Math. **152**(2000), 288–313.

[25] T. Morimoto, *Markov processes and the H-theorem*. J. Phys. Soc. Japan **18**(1963), 328–331.

[26] G. Paouris and E. M. Werner, *Relative entropy of cone measures and L_p centroid bodies*. Proc. Lond. Math. Soc. (3) **104**(2012), 253–286.

[27] C. Petty, *Affine isoperimetric problem, discrete geometry and convexity*. Ann. N. Y. Acad. Sci. **441**(1985), 113–127.

[28] A. Rényi, *On measures of entropy and information*. Proc. 4th Berkeley Symp. Prob. Theor. Math. Stat. **1**(1961), 547–561.

[29] R. Schneider, *Convex bodies: The Brunn–Minkowski theory*. 2nd expanded ed., Cambridge University Press, Cambridge, 2014.

[30] F. Schuster, *Crofton measures and Minkowski valuations*. Duke Math. J. **154**(2010), 1–30.

[31] C. Schütt, *On the affine surface area*. Proc. Amer. Math. Soc. **118**(1990), 275–290.

[32] C. Schütt and E. M. Werner, *Polytopes with vertices chosen randomly from the boundary of a convex body*. In: GAFA seminar notes, Lecture Notes in Mathematics, 1807, Springer, 2003, pp. 241–422.

[33] C. Schütt and E. M. Werner, *Surface bodies and p-affine surface area*. Adv. Math. **187**(2004), 98–145.

[34] K. Tatarko and E. M. Werner, *A Steiner formula in the L_p Brunn Minkowski theory*. Adv. Math. **355**(2019), 106772.

[35] K. Tatarko and E. M. Werner, *L_p-Steiner quermassintegrals*. Preprint, 2021. arXiv:2110.08659

[36] E. M. Werner, *The p-affine surface area and geometric interpretations*. Rend. Circ. Mat. Palermo (2) (2002), Suppl. no. 70(part II), 367–382.

[37] E. M. Werner, *Rényi Divergence and L_p-affine surface area for convex bodies*. Adv. Math. **230**(2012), 1040–1059.

[38] E. M. Werner, *F-divergence for convex bodies*. In: *Proceedings of the "asymptotic geometric analysis" workshop*, Fields Institute, Toronto, 2012.

[39] E. M. Werner and D. Ye, *New L_p affine isoperimetric inequalities*. Adv. Math. **218**(2008), 762–780.

[40] E. M. Werner and D. Ye, *Mixed f-divergence for multiple pairs of measures*. Can. Math. Bull. **60**(2017), 641–654.

[41] D. Ye, *Inequalities for general mixed affine surface areas*. J. Lond. Math. Soc. **85**(2012), 101–120.
[42] S. Xing and D. Ye, *On the general dual Orlicz–Minkowski problem*. Indiana Univ. Math. J. **69**(2020), 621–655.
[43] G. Zhang, *New affine isoperimetric inequalities*. ICCM II(2007), 239–267.
[44] Y. Zhao, *The dual Minkowski problem for negative indices*. Calc. Var. Partial Differ. Equ. **56**(2017), no. 2, 18.

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