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SOME PROPERTIES OF FATOU AND JULIA SETS OF TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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The radial distribution of Julia sets and non-existence of unbounded Fatou components of transcendental meromorphic functions are investigated in this paper.

1. INTRODUCTION AND MAIN RESULTS

Let $f : \mathbf{C} \to \overline{\mathbf{C}}$ be a transcendental meromorphic function, where $\mathbf{C}$ is the complex plane and $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. $f^n(z)$ denotes the n-th iterate of $f(z)$, that is, $f^0(z) = z, f^1(z) = f(z), \ldots, f^n(z) = f\big(f^{n-1}(z)\big)$, n is a non-negative integer. $f^n(z)$ is well defined for all $z \in \mathbf{C}$, possibly except for an (at most) countable set of poles of $f(z), f^2(z), \ldots, f^{n-1}(z)$. Denote by F_f the set of those points in $\mathbf{C}$ such that $\big\{f^n(z)\big\}_{n=1}^{\infty}$ is well defined and forms a normal family in some neighbourhood of z. F_f is called the Fatou set of $f(z)$ and its complement J_f the Julia set of $f(z)$. F_f is open and J_f is non-empty closed.

Nevanlinna theory is an important tool in the discussion of this paper, some standard notations of which, such as the Nevanlinna deficiency $\delta(\infty, f)$ with respect to ∞ and the characteristic function $T(r, f)$ of a meromorphic function $f(z)$ and so on, come mainly from [**7**]. The lower order $\mu(f)$ of a meromorphic function $f(z)$ is defined as follows:

$$\mu(f) := \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

Our first result is about the radial distribution of the Julia sets of transcendental meromorphic functions. In the theory of meromorphic functions, a great deal of work on the relations between the growth in terms of the order and the radial distribution of some value-points of a transcendental meromorphic function were made, for references see [**4, 5, 10, 13**].

For a $\theta \in [0, 2\pi)$, we say that the Julia set J_f has the radial distribution with respect to the radial $\arg z = \theta$, if for any small positive number $\varepsilon > 0$, $\Omega(\theta - \varepsilon, \theta + \varepsilon) \cap J_f$ is unbounded, where

$$\Omega(\theta - \varepsilon, \theta + \varepsilon) = \big\{z \in \mathbf{C} : \arg z \in (\theta - \varepsilon, \theta + \varepsilon)\big\}.$$

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Define

$$E := \{\theta \in [0, 2\pi) : \ J_f \text{ has the radial distribution with respect to } \arg z = \theta\}.$$

It is easy to see that E is closed. By $\operatorname{mes} E$ we stands for the linear measure of E.

THEOREM 1.1. *Let $f(z)$ be a transcendental meromorphic function in $\mathbb{C}$ with $\mu = \mu(f) < \infty$ and $\delta = \delta(\infty, f) > 0$. If $\mu = 0$, then $E = [0, 2\pi)$; If $\mu > 0$ and J_f has an unbounded component, then*

$$\operatorname{mes} E \geqslant \min\left\{2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta}{2}}\right\}.$$

We make some remarks on Theorem 1.1.

(1) If J_f has only bounded components, we do not know if Theorem 1.1 holds. In this case, F_f has at most an unbounded component. If F_f has no unbounded components, it is obvious that $E = [0, 2\pi)$. If F_f has only an unbounded component U, and if U is wandering or periodic of period at least two, then f is bounded in U, from the proof of Theorem 1.1 it follows that Theorem 1.1 holds. Then we are left with the case when U is invariant. In this special case, if for some $a \in J_f$, $C_{F_f}(a) > 0$ (please see the statement before Lemma 2.2 for its definition), then Theorem 1.1 still follows from Lemma 2.2 and the proof of Theorem 1.1.

(2) The condition that $\delta(\infty, f) > 0$ is necessary. Observe $f(z) = \lambda \tan z$, $\lambda \in \Re$, the real axis. It is easy to get $\mu(f) = 1$ and $\delta(\infty, f) = 0$. It was proved in [**3**] that when $\lambda > 1$, $J_f = \Re$, then $E = \{0, \pi\}$, and $\operatorname{mes} E = 0$.

When $0 < \lambda < 1$, the Julia set of $f(z) = \lambda \tan z$ is a Cantor set and the Fatou set consists of one unbounded component, but since $f(z)$ has only two singularity values, it was proved in [**11**] that for any $a \in J_f$, $C_{F_f}(a) > 0$.

(3) Baker [**2**] investigated the radial distribution of the Julia set of a transcendental entire function and constructed an entire function with infinite lower order whose Julia set lies in a horizontal strip. It is well known that an entire function f may only have unbounded simply connnected components of the Fatou set and $\delta(\infty, f) = 1$. Therefore, the condition that f has a finite lower order is necessary in Theorem 1.1. A further discussion on this subject of entire functions with finite lower order was made in [**9**] after Baker [**2**]. Their methods are not available for the case of meromorphic functions.

Next we consider when $\operatorname{mes} E = 2\pi$. If $\operatorname{mes} E < 2\pi$, then F_f must contain unbounded angle domains. Now [**12**, Theorem 3] says that F_f contains no unbounded angle domains, if for arbitrary positive integer m, the following holds

$$\limsup_{r \to \infty} \frac{L(r, f)}{r^m} = \infty, \tag{1}$$

where $L(r, f) := \min_{|z|=r} \{|f(z)|\}$. Thus we have

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Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function in $\mathbf{C}$ satisfying (1). Then $E = [0, 2\pi)$.*

Remark. (1) above suggests a further discussion of non-existence of the unbounded periodic components of F_f, which was investigated in Zheng [**11, 12**].

Theorem 1.3. *Let f_j $(j = 1, 2, \dots, N)$ be transcendental meromorphic functions. Assume that there exists a sequence $\{r_n\}$ of positive numbers which tends to infinity such that*

$$\lim_{n\to\infty} \frac{L(r_n, f_1)}{r_n} = \infty, \tag{2}$$

and for each j and sufficiently large n, there is a $R_{j,n} \leqslant r_n$, such that

$$L(R_{j,n}, f_j) > r_n, \quad j = 2, \dots, N. \tag{3}$$

Define $g(z) = f_1 \circ \cdots \circ f_N(z)$. Let D be a hyperbolic domain in $\mathbf{C}$ such that for $p > 0$, $g^p(z) : D \to D$ is analytic. If for some $a \in D$, $g^{np}(a) \to b \in \partial D$, assume, in addition, that b is not an essential singularity point of $g(z)$. Then D is bounded.

We make remarks on Theorem 1.3.

(i) Theorem 1.3 is a generalisation of results in [**12**]. For example it was proved in [**12**] that a transcendental meromorphic function has no unbounded (pre)periodic Fatou components if it satisfies (2).

(ii) If f is a transcendental meromorphic function of order $\lambda = \lambda(f) < 1/2$ and $\delta(\infty, f) > 1 - \cos\pi\lambda$, then for arbitrarily large $\tilde{r} > 0$, we have a $R \leqslant \tilde{r}$ such that $L(R, f) > \tilde{r}$. In fact, we can take $\lambda(f) < \alpha < 1/2$ such that $\delta(\infty, f) > 1 - \cos\pi\alpha$. From [**6**], the set

$$F := \left\{ r > 1 : \log L(r, f) > \frac{\pi\alpha}{\sin\pi\alpha}\bigl(\cos\pi\alpha + \delta(\infty, f) - 1\bigr)T(r, f) \right\}$$

has lower logarithmic density at least $1 - \bigl(\lambda(f)\bigr)/\alpha > 0$. Therefore, for all sufficiently large $r > 0$, there exists a $R \in (r^{1/d}, r)$ such that

$$L(R, f) > e^{\beta T(R,f)} > R^d \geqslant r,$$

where $\beta = \bigl[(\pi\alpha)/(\sin\pi\alpha)\bigr]\bigl(\cos\pi\alpha + \delta(\infty, f) - 1\bigr) > 0$ and $\bigl(\alpha/\lambda(f)\bigr) < d < +\infty$.

This paper was mainly completed by the first author.

2. Proof of Theorem 1.1

In order to prove the Theorem 1.1, we need the following results. The first result we need is a special version of the main result in [**1**].

Lemma 2.1. *Let $f(z)$ be transcendental and meromorphic in $\mathbf{C}$ with finite positive lower order $\mu = \mu(f)$ and such that $\delta = \delta(\infty, f) > 0$. Define for $r > 0$*

$$D(r) := \left\{\theta \in [0, 2\pi) : \log^{+}\left|f(re^{i\theta})\right| > \frac{1}{\log r}T(r, f)\right\}. \tag{4}$$

Then there exists an unbounded sequence $\{r_j\}$ of r such that for sufficiently small $\varepsilon > 0$ we have a j_0 such that when $j \geqslant j_0$,

$$\operatorname{mes} D(r_j) \geqslant \min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta}{2}}\right\} - \varepsilon. \tag{5}$$

An open set is called hyperbolic if it has at least three boundary points in $\overline{\mathbf{C}}$. We define the hyperbolic metric on an open set by the hyperbolic metrics of its components. Let W be a hyperbolic open set in $\mathbf{C}$. For an $a \in \mathbf{C}\backslash W$, define

$$C_W(a) = \inf\left\{\lambda_W(z)|z - a| : \forall z \in W\right\},$$

where $\lambda_W(z)$ is the hyperbolic density on W. It is well-known that $\lambda_W(z)\delta_W(z) \leqslant 1$, $z \in W$, where $\delta_W(z)$ is the Euclidean distance of z to ∂W and if every component of W is simply connected, then $C_W(a) \geqslant 1/2$. For $r > 0$ and $\theta_1, \theta_2 \in [0, 2\pi), \theta_1 < \theta_2$, define

$$\Omega(r; \theta_1, \theta_2) := \left\{z : \arg z \in (\theta_1, \theta_2),\ |z| > r\right\}.$$

Lemma 2.2. *Let $f(z)$ be analytic in $\Omega(r_0; \theta_1, \theta_2)$, U a hyperbolic domain and*

$$f : \Omega(r_0; \theta_1, \theta_2) \to U.$$

If there exists a point $a \in \partial U\backslash\{\infty\}$, such that $C_U(a) > 0$, then there exists a constant $d > 0$ such that for sufficiently small $\varepsilon > 0$, we have

$$\left|f(z)\right| = O\left(|z|^d\right), z \to \infty,\ z \in \Omega(r_0; \theta_1 + \varepsilon, \theta_2 - \varepsilon). \tag{6}$$

Proof: Write $\Omega = \Omega(r_0; \theta_1, \theta_2)$. Since $f(\Omega) \subset U$, from the Schwarz-Pick Lemma we have

$$\lambda_U\left(f(z)\right)\left|f'(z)\right| \leqslant \lambda_\Omega(z),\ z \in \Omega. \tag{7}$$

From the definition of $C_U(a)$, we have

$$\lambda_U\left(f(z)\right)\left|f'(z)\right| \geqslant C_U(a)\frac{\left|f'(z)\right|}{\left|f(z) - a\right|},\ z \in \Omega. \tag{8}$$

On the other hand, since for $\varepsilon > 0$ and $z \in \Omega_0 = \Omega(r_0; \theta_1 + \varepsilon, \theta_2 - \varepsilon)$, $\delta_\Omega(z) \geqslant |z|\sin\varepsilon$, we have

$$\lambda_\Omega(z) \leqslant \left[|z|\sin\varepsilon\right]^{-1},\ z \in \Omega_0. \tag{9}$$

Combining (7), (8) and (9) gives

$$c\frac{|f'(z)||dz|}{|f(z)-a|} \leqslant \frac{1}{|z|}|dz|,\ z \in \Omega_0, \tag{10}$$

where $c = C_U(a)\sin\varepsilon$. We draw a curve γ in Ω_0 from a fixed point b to z by connecting b and $|b|e^{i\theta}$, $\theta = \arg z$ along the circle $\{w : |w| = |b|\}$ and $|b|e^{i\theta}$ and z along the radial $\{w : \arg w = \theta\}$. Then integrating the both sides of (10) along γ implies the desired inequality (6). □

PROOF OF THEOREM 1.1: If $\mu(f) = 0$, from the remark (ii) on Theorem 1.3 we know that f satisfies (1) and therefore from Theorem 1.2 we have $E = [0, 2\pi)$. Now we assume $\mu > 0$. For the sake of convenience, put

$$\sigma = \min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta}{2}}\right\}.$$

Now we conversely suppose that $\operatorname{mes} E < \sigma$. Take a $t > 0$ such that $\sigma - \operatorname{mes} E > t > 0$. Since E is closed, $S = [0, 2\pi) \setminus E$ consists of (at most countablely many) open intervals I from which we can then find finitely many open intervals I_i $(i = 1, 2, \ldots, m)$ such that $\operatorname{mes}\left(S \setminus \bigcup_{i=1}^{m} I_i\right) < K/2$, where $K = \sigma - \operatorname{mes} E - t > 0$. Under the assumption of Theorem 1.1, an application of Lemma 2.1 implies that there exists a sequence $\{r_j\}$ of positive numbers such that $\operatorname{mes} D(r_j) > \sigma - t > 0$, where $D(r_j)$ is defined as in (4). Obviously

$$\operatorname{mes}\bigl(D(r_j) \cap S\bigr) = \operatorname{mes}\bigl(D(r_j)\bigr) \setminus \bigl(E \cap D(r_j)\bigr) \geqslant \operatorname{mes} D(r_j) - \operatorname{mes} E \geqslant K > 0.$$

Thus for each j we have

$$\begin{aligned}\operatorname{mes}\left(\left(\bigcup_{i=1}^{m} I_i\right) \cap D(r_j)\right) &= \operatorname{mes}(S \cap D(r_j)) - \operatorname{mes}\left(\left(S \setminus \bigcup_{i=1}^{m} I_i\right) \cap D(r_j)\right) \\ &> K - \frac{K}{2} = \frac{K}{2},\end{aligned}$$

so there exists an open interval $I = I_{i_0} \subset S$ such that for infinitely many j,

$$\operatorname{mes}\bigl(D(r_j) \cap I\bigr) > \frac{K}{2m} > 0. \tag{11}$$

It is easy to see that we can assume that for each j, (11) holds. Write $I = (\alpha, \beta)$. Take a positive number ε such that

$$\operatorname{mes}\bigl(D(r_j) \cap I_{2\varepsilon}\bigr) > \frac{K}{3m} > 0\ (j = 1, 2, \ldots), \tag{12}$$

where we denote by I_d the interval $(\alpha + d, \beta - d)$ for $0 < 2d < \beta - \alpha$. It is easy to see from $I \cap E = \emptyset$ that there exists a positive R such that $\Omega(R; I_\varepsilon) = \bigl\{z \in C : |z| \geqslant R \text{ and } \arg z \in I_\varepsilon\bigr\} \subset F_f$. Now by applying Lemma 2.2 to f in $\Omega(R; I_\varepsilon)$, we have

$$\bigl|f(z)\bigr| \leqslant C_0|z|^p,\ z \in \Omega(R; I_{2\varepsilon}), \tag{13}$$

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where C_0 and p are both positive constants. Then

$$\int_{I_{2\varepsilon}\cap D(r_j)} \log^+|f(r_je^{i\theta})|d\theta \leqslant \int_I p\log r_j d\theta + O(1) = O(\log r_j). \tag{14}$$

On the other hand, applying Lemma 2.1 to f gives

$$\int_{I_{2\varepsilon}\cap D(r_j)} \log^+|f(r_je^{i\theta})|d\theta \geqslant \operatorname{mes}\big(I_{2\varepsilon}\cap D(r_j)\big)\frac{T(r_j,f)}{\log r_j} > \frac{K}{3m}\frac{T(r_j,f)}{\log r_j}. \tag{15}$$

Combining (14) with (15) gives

$$T(r_j,f) = O\big((\log r_j)^2\big).$$

Then $\mu(f) = 0$. We get a contradiction. □

Theorem 1.1 follows.

3. Proof of Theorem 1.3

In order to prove Theorem 1.3, we need the following result.

Lemma 3.1. *Let D be a domain with at least three boundary points and f be meromorphic in $\mathbf{C}$ except possibly at most countably many essential sigularity points such that $f(D) \subset D$. Then one of the following mutually exclusive possibilities can occur.*

(1) *There is a subsequence of $\{f^n(z)\}$ which converges to z in D.*

(2) *$f^n(z) \to b \in \overline{D}$, $\overline{D}$ is the closure of D.*

Lemma 3.1 can be proved from the arguments of Heins [**8**]. We also need the following result, which is due to Zheng [**12**] and of independent significance.

Lemma 3.2. *Let U be an unbounded hyperbolic domain and $f : U \to U$ analytic. If*

$$f^n(z)|_U \to \infty, \quad n \to \infty,$$

then there exists a curve γ tending to infinity and a constant $\mathcal{L} > 1$, such that $f(\gamma) \subset \gamma$ and

$$\frac{|z|}{\mathcal{L}} \leqslant |f(z)| \leqslant \mathcal{L}|z|, \ \forall z \in \gamma.$$

Proof of Theorem 1.3: Suppose that D is unbounded. Take a point $a \in D$ and a positive number M such that

$$|a| < M \text{ and } \left|f_j \circ \cdots \circ f_N\big(g^i(a)\big)\right| < M, \text{ for } j = 1,2,\ldots,N \text{ and } i = 0,1,\ldots,p. \tag{16}$$

From (2) and (3) for arbitrarily large $K > 3$ we can take a sufficiently large $R > M$ such that

$$L(R,f_1) > KR \tag{17}$$

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and for each j there exists a $M \leqslant R_j \leqslant R$ such that

$$L(R_j, f_j) > R. \tag{18}$$

Now we draw in D a curve γ connecting a and a point in $\{|z| = R\}$ such that $\gamma \subset \{|z| \leqslant R\}$. It is easy to see that $\{|z| = R_N\} \cap \gamma \neq \emptyset$ and from (16), $f_N(\gamma) \cap \{|z| < M\} \neq \emptyset$, and then from (18) when $j = N$ we have

$$f_N(\gamma) \cap \{|z| = R\} \neq \emptyset. \tag{19}$$

Obviously from (16), $f_{N-1} \circ f_N(\gamma) \cap \{|z| < M\} \neq \emptyset$ and from (19), $f_N(\gamma) \cap \{|z| = R_{N-1}\} \neq \emptyset$. Again applying (18) when $j = N - 1$ gives that $f_{N-1} \circ f_N(\gamma) \cap \{|z| = R\} \neq \emptyset$. Inductively we get

$$f_2 \circ \cdots \circ f_{N-1} \circ f_N(\gamma) \cap \{|z| = R\} \neq \emptyset. \tag{20}$$

Then there exists a point $z_1 \in \gamma$ such that $|f_2 \circ \cdots \circ f_{N-1} \circ f_N(z_1)| = R$, and from (17) we have $|g(z_1)| > KR$. This implies that $g(\gamma) \cap \{|z| = R\} \neq \emptyset$. Take a segment of $g(\gamma)$, denoted by γ_1, from $g(a)$ to $\{|z| = R\}$ such that $\gamma_1 \subset \{|z| \leqslant R\}$. Repeating the above process to γ_1 gives that $g(\gamma_1) \cap \{|z| = R\} \neq \emptyset$, so $g^2(\gamma) \cap \{|z| = R\} \neq \emptyset$. Inductively we obtain $g^{p-1}(\gamma) \cap \{|z| = R\} \neq \emptyset$. Now take a curve γ_{p-1} from $g^{p-1}(\gamma)$ connecting $g^{p-1}(a)$ and a point in $\{|z| = R\}$ such that $\gamma_{p-1} \subset \{|z| \leqslant R\}$. As we did for γ in the first step, we have (20) for γ_{p-1}, so that we have (20) for $g^{p-1}(\gamma)$. Then there exists a point $z^* \in \gamma$ such that $\left|f_2 \circ \cdots \circ f_{N-1} \circ f_N(g^{p-1}(z^*))\right| = R$, and from (17) we have $|g^p(z^*)| > KR$.

If $\{g^n(a)\}$ is unbounded, then for some $0 \leqslant m \leqslant p-1$, $g^{np+m}|_D \to \infty (n \to \infty)$. It is easy to see from unboundedness of D and (2) and (3) that $g^m(D)$ is also unbounded. Since $g^p(g^m(D)) \subset g^m(D)$, applying Lemma 3.2 to g^p in $g^m(D)$ gives the existence of Γ tending to ∞ from $g^m(a)$ in $g^m(D)$ such that $|g^p(z)| \leqslant L|z|$, $z \in \Gamma$, where L is a positive constant. In the above discussion, we replace γ by a segment curve of Γ in $\{|z| \leqslant R\}$, D by $g^m(D)$ and K by $2L$, then we have a point $z^* \in \Gamma$ such that $|g^p(z^*)| > 2LR$. On the other hand, $|g^p(z^*)| \leqslant L|z^*| \leqslant LR$, so that $LR \geqslant 2LR$. We get a contradiction. Then $\{g^n(a)\}$ is bounded.

If $\{g^n|_D\}$ has only constant limit functions, then from Lemma 3.1 $g^{np+m}(a) \to p_m$ $(n \to \infty), 0 \leqslant m \leqslant p-1$. If $p_0 \in D$, then $p_m \in g^m(D), 0 \leqslant m \leqslant p-1$ so that we can reqiure that (16) also holds for $i > p$. If $p_0 \in \partial D$, we can also do this under the assumption of Theorem 1.3 about this case. Thus it is clear that $g^{np}(\gamma)$ cannot tend to p_0. A contradiction we get gives that $\{g^n|_D\}$ has a non-constant limit function. Then all the limit points of $\{g^n(a)\}$ must be analytic points of $f_j \circ \cdots \circ f_N(z)$, $(j = 1, 2, \ldots, p)$ so that we can make (16) hold for $i > p$. Applying Lemma 3.1 implies that there is a subsequence $\{g^{n_k p}\}$ of $\{g^{np}\}$ such that

$$g^{n_k p}(z) \to z, \ (k \to \infty)$$

in D. When n_k is sufficiently large, we have

$$|g^{n_k p}(z)| < (K-1)|z|, \; z \in \gamma. \tag{21}$$

On the other hand, from (21) and the discussion in the first paragraph, there is a $z_{n_k} \in \gamma$ such that

$$KR < |g^{n_k p}(z_{n_k})| < (K-1)|z_{n_k}| \leqslant (K-1)R.$$

This is impossible. □

Theorem 1.3 follows.

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