THE BIANCHI IDENTITIES IN THE GENERALIZED THEORY OF GRAVITATION

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1. General remarks. The heuristic strength of the general principle of relativity lies in the fact that it considerably reduces the number of imaginable sets of field equations; the field equations must be covariant with respect to all continuous transformations of the four coordinates. But the problem becomes mathematically well-defined only if we have postulated the dependent variables which are to occur in the equations, and their transformation properties (field-structure). But even if we have chosen the field-structure (in such a way that there exist sufficiently strong relativistic field-equations), the principle of relativity does not determine the field-equations uniquely. The principle of "logical simplicity" must be added (which, however, cannot be formulated in a non-arbitrary way). Only then do we have a definite theory whose physical validity can be tested *a posteriori.*

The relativistic theory of gravitation bases its field-structure on a symmetric tensor g_{ik}. The most important physical reason for this is that in the special theory we are convinced of the existence of a "light-cone" ($g_{ik}dx^i dx^k = 0$) at each world-point, which separates space-like line-elements from time-like ones. What is the most natural way of generalizing this field-structure? The use of a non-symmetric tensor seems to be the simplest possibility, although this cannot be justified convincingly from a physical standpoint. But the following formal reason seems to me important. For the general theory of gravitation it is essential that we can associate with the covariant tensor g_{ik} a contravariant g^{ik}, through the relation $g_{is}g^{ks} = \delta_i^{\ k} = g_{si}g^{sk}$ (normalized cofactors). This association can be carried over to the non-symmetric case directly. So it is natural to try to extend the theory of gravitation to non-symmetric g_{ik}-fields.

The main difficulty in this attempt lies in the fact that we can build many more covariant equations from a non-symmetric tensor than from a symmetric one. This is due to the fact that the symmetric part, $g_{\underline{ik}}$, and the antisymmetric part, $g_{\overset{\vee}{ik}}$, are tensors independently. Is there a formal point of view which makes one of the many possibilities seem most natural? It seems to me that there is. In the case of the gravitational theory it is essential that besides the g_{ik} tensor we also have the symmetric infinitesimal displacement $\Gamma_{ik}^{\ \ l}$. This is connected with g_{ik} by the equation

$$g_{ik,l} - g_{sk}\Gamma_{il}^{\ \ s} - g_{is}\Gamma_{lk}^{\ \ s} = 0. \tag{1}$$

But in the symmetric case the order of indices does not matter. How shall we generalize (1) to our case? We make use of the following postulate: there

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is a tensor g_{ki}, the "conjugate" of g_{ik}, and a $\Gamma_{ki}{}^{l}$ "conjugate" of $\Gamma_{ik}{}^{l}$. It seems reasonable that conjugates should play equivalent roles in the field-equations. So we require that if in any field-equation we replace g and Γ by their conjugates, we should get an equivalent equation. This requirement replaces symmetry in our system. (See sec. **2**.) If we require that the set of equations (1) should go over into itself under this operation of "conjugation," then the order of indices must be as in (1).

Our main task now is to find out whether there is a sufficiently convincing method of finding a unique set of field-equations for the non-symmetric fields with the above structure. In both previous publications[1] this was solved by forming a variational principle in close analogy to the symmetric case. This way we make sure that the resulting equations will be compatible. The only reason why this derivation may seem not completely satisfactory is that we subject the field *a priori* to two conditions, for reasons of logical simplicity:

$$\Gamma_{\underset{\vee}{is}}{}^{s} = \tfrac{1}{2}(\Gamma_{is}{}^{s} - \Gamma_{si}{}^{s}) = 0, \tag{2}$$

$$\mathfrak{g}^{\underset{\vee}{is}}{}_{,s} = \tfrac{1}{2}(\mathfrak{g}^{is} - \mathfrak{g}^{si})_{,s} = 0; \quad (\mathfrak{g}^{is} = g^{is}(-\det g_{ab})^{\frac{1}{2}}). \tag{3}$$

These side-conditions make the derivation more complex than in the gravitational theory, and their formal justification has not been accomplished in a fully satisfactory manner so far.[2]

In the theory of symmetric fields there is a second method of ensuring the compatibility of the field-equations ($R_{ik} = 0$). We must have four identities connecting the equations. These can be derived by contracting the Bianchi-identities which hold for the curvature tensor:

$$R_{ik\dot{l}\dot{m};\dot{n}} = R_{iklm;n} + R_{ikmn;l} + R_{iknl;m} = 0.$$

In this article we shall show that an analogous argument can be used for the justification of the field-equations also in our case. This will give a deeper insight into the structure of non-symmetric fields, and it will demonstrate in a new way that the field-equations chosen for the non-symmetric fields are really the natural ones.

2. Non-symmetric tensors. For the sake of convenient reference we shall sum up the main facts of the calculus of non-symmetric tensors.

Given any tensor A_{ik}, it can be written as the sum of a symmetric tensor $A_{\underline{ik}}$ and an antisymmetric $A_{\underset{\vee}{ik}}$. These are uniquely determined by the relations:

$$A_{\underline{ik}} = \tfrac{1}{2}(A_{ik} + A_{ki}), \tag{4}$$

$$A_{\underset{\vee}{ik}} = \tfrac{1}{2}(A_{ik} - A_{ki}). \tag{5}$$

A complication is introduced into this theory by the fact that besides the fundamental tensor g_{ik} we also have its conjugate

$$\tilde{g}_{ik} = g_{ki}. \tag{6}$$

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[1]*Ann. of Math.*, vol. 46 (1945), no. 4; vol. 47(1946), no. 4.

[2]It is a consequence of (1) that (2) and (3) are equivalent. This will be proven in sec. **5**.

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The other tensors of our theory are defined in terms of g_{ik}. Given a tensor A_{ik}, by its *conjugate* $\tilde{A}_{ik}$ we mean[3] the tensor we get by replacing g_{ik} in the definition of A_{ik} by g_{ki}. (This definition agrees with (6) in particular.) We shall be particularly interested in tensors in whose definition g and $\tilde{g}$ play analogous roles; more precisely those tensors for which replacing g_{ik} by g_{ki} merely changes A_{ik} into A_{ki}, or for which

$$\tilde{A}_{ik} = A_{ki}. \tag{7}$$

A tensor having the property (7) is called *Hermitian*.[3] More generally any function $A \ldots {}_{ik} \ldots$ of the g_{ik} is Hermitian in (ik) if

$$\tilde{A} \ldots {}_{ik} \ldots = A \ldots {}_{ki} \ldots \tag{7a}$$

If $\Gamma_{ik}{}^l$ is defined by (1), then Γ is Hermitian in (ik). This is another way of stating the principle by which we chose the order of indices in (1).[4]

We say that $A \ldots {}_{ik} \ldots$ is anti-Hermitian if

$$\tilde{A} \ldots {}_{ik} \ldots = - A \ldots {}_{ki} \ldots \tag{8}$$

In analogy to (4), (5), we can decompose any tensor uniquely into

$$A_{ik} = \tfrac{1}{2}(A_{ik} + \tilde{A}_{ki}) + \tfrac{1}{2}(A_{ik} - \tilde{A}_{ki}). \tag{9}$$

The first term is the Hermitian, the second the anti-Hermitian part of A_{ik}.

Covariant derivatives still have to be generalized. In the symmetric theory, if $A \ldots {}^i{}_k \ldots$ is any tensor, then

$$A \ldots {}^i{}_k \ldots {}_{;l} = A \ldots {}^i{}_k \ldots {}_{,l} \pm \cdots + A \ldots {}^s{}_k \ldots \Gamma_{sl}{}^i - A \ldots {}^i{}_s \ldots \Gamma_{kl}{}^s \pm \cdots$$

is also a tensor. This is true in our theory also, but we can order the two lower indices of Γ in two ways, in each term (after the first one). If the differentiation index l is to be on the right in a certain term, we put $+$ under the corresponding tensor-index; if on the left, put $-$ under the index. As an illustration we give a new form of (1):

$$g_{\underset{+}{i}\underset{-}{k};l} = g_{ik,l} - g_{sk}\Gamma_{il}{}^s - g_{is}\Gamma_{lk}{}^s = 0. \tag{1a}$$

The theorems about covariant differentiation can be taken over from the symmetric theory, if we are careful to distinguish the two kinds of derivatives. By raising the indices i and k in (1a) we have:

$$g^{\underset{+}{i}\underset{-}{k}}{}_{;l} = g^{ik}{}_{,l} + g^{sk}\Gamma_{sl}{}^i + g^{is}\Gamma_{ls}{}^k = 0. \tag{1b}$$

Sometimes it is even convenient to write things like

$$A_{\underset{-}{i}\underset{+}{k}lm;n} = A_{iklm,n} - A_{sklm}\Gamma_{ni}{}^s - A_{islm}\Gamma_{kn}{}^s$$

[3]The names "conjugate" and "Hermitian" can be justified as follows: an interesting possibility is to choose $g_{\overset{\vee}{ik}}$ imaginary. Then $\tilde{g}$ is really the conjugate of g. Hence $\tilde{A}$ is the conjugate of A, and the definition of "Hermitian" agrees with the usual one.

[4]Thus in our theory the condition of symmetry is generalized to that of being Hermitian. g_{ik}, $\Gamma_{ik}{}^l$, R_{ik} are all Hermitian in (ik).

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but it must be remembered that such expressions are not always tensors, unless + or − occurs under *each* tensor subscript.

If we let $\mathfrak{g}$ stand for the square-root of the negative determinant of g_{ik}, then $\mathfrak{g}$ is a scalar density. We can describe a tensor density as a product of $\mathfrak{g}$ and a tensor. Let us study these densities. Multiply (1) by g^{ik} and sum:

$$\frac{(\det g_{ik})_{,l}}{(\det g_{ik})} - \Gamma_{sl}{}^{s} - \Gamma_{ls}{}^{s} = 0,$$

$$\frac{(\mathfrak{g}^2)_{,l}}{\mathfrak{g}^2} - 2\Gamma_{\underline{ls}}{}^{s} = 0,$$

(10) $$\mathfrak{g}_{,l} - \mathfrak{g}\Gamma_{\underline{ls}}{}^{s} = 0.$$

It is, therefore, natural to define[5] $\mathfrak{g}_{;l}$ as $\mathfrak{g}_{,l} - \mathfrak{g}\Gamma_{\underline{ls}}{}^{s}$.

If (1a) is satisfied, then $\mathfrak{g}_{;l} = 0$. If we do not assume (1), then $g_{\underset{+}{i}\underset{-}{k};l}$ and $g^{\underset{-}{i}\underset{+}{k}}{}_{;l}$ do not vanish but they have tensorial character. Also $\mathfrak{g}_{;l}$ has then the character of a vector density.

We can now calculate the covariant derivative of a tensor density from the rule for differentiating a product. For example:

$$\mathfrak{g}^{ik}{}_{;l} = (\mathfrak{g}g^{ik})_{;l} = \mathfrak{g}_{;l}\, g^{ik} + \mathfrak{g}g^{ik}{}_{;l}.$$

This vanishes, if (1) is satisfied. More explicitly:

$$\begin{aligned}\mathfrak{g}^{ik}{}_{;l} &= (\mathfrak{g}_{,l} - \mathfrak{g}\Gamma_{\underline{ls}}{}^{s})\, g^{ik} + \mathfrak{g}(g^{ik}{}_{,l} + g^{sk}\Gamma_{sl}{}^{i} + g^{is}\Gamma_{ls}{}^{k})\\ &= \mathfrak{g}^{ik}{}_{,l} + \mathfrak{g}^{sk}\Gamma_{sl}{}^{i} + \mathfrak{g}^{is}\Gamma_{ls}{}^{k} - \mathfrak{g}^{ik}\Gamma_{\underline{ls}}{}^{s}.\end{aligned}$$

Therefore we have:

$$\mathfrak{g}^{\underset{+}{i}\underset{-}{k}}{}_{;l} = \mathfrak{g}g^{\underset{+}{i}\underset{-}{k}}{}_{;l} = 0.$$

For completeness we include the following abbreviation:

$$A_{\underset{\cdots}{ikl}} = A_{ikl} + A_{kli} + A_{lik}.$$

3. Properties of the generalized curvature. We start with a non-symmetric Γ and build the curvature tensor as usual by parallel translation of a vector around an infinitesimal area element:

(11) $$R^{i}{}_{klm} = \Gamma_{kl}{}^{i}{}_{,m} - \Gamma_{km}{}^{i}{}_{,l} - \Gamma_{sl}{}^{i}\Gamma_{km}{}^{s} + \Gamma_{sm}{}^{i}\Gamma_{kl}{}^{s}.$$

A direct computation shows that the tensor satisfies the identities:

(12) $$R^{\underset{+}{i}}{}_{\underset{\cdots}{klm};n} = R^{i}{}_{\underset{\cdots}{klm},n} + R^{s}{}_{\underset{\cdot\cdot}{klm}}\Gamma_{\underset{\cdot}{sn}}{}^{i} - R^{i}{}_{\underset{\cdot\cdot}{slm}}\Gamma_{\underset{\cdot}{kn}}{}^{s} = 0.$$

From (11) we can form the covariant curvature tensor in analogy to the symmetric case,

(13) $$R_{iklm} = g_{si}R^{s}{}_{klm}.$$

The choice of g_{si} instead of g_{is} may seem arbitrary, but this is not really so. We have to lower the index i in the identities (12). The contravariant index i has the + differentiation character, so it must be summed with a similar

[5]Since $\Gamma_{ls}{}^{s} = \Gamma_{sl}{}^{s}$, the two kinds of differentiation coincide when applied to $\mathfrak{g}$. This must be so since there is no index which could have a + or − character.

index, i.e. the first index of g. Only this way can we lower the index i in (12) without introducing additional terms. Thus we get the covariant identities

(14)
$$g_{si}R^{s}{}_{\underset{+}{k}\underset{\cdot}{l}\underset{\cdot}{m};\underset{\cdot}{n}} = (g_{s\underset{-}{i}}R^{s}{}_{\underset{+}{k}\underset{\cdot}{l}\underset{\cdot}{m}})_{;\underset{\cdot}{n}} = R_{\underset{-}{i}\underset{+}{k}\underset{\cdot}{l}\underset{\cdot}{m};\underset{\cdot}{n}} = 0.$$

For what follows we must also find the symmetry properties of R_{iklm}. From (11) it is clear that $R^{i}{}_{klm}$ is antisymmetric in (lm). From (13) we see that R_{iklm} has the same property:

(15)
$$R_{iklm} = -R_{ikml}.$$

If we differentiate (1) with respect to m and antisymmetrize with respect to l and m, we have

$$(g_{ik,l} - g_{sk}\Gamma_{il}{}^{s} - g_{is}\Gamma_{lk}{}^{s})_{,m} - (g_{ik,m} - g_{sk}\Gamma_{im}{}^{s} - g_{is}\Gamma_{mk}{}^{s})_{,l} = 0$$

or

$$\begin{aligned} &- g_{sk,m}\Gamma_{il}{}^{s} - g_{is,m}\Gamma_{lk}{}^{s} + g_{sk,l}\Gamma_{im}{}^{s} + g_{is,l}\Gamma_{mk}{}^{s} \\ &- g_{sk}(\Gamma_{il}{}^{s}{}_{,m} - \Gamma_{im}{}^{s}{}_{,l}) - g_{is}(\Gamma_{lk}{}^{s}{}_{,m} - \Gamma_{mk}{}^{s}{}_{,l}) = 0. \end{aligned}$$

Using (1) again on the first four terms and then collecting terms,

$$\begin{aligned} &-g_{sk}(\Gamma_{il}{}^{s}{}_{,m} - \Gamma_{im}{}^{s}{}_{,l} - \Gamma_{tl}{}^{s}\Gamma_{im}{}^{t} + \Gamma_{tm}{}^{s}\Gamma_{il}{}^{t}) \\ &\qquad - g_{is}(\Gamma_{lk}{}^{s}{}_{,m} - \Gamma_{mk}{}^{s}{}_{,l} - \Gamma_{lt}{}^{s}\Gamma_{mk}{}^{t} + \Gamma_{mt}{}^{s}\Gamma_{lk}{}^{t}) = 0 \end{aligned}$$

or, using (11), (13), we have

(16)
$$R_{kilm} = -\tilde{R}_{iklm}.$$

This expresses that R_{iklm} is anti-Hermitian in (ik); this is the manner in which the antisymmetry of R_{iklm} (in the gravitational theory) generalizes to our case.

In (14) it is not immediately clear that $R_{\underset{-}{i}\underset{+}{k}\underset{\cdot}{l}\underset{\cdot}{m};\underset{\cdot}{n}}$ is a tensor. We are now in a position to give a more useful form for (14) in which this is obvious.

$$\begin{aligned} R_{\underset{-}{i}\underset{+}{k}\underset{-}{l}\underset{+}{m};n} + R_{\underset{-}{i}\underset{+}{k}\underset{+}{m}\underset{+}{n};l} + R_{\underset{-}{i}\underset{+}{k}\underset{-}{n}\underset{-}{l};m} &= R_{\underset{-}{i}\underset{+}{k}\underset{\cdot}{l}\underset{\cdot}{m};\underset{\cdot}{n}} - R_{iksm}\Gamma_{nl}{}^{s} - R_{ikls}\Gamma_{mn}{}^{s} \\ &\quad - R_{iksn}\Gamma_{ml}{}^{s} - R_{ikms}\Gamma_{nl}{}^{s} - R_{iksl}\Gamma_{mn}{}^{s} - R_{ikns}\Gamma_{ml}{}^{s}. \end{aligned}$$

The first term on the right side of the equation vanishes by (14), the last six cancel out due to (15). Therefore,

(14a)
$$R_{\underset{-}{i}\underset{+}{k}\underset{-}{l}\underset{+}{m};n} + R_{\underset{-}{i}\underset{+}{k}\underset{+}{m}\underset{+}{n};l} + R_{\underset{-}{i}\underset{+}{k}\underset{-}{n}\underset{-}{l};m} = 0.$$

4. The field-equations. We are now in a position to carry out the derivation of the identities for the field equations. In analogy to the gravitational theory, we contract (14a) by $g^{mi}g^{kl}$. (Note that the order of the indices is determined by the differentiation character of the corresponding indices in (14a).) Making use of (15), we get

$$g^{mi}g^{kl}[R_{\underset{-}{i}\underset{+}{k}\underset{-}{l}\underset{+}{m};n} - R_{\underset{-}{i}\underset{+}{k}\underset{+}{n}\underset{+}{m};l} - R_{\underset{-}{i}\underset{+}{k}\underset{-}{l}\underset{-}{n};m}] = 0$$

or using (1a),

(17)
$$g^{kl}[g^{mi}R_{\underset{+}{i}\underset{-}{k}lm}]_{;n} - g^{kl}[g^{mi}R_{\underset{+}{i}\underset{+}{k}nm}]_{;l} - g^{mi}[g^{kl}R_{\underset{-}{i}kl\underset{-}{n}}]_{;m} = 0.$$

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Let us define

$$(18) \qquad R_{kl} = g^{mi}R_{iklm}$$

$$(19) \qquad S_{mi} = g^{kl}R_{iklm}$$

where

$$(18a) \qquad R_{kl} = g^{mi}g_{si}R^{s}{}_{klm} = \delta_s{}^{m}R_{klm}{}^{s} = \Gamma_{kl}{}^{s}{}_{,s} - \Gamma_{ks}{}^{s}{}_{,l} - \Gamma_{tl}{}^{s}\Gamma_{ks}{}^{t} + \Gamma_{ts}{}^{s}\Gamma_{kl}{}^{t}.$$

Then we have

$$(17a) \qquad g^{kl}[R_{\underset{+}{k}\,\underset{-}{l}\,;n} - R_{\underset{+}{k}\,\underset{+}{n};l} - S_{\underset{-}{n}\,\underset{-}{l}\,;k}] = 0.$$

We need some connection between R and S. From (15), (16) we see that

$$R_{kiml} = \tilde{R}_{iklm}.$$

Multiply by $g^{im}(=\tilde{g}^{mi})$ and sum (i.e. contract):

$$(20) \qquad S_{lk} = \tilde{R}_{kl}.$$

If R were Hermitian, R and S would be identical. Hence we have a new reason for requiring that R_{kl} should be Hermitian. But from (18a) we see that R_{kl} has an anti-Hermitian part (compare with (9)):

$$(21) \qquad \tfrac{1}{2}(R_{kl} - \tilde{R}_{lk}) = \tfrac{1}{2}[(\Gamma_{sl}{}^{s}{}_{,k} - \Gamma_{ks}{}^{s}{}_{,l}) - \Gamma_{kl}{}^{t}(\Gamma_{st}{}^{s} - \Gamma_{ts}{}^{s})].$$

From (10) we see that

$$(22) \qquad \Gamma_{\underset{\smile}{ls}}{}^{s} = \frac{g_{,l}}{g} = (\tfrac{1}{2}\log|\det g_{ik}|)_{,l}.$$

Therefore,

$$\Gamma_{sl}{}^{s}{}_{,k} - \Gamma_{ks}{}^{s}{}_{,l} = \Gamma_{\underset{\vee}{sl}}{}^{s}{}_{,k} - \Gamma_{\underset{\vee}{ks}}{}^{s}{}_{,l}$$

$$(21a) \qquad \tfrac{1}{2}(R_{kl} - \tilde{R}_{lk}) = -\tfrac{1}{2}(\Gamma_{\underset{\vee}{ls}}{}^{s}{}_{,k} + \Gamma_{\underset{\vee}{ks}}{}^{s}{}_{,l} - \Gamma_{kl}{}^{t}\Gamma_{\underset{\vee}{ts}}{}^{s}).$$

From this we see that R_{kl} is Hermitian if we subject the field to the four conditions

$$(2) \qquad \Gamma_{\underset{\vee}{is}}{}^{s} = 0.$$

It then follows from (20) that

$$(20a) \qquad S_{lk} = R_{lk},$$

and (17a) becomes

$$(17b) \qquad g^{kl}[R_{\underset{+}{k}\,\underset{-}{l}\,;n} - R_{\underset{+}{k}\,\underset{+}{n};l} - R_{\underset{-}{n}\,\underset{-}{l}\,;k}] = 0.$$

These identities hold for all fields where Γ is defined by (1) and is subject to (2). We might jump to the conclusion that the field equations should stipulate the vanishing of all R_{kl}. This set, together with (1) and (2) would, however, be overdetermined. We can get a weaker set of equations by observing how $R_{\underset{\vee}{kl}}$ enters (17b). The contribution of $R_{\underset{\vee}{kl}}$ to the equations is:

$$g^{kl}[R_{\underset{\vee}{\underset{+}{k}\,\underset{-}{l}}\,;n} - R_{\underset{\vee}{\underset{+}{k}\,\underset{+}{n}}\,;l} - R_{\underset{\vee}{\underset{-}{n}\,\underset{-}{l}}\,;k}]$$

which can be written as

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$$g^{kl}[R_{\underset{\vee}{kl},n} - R_{\underset{\vee}{sl}}\Gamma_{kn}{}^{s} - R_{\underset{\vee}{ks}}\Gamma_{nl}{}^{s} - R_{\underset{\vee}{kn},l}$$
$$+ R_{\underset{\vee}{sn}}\Gamma_{kl}{}^{s} + R_{\underset{\vee}{ks}}\Gamma_{nl}{}^{s} - R_{\underset{\vee}{nl},k} + R_{\underset{\vee}{sl}}\Gamma_{kn}{}^{s} + R_{\underset{\vee}{ns}}\Gamma_{kl}{}^{s}]$$
$$= g^{kl}[R_{\underset{\vee}{kl},n} - R_{\underset{\vee}{kn},l} - R_{\underset{\vee}{nl},k}]$$
$$= g^{kl}[R_{\underset{\vee}{kl},n} + R_{\underset{\vee}{nk},l} + R_{\underset{\vee}{ln},k}]$$
$$= g^{kl}\, R_{\underset{\dot{\vee}\cdot}{kl,n}}.$$

Since we see that R_{kl} enters the equations only in the combination $R_{\underset{\dot{\vee}\cdot}{kl,n}}$, it is natural to choose the field equations for $R_{\underset{\vee}{kl}}$ as

(23) $$R_{\underset{\dot{\vee}\cdot}{kl,n}} = 0$$

instead of $R_{kl} = 0$. So we get the field equations

(2) $$\Gamma_{\underset{\vee}{is}}{}^{s} = 0$$

(24) $$R_{\underset{-}{kl}} = 0$$

(23) $$R_{\underset{\dot{\vee}\cdot}{kl,n}} = 0,$$

where the $\Gamma_{ik}{}^{l}$ are defined by:

(1a) $$g_{\underset{+\,-}{i\;k};l} = 0.$$

The foregoing derivation shows how naturally we can extend general relativity theory to a non-symmetric field, and that the field-equations previously published are really the natural generalizations of the gravitational equations. If we were sure that a non-symmetric tensor g_{ik} is the right means for describing the structure of the generalized field, then we could hardly doubt that the above equations are the correct ones.

5. The variational principle. For comparison we include a derivation of the equations based on a variational principle. This is formally simpler than the previous derivation, but it has the disadvantage of making use of two apparently arbitrary restrictions of the $g - \Gamma$ field:

(2) $$\Gamma_{\underset{\vee}{is}}{}^{s} = 0,$$

(3) $$\mathfrak{g}^{\underset{\vee}{is}}{}_{,s} = 0.$$

On the other hand the equations (1) are deduced from the variation; we need not postulate them. It is advantageous to make use of Palatini's method in this derivation. As in sec. **3**, we form the curvature tensor:

(11) $$R^{i}{}_{klm} = \Gamma_{kl}{}^{i}{}_{,m} - \Gamma_{km}{}^{i}{}_{,l} - \Gamma_{sl}{}^{i}\Gamma_{km}{}^{s} + \Gamma_{sm}{}^{i}\Gamma_{kl}{}^{s}.$$

By generalizing Palatini's method to the non-symmetric case, it is easy to verify that

(25) $$\delta R^{i}{}_{klm} = (\delta\Gamma_{\underset{+\,-}{k\;l}}\overset{i}{+})_{;m} - (\delta\Gamma_{\underset{+\,+}{k\;m}}\overset{i}{+})_{;l}.$$

We choose the Hamiltonian function

(26) $$\mathfrak{H} = \mathfrak{g}^{kl}R_{kl}$$

(26a) $$\mathfrak{H} = \delta_{i}{}^{m}\mathfrak{g}^{kl}R^{i}{}_{klm}.$$

We vary (26a) relative to the Γ's:

$$(27)\qquad \begin{aligned}\delta\mathfrak{H} &= \delta_i{}^m\mathfrak{g}^{kl}(\delta R^i{}_{klm})\\ &= \delta_i{}^m\mathfrak{g}^{kl}[(\delta\Gamma_{\underset{+}{k}\,l}{}^{\underset{+}{i}})_{;m} - (\delta\Gamma_{\underset{+}{k}\,\underset{+}{m}}{}^{\underset{+}{i}})_{;l}].\end{aligned}$$

For brevity we set

$$(28)\qquad \mathfrak{A}^m = \delta_i{}^m\mathfrak{g}^{kl}(\delta\Gamma_{kl}{}^i) = \mathfrak{g}^{kl}(\delta\Gamma_{kl}{}^m)$$

$$(29)\qquad \mathfrak{B}^l = \delta_i{}^m\mathfrak{g}^{kl}(\delta\Gamma_{km}{}^i) = \mathfrak{g}^{kl}(\delta\Gamma_{km}{}^m).$$

Then we can write (27) as

$$\begin{aligned}\delta\mathfrak{H} = {}& \mathfrak{A}^{\underset{+}{m}}{}_{;m} - (\delta_{\underset{+}{i}}{}^{\underset{+}{m}}\mathfrak{g}^{\underset{+}{k}\,\underset{+}{l}})_{;m}(\delta\Gamma_{kl}{}^i)\\ & - \mathfrak{B}^{\underset{-}{l}}{}_{;l} + (\delta_{\underset{+}{i}}{}^{\underset{+}{m}}\mathfrak{g}^{\underset{+}{k}\,\underset{-}{l}})_{;l}(\delta\Gamma_{km}{}^i).\end{aligned}$$

We have to form the integral of $\delta\mathfrak{H}$. Let us see what $\mathfrak{A}^{\underset{+}{m}}{}_{;m}$ contributes to the integral. (See sec. **2**.)

$$(30)\qquad \begin{aligned}\mathfrak{A}^{\underset{+}{m}}{}_{;m} &= \mathfrak{A}^m{}_{,m} + \mathfrak{A}^s\Gamma_{sm}{}^m - \mathfrak{A}_{\underline{ms}}{}^s\\ &= \mathfrak{A}^m{}_{,m} + \mathfrak{A}^m\Gamma_{\underset{\vee}{ms}}{}^s.\end{aligned}$$

The first term is an ordinary divergence, and hence contributes nothing to the integral. We see that we need (2) to make the second term vanish. By subjecting the field to (2) we make sure that $\mathfrak{A}^{\underset{+}{m}}{}_{;m}$ (and similarly $\mathfrak{B}^{\underset{-}{l}}{}_{;l}$) contributes nothing to the integral. So we may omit these from (27a) and write:

$$(27b)\qquad \delta\mathfrak{H} = [-(\delta_{\underset{+}{i}}{}^{\underset{+}{m}}\mathfrak{g}^{\underset{+}{k}\,\underset{-}{l}})_{;m} + (\delta_{\underset{+}{i}}{}^{\underset{+}{l}}\mathfrak{g}^{\underset{+}{k}\,\underset{-}{m}})_{;m}](\delta\Gamma_{kl}{}^i).$$

Or since $\delta_{\underset{+}{i}}{}^{\underset{+}{l}}{}_{;m}$ vanishes:

$$(27c)\qquad \delta\mathfrak{H} = [-\,\mathfrak{g}^{\underset{+}{k}\,\underset{-}{l}}{}_{;m} + \mathfrak{g}^{\underset{+}{k}\underset{-}{m}}{}_{;m}\delta_{\underset{+}{i}}{}^{\underset{+}{l}}](\delta\Gamma_{kl}{}^i).$$

We cannot conclude yet that the quantity in brackets vanishes, because the $\Gamma_{kl}{}^i$ are not independent but satisfy (2). But we could conclude the vanishing of these quantities if they depended on only 60 parameters instead of the 64 $\mathfrak{g}^{\underset{+}{k}\,\underset{-}{l}}{}_{;\underset{+}{i}}$. This is actually so, for the following reason: we have

$$(31)\qquad \tfrac{1}{2}(\mathfrak{g}^{\underset{+}{k}\,\underset{-}{l}}{}_{;l} - \mathfrak{g}^{\underset{+}{l}\,\underset{-}{k}}{}_{;l}) = \mathfrak{g}^{\underset{\vee}{kl}}{}_{,l} - \mathfrak{g}^{\underline{kl}}\Gamma_{\underset{\vee}{ls}}{}^s.$$

By subjecting the field to (2) and (3), we make sure that these four quantities vanish. Hence only 60 of the $\mathfrak{g}^{\underset{+}{k}\,\underset{-}{l}}{}_{;i}$ are independent. The same must be true of the square bracketed quantities in (27c). Thus we can conclude from (27c) that all these vanish. Contracting with respect to l and i we have $\mathfrak{g}^{\underset{+}{k}\,\underset{-}{m}}{}_{;m} = 0$. Hence all the $\mathfrak{g}^{\underset{+}{k}\,\underset{-}{l}}{}_{;i}$ vanish. Therefore also the $g_{\underset{+}{k}\,\underset{-}{l}\,;i}$. (See (1b), (1c).) Thus we have derived that

$$(1a)\qquad g_{\underset{+}{i}\,\underset{-}{k}\,;l} = 0.$$

(It follows from these and (31) that either of the conditions (2) and (3) implies the other one.) We still have to vary (21) relative to $\mathfrak{g}^{ik}$. But we must remember that the $\mathfrak{g}^{ik}$ satisfy (3). This can be done most easily by setting

$$\mathfrak{g}^{ik}_{\vee} = \mathfrak{g}^{iks}{}_{,s} \tag{32}$$
$$\mathfrak{g}^{ik} = \mathfrak{g}^{\underline{ik}} + \mathfrak{g}^{iks}{}_{,s}$$

and varying with respect to $\mathfrak{g}^{\underline{ik}}$ and $\mathfrak{g}^{iks}$, which are independent. ($\mathfrak{g}^{iks}$ is a tensor density antisymmetric in each pair of indices.) We get the equations

$$R_{\underset{\vee}{kl};n} = 0, \tag{23}$$

$$R_{\underline{kl}} = 0. \tag{24}$$

This completes the derivation of the field-equations.

We can further justify the *a priori* assumption of (2) by the fact that this equation is necessary and sufficient to make R_{kl} a Hermitian tensor. (See (21a).)

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