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Well-posedness for nonlinear SPDEs with strongly continuous perturbation - ERRATUM

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The authors would like to correct an error in their original article.

Well-posedness for nonlinear SPDEs with strongly continuous perturbation, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, vol. 151, no. 1, pp. 265–295, 2021. doi:10.1017/prm.2020.13, see [1].

Although the results of our work are correct, recently we have realized that some arguments in Eq. (2.5) as well as in lemma 2.5 and its consequences need some modifications. We sincerely apologise for our mistake.

In the sequel, when we mention page numbers, equations, lemmas, definitions, or remarks, they will be systematically compatible with the reference to our article [1].

- p. 269, Eq. (2.5): thanks to the condition $p > \frac{2d}{d+1}$, with $d \in \mathbb{N}^*$ being the space dimension, it follows that, for $1 < p < d$,

$$p^* := \frac{dp}{d-p} > p' = \frac{p}{p-1}$$

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and therefore $L^{p^*}(D)$ is continuously embedded into $L^{p'}(D)$ if $d > p > \frac{2d}{d+1}$. Now, using the Sobolev embedding theorem, we may conclude that $W_0^{1,p}(D) \hookrightarrow L^{p'}(D)$ for all $p > \frac{2d}{d+1}$. Thus, for any $u \in V \subset \overline{B}_{W_0^{1,p}(D)}(0, C_V)$ (cf. [1, p. 269]) and $v \in W_0^{1,p}(D)$ such that $\|v\|_{W_0^{1,p}(D)} \leq 1$, using Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$ for $a, b \geq 0$, we get

$$I_3 = \tau \int_D |F(u)||\nabla v|\, dx \leq \frac{\tau}{p'} \int_D |F(u)|^{p'}\, dx + \frac{\tau}{p}\|v\|^p_{W_0^{1,p}(D)} \leq \frac{\tau}{p'} L^{p'} \|u\|^{p'}_{p'} + \frac{\tau}{p}$$
$$\leq \tau(L^{p'}\|u\|^{p'}_{p'} + 1) \leq \tau C_4(\|u\|^{p'}_{W_0^{1,p}(D)} + 1) \leq \tau C_4(C_V^{p'} + 1) \tag{2.5}$$

with $L > 0$ being the Lipschitz constant of F and recalling that $F(0) = 0$.

- p. 275, lemma 2.5:

LEMMA. *Let* $q := \min(p, p')$ *and* $\widehat{u}_N$, $\widehat{M}_N$ *be the piecewise affine functions defined in definition 2.1. There exists* $K \geq 0$ *not depending on* $N \in \mathbb{N}^*$ *such that*

$$\mathbb{E} \int_0^T \|\frac{d}{dt}(\widehat{u}_N - \widehat{M}_N)\|^q_{W^{-1,p'}(D)}\, dt \leq K.$$

- Proof of lemma 2.5, p. 275 after Eq. (2.25):
recalling that $p' = \frac{p}{p-1}$, using the inequality $(a+b+c)^r \leq 3^{r-1}(a^r + b^r + c^r)$ for $r > 1$ and $a, b, c \geq 0$ as well as Hölder's inequality, we find $C_1, C_2 \geq 0$ such that for any $v \in W_0^{1,p}(D)$,

$$I_1 = \int_D (C_a^2 |\nabla u^{k+1}|^{p-1} + C_a^3 |u^{k+1}|^{p-1} + g(x))|\nabla v|\, dx$$
$$\leq \left(\int_D \left(C_a^2 |\nabla u^{k+1}|^{p-1} + C_a^3 |u^{k+1}|^{p-1} + |g(x)|\right)^{p'}\, dx\right)^{1/p'} \|v\|_{W_0^{1,p}(D)}$$
$$\leq \left(3^{p'-1} \int_D C_a^{2p'} |\nabla u^{k+1}|^p + C_a^{3p'} |u^{k+1}|^p + |g(x)|^{p'}\, dx\right)^{1/p'} \|v\|_{W_0^{1,p}(D)}$$
$$\leq C_1 \left(\|\nabla u^{k+1}\|_p^p + \|u^{k+1}\|_p^p + \|g\|^{p'}_{p'}\right)^{1/p'} \|v\|_{W_0^{1,p}(D)}$$

and, using Poincaré's inequality, we may conclude

$$I_1 \leq C_2 \left(\|u^{k+1}\|^p_{W_0^{1,p}(D)} + \|g\|^{p'}_{p'}\right)^{1/p'} \|v\|_{W_0^{1,p}(D)}.$$

Now, we turn our attention to

$$I_2 = L \int_D |u^{k+1}||\nabla v|\, dx$$

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with $L > 0$ being the Lipschitz constant of F. By Hölder's inequality,

$$I_2 \leq L\|u^{k+1}\|_{p'}\|\nabla v\|_p.$$

Recalling that, for all $p > \frac{2d}{d+1}$ with $d \in \mathbb{N}^*$ being the space dimension, we have the continuous embedding

$$W_0^{1,p}(D) \hookrightarrow L^{p'}(D).$$

Using this embedding together with $u^{k+1} \in W_0^{1,p}(D)$ in the above inequality, it follows that there exists a constant $C_3 \geq 0$ such that

$$I_2 \leq C_3\|u^{k+1}\|_{W_0^{1,p}(D)}\|v\|_{W_0^{1,p}(D)}. \tag{2.27}$$

Choosing $v \in W_0^{1,p}(D)$ such that $\|v\|_{W_0^{1,p}(D)} \leq 1$, from (2.26) and (2.27) for all $t \in (t_k, t_{k+1})$, it follows that

$$\begin{aligned} &\left\|\frac{d}{dt}(\widehat{u}_N - \widehat{M}_N)(t)\right\|_{W^{-1,p'}(D)} \\ &\quad \leq C_2\left(\|u^{k+1}\|^p_{W_0^{1,p}(D)} + \|g\|^{p'}_{p'}\right)^{1/p'} + C_3\|u^{k+1}\|_{W_0^{1,p}(D)}. \end{aligned}$$

For $q = \min(p, p')$, we have $q \leq p$ and $q/p' \leq 1$. Consequently, using the inequality $(a+b)^r \leq 2^{r-1}(a^r + b^r)$ for $r > 1$ and $a, b \geq 0$, it follows that there exist constants $C_4, C_5 \geq 0$ such that for all $t \in (t_k, t_{k+1})$ we have

$$\begin{aligned} &\left\|\frac{d}{dt}(\widehat{u}_N - \widehat{M}_N)(t)\right\|^q_{W^{-1,p'}(D)} \\ &\quad \leq 2^{q-1}\left(C_2^q\left(\|u^{k+1}\|^p_{W_0^{1,p}(D)} + \|g\|^{p'}_{p'}\right)^{q/p'} + C_3^q\|u^{k+1}\|^q_{W_0^{1,p}(D)}\right) \\ &\quad \leq C_4\left(\|u^{k+1}\|^p_{W_0^{1,p}(D)} + \|g\|^{p'}_{p'} + 1 + \|u^{k+1}\|^q_{W_0^{1,p}(D)}\right) \\ &\quad \leq C_5\left(\|u^{k+1}\|^p_{W_0^{1,p}(D)} + \|g\|^{p'}_{p'} + 1\right). \end{aligned}$$

Now we integrate this inequality over (t_k, t_{k+1}), sum over $k = 0, \ldots, N-1$, and take expectation to arrive at

$$\begin{aligned} &\mathbb{E}\int_0^T \left\|\frac{d}{dt}(\widehat{u}_N - \widehat{M}_N)\right\|^q_{W^{-1,p'}} dt \\ &\quad \leq C_5\left(\mathbb{E}\int_0^T \|\nabla u_N^r\|^p_p\, dt + T(\|g\|^{p'}_{p'} + 1)\right) \end{aligned}$$

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where u_N^r is the right-continuous step function defined in definition 2.1. Now, the assertion follows from the bound given in (2.16) given in lemma 2.3. □

- p. 278, line 23:
from lemma 2.4 and lemma 2.5, it follows that $(\widehat{u}_N - \widehat{M}_N)_N$ is bounded in $L^2(\Omega; \mathcal{C}([0,T]; L^2(D)))$ and $(\frac{d}{dt}(\widehat{u}_N - \widehat{M}_N))_N$ is bounded in $L^q(\Omega \times (0,T); W^{-1,p'}(D))$, with $q = \min(p,p')$. Consequently, $(\widehat{u}_N - \widehat{M}_N)_N$ is bounded in $L^q(\Omega; \mathcal{W})$, where

$$\mathcal{W} := \{u \in L^2(0,T; L^2(D)) \mid \frac{d}{dt}u \in L^q(0,T; W^{-1,p'}(D))\}.$$

- p. 280, §2.3.2, Compactness:
now, we apply the theorem of Skorokhod without changing the notation of random variables with the same law in order not to overload the presentation: there exists a new probability space $(\Omega', \mathcal{F}', P')$, such that, passing to a subsequence if necessary,

$$Y_N = (u_N^r - \widehat{u}_N, \widehat{u}_N, \widehat{u}_N - \widehat{M}_N, M_N, \widehat{M}_N, \Phi_N, W_N)$$

converges almost surely in Ω'. Moreover, the semi-implicit Euler scheme (2.1) is satisfied on Ω'. Then, remark $A.1$ and in particular the *a priori* estimates developed in lemma 2.2 to lemma 2.6 hold true on $(\Omega', \mathcal{F}', P')$. Thus, on $(\Omega', \mathcal{F}', P')$,
  - p. 280, 6th bullet point:
  further, there exists a $\mathcal{C}([0,T]; W^{-1,p'}(D))$-valued random variable B_∞ on Ω' such that $\mathcal{L}(B_\infty) = \nu_2$ and $\lim_{N\to\infty}(\widehat{u}_N - \widehat{M}_N) = B_\infty$ in $\mathcal{C}([0,T]; W^{-1,p'}(D))$ a.s. in Ω' and in $L^\ell(\Omega, \mathcal{C}([0,T]; W^{-1,p'}(D)))$ for any $1 \le \ell < 2$ by Vitali's theorem. Thanks to the previous convergence results, $(\widehat{u}_N) = (\widehat{u}_N - \widehat{M}_N + \widehat{M}_N)$ converges a.s. in $\mathcal{C}([0,T]; W^{-1,p'}(D))$. Thus, $B_\infty = u_\infty - M_\infty$, $u_\infty \in \mathcal{C}([0,T]; W^{-1,p'}(D))$ and lemma 2.4, and Vitali's theorem yield the convergence of $\widehat{u}_N$ to u_∞ in $L^\ell(\Omega'; \mathcal{C}([0,T]; W^{-1,p'}(D)))$ for all $1 \le \ell < 2$.
- p. 282, line 13:
since $\widehat{u}_N - \widehat{M}_N$ converges to $u_\infty - M_\infty$ in $L^\ell(\Omega'; \mathcal{C}([0,T]; W^{-1,p'}(D)))$ for all $1 \le \ell < 2$, it follows that

$$\lim_{N\to\infty} I_1 = \int_A \int_0^T \langle (\widehat{u}_\infty - \widehat{M}_\infty)(t), \psi \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \xi_t \, dx \, dt \, dP'.$$

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- p. 283, line 11:
  from (2.44), it follows that

$$\frac{d}{dt}(u_\infty - M_\infty) - \mathrm{div}\,(G + F(u_\infty)) = 0$$

in $L^q(0,T;W^{-1,p'}(D))$ a.s. in Ω' with $q = \min(p,p')$.

References

[1] G. Vallet and A. Zimmermann. Well-posedness for nonlinear SPDEs with strongly continuous perturbation. *Proc. Roy. Soc. Edinburgh Sect. A* **151** (2021), 265–295. doi:10.1017/prm.2020.13.

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