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EXACTNESS OF CUNTZ–PIMSNER C^*-ALGEBRAS

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Abstract Let H be a full Hilbert bimodule over a C^*-algebra A. We show that the Cuntz–Pimsner algebra associated to H is exact if and only if A is exact. Using this result, we give alternative proofs for exactness of reduced amalgamated free products of exact C^*-algebras. In the case in which A is a finite-dimensional C^*-algebra, we also show that the Brown–Voiculescu topological entropy of Bogljubov automorphisms of the Cuntz–Pimsner algebra associated to an A, A Hilbert bimodule is zero.

Keywords: Hilbert bimodules; non-commutative topological entropy; exact C^*-algebras; Bogljubov automorphisms

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1. Introduction and description of results

A C^*-algebra A is said to be *exact* if the functor $B \mapsto B \otimes_{\min} A$ preserves short exact sequences of C^*-algebras and $*$-homomorphisms. Recently there has been great progress in understanding exact C^*-algebras, much of it due to Kirchberg and to Kirchberg and Wassermann [**15**–**19**]. A good general reference for exact C^*-algebras is Wassermann's monograph [**34**].

In [**23**], Pimsner introduced a construction of C^*-algebras $E(H)$ and $O(H)$, respectively, called the *extended Cuntz–Pimsner algebra* and the *Cuntz–Pimsner algebra* of a Hilbert C^*-bimodule H over a C^*-algebra B; we will review his construction at the beginning of § 3. (In this paper we always assume that the Hilbert module H is full as a right B-module, i.e. that $\overline{\operatorname{span}}\,\{\langle \xi, \eta\rangle \mid \xi, \eta \in H\} = B$; for a good general reference on Hilbert C^*-modules, see the monograph of Lance [**20**].) The C^*-algebras $E(H)$ and $O(H)$ are quite important and appear in several areas of operator theory. They are central in the work of Muhly and Solel [**22**] on triangular operator algebras analogous to the algebra of analytic Toeplitz operators on the circle, and their ideal structures have been studied in [**8**, **13**] and [**21**] (see also [**24**]).

Moreover, the C^*-algebra $E(H)$ is related to freeness in the sense of Voiculescu [**31**] (see also [**33**]). For example, Speicher [**29**] has proved that if $H = H_1 \oplus H_2$, then $E(H)$ is

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isomorphic to the reduced amalgamated free product of C^*-algebras $E(H_1)$ and $E(H_2)$, amalgamating over B with respect to the canonical conditional expectations $E(H_\iota) \to B$. The algebras $E(H)$ are also the natural setting for operator-valued analogues of the Gaussian functor. The extended Cuntz–Pimsner algebras have been important in work related to freeness [**26**, **27**] by D.S.

In § 3 of this paper we show that if B is an exact C^*-algebra and if H is any Hilbert B, B-bimodule, then the C^*-algebras $E(H)$ and $O(H)$ are exact. The inspiration for our investigation came from the recent result of K.J.D. [**9**] that every C^*-algebra arising as the reduced amalgamated free product of exact C^*-algebras is exact. Moreover, our proof here of exactness of $E(H)$ resembles at its core the proof found in [**9**] of exactness of reduced amalgamated free products; in both cases, exactness of the algebra is proved by reducing the problem to the question of exactness of algebras generated by words of length $\leqslant n$; this question is then resolved by using a chain of ideals of length $n + 1$.

This paper's main result on exactness of $E(H)$ can be used to give a new proof of the main result of [**9**]. More specifically, in § 5 we show that if $(A, \phi) = (A_1, \phi_1) *_B (A_2, \phi_2)$ is a reduced amalgamated free product of C^*-algebras (amalgamating over B with respect to conditional expectations $\phi_\iota : A_\iota \to B$), then A is a quotient of a subalgebra of $E(H)$, for some Hilbert C^*-bimodule H over $A_1 \oplus A_2$; if A_1 and A_2 are exact, then from the exactness of $E(H)$ we may conclude that A is exact. Before giving this argument, however, we consider in § 4 a special case and give an easier argument showing that if $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$ is a reduced free product (i.e. amalgamating over only the scalars $\boldsymbol{C}$), then A can be embedded in $E(H)$ for some Hilbert C^*-bimodule over $A_1 \otimes_{\min} A_2$; this in turn implies the special case of the main result of [**9**] that the class of unital exact C^*-algebras is closed under taking reduced free products. (See [**11**] for a simpler version of the argument of [**9**] in a special case.)

Topological entropy for automorphisms of unital nuclear C^*-algebras was invented by Voiculescu [**32**] and was extended by Brown [**2**] to apply to automorphisms of exact C^*-algebras. It has many natural properties, and when applied to automorphisms of commutative C^*-algebras gives the usual topological entropy of a homeomorphism. In § 6 we examine the topological entropy of Bogljubov automorphisms of the algebras $E(H)$; we show (Theorem 6.4) that if H is a Hilbert bimodule over B, where B is finite dimensional, then every Bogljubov automorphism of $E(H)$ has topological entropy zero. The fact that we are only able to give results of this sort when B is finite dimensional parallels the current state of knowledge about the topological entropy of automorphisms of reduced amalgamated free product C^*-algebras (see the results and questions in [**10**]).

We have outlined the contents of the entire paper except for § 2 above; there we collect some results about exactness of crossed product C^*-algebras and the topological entropy of some automorphisms of them.

1.1. Standard notation

We will use the convention $\mathbb{N} = \{0, 1, 2, \dots\}$. The words endomorphism, homomorphism, automorphism and representation when applied to C^*-algebras will mean $*$-endomorphism, $*$-homomorphism, $*$-automorphism and $*$-representation.

2. Preliminaries on crossed products

In this section, we will describe and prove some results about topological entropy in crossed product C^*-algebras. These results are applications of Proposition 2.6 of Brown and Choda's paper [**3**], which is in turn based on work of Sinclair and Smith [**28**]. It has come to our attention that Choda [**4**] has independently proved Proposition 2.2, but for completeness we will provide a proof here.

Let us begin by recalling the construction of reduced crossed products, thereby introducing the notation we will use. Let A be a C^*-algebra with a faithful and non-degenerate representation $\sigma : A \to B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Let G be a group taken with discrete topology and let $G \ni g \mapsto \alpha_g \in \mathrm{Aut}(A)$ be an action of G on A via automorphisms. Let $\pi : A \to B(\ell^2(G, \mathcal{H}))$ be the representation given by

$$(\pi(a)\xi)(h) = \sigma(\alpha_{h^{-1}}(a))(\xi(h)) \quad (a \in A,\ \xi \in \ell^2(G, \mathcal{H}),\ h \in G),$$

and let λ be the unitary representation of G on $\ell^2(G, \mathcal{H})$ given by

$$(\lambda_g \xi)(h) = \xi(g^{-1}h) \quad (\xi \in \ell^2(G, \mathcal{H}),\ g, h \in G).$$

We then have

$$\lambda_{g^{-1}} \pi(a) \lambda_g = \pi(\alpha_g(a)) \quad (a \in A,\ g \in G),$$

and the *reduced crossed product C^*-algebra* $\hat{A} = A \rtimes_\alpha G$ is the norm closure of the linear span of $\{\pi(a)\lambda_g \mid a \in A,\ g \in G\}$. It is known that the C^*-algebra $\hat{A}$ is independent of the choice of σ.

Lemma 2.1. *Let $\hat{A} = A \rtimes_\alpha G$ be a reduced crossed product C^*-algebra, as described above. Suppose $\beta \in \mathrm{Aut}(A)$ and β commutes with α_g for every $g \in G$. Then there is a unique automorphism $\hat{\beta} \in \mathrm{Aut}(\hat{A})$ such that $\hat{\beta}(\pi(a)\lambda_g) = \pi(\beta(a))\lambda_g$ for every $a \in A$ and $g \in G$.*

Proof. Uniqueness is clear. In the notation used above, we may without loss of generality take the representation $\sigma : A \to B(\mathcal{H})$ to be so that the automorphism β is spatially implemented, i.e. so that there is a unitary $V \in B(\mathcal{H})$ with $V^*\sigma(a)V = \sigma(\beta(a))$ for every $a \in A$. Then, equating $\ell^2(G, \mathcal{H})$ with $\ell^2(G) \otimes \mathcal{H}$ we have the unitary $1 \otimes V$ and $(1 \otimes V)^* \pi(a)\lambda_g(1 \otimes V) = \pi(\beta(a))\lambda_g$ for every $a \in A$ and $g \in G$. Let $\hat{\beta} = \mathrm{Ad}_{1\otimes V}$. □

The following proposition concerns the topological entropy, ht, of certain automorphisms of crossed product C^*-algebras. We refer to [**2**] for notation and definitions used in the proof. Let us note that the C^*-algebra crossed product $A \rtimes_\alpha G$ of an exact C^*-algebra A by an action of an amenable countable group G is exact by [**16**, Proposition 7.1].

Proposition 2.2. *Let A be an exact C^*-algebra, let G be an amenable countable group, let $g \mapsto \alpha_g$ be an action of G on A via automorphisms and let $\hat{A} = A \rtimes_\alpha G$ be the (reduced) crossed product C^*-algebra. Suppose that $\beta \in \mathrm{Aut}(A)$ and that β commutes with α_g for every $g \in G$. Let $\hat{\beta} \in \mathrm{Aut}(\hat{A})$ be the automorphism found in Lemma 2.1. Then $\mathrm{ht}(\hat{\beta}) = \mathrm{ht}(\beta)$.*

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Proof. Because $\hat{\beta}\circ\pi = \pi\circ\beta$, using Brown's result [**2**, Proposition 2.1] that ht is monotone, we have $\mathrm{ht}(\hat{\beta}) \geqslant \mathrm{ht}(\beta)$. We will use the notation from the beginning of this section and we will denote by $\hat{\pi} : \hat{A} \to B(\ell^2(G, \mathcal{H}))$ the inclusion arising from the construction. In order to show that $\mathrm{ht}(\hat{\beta}) \leqslant \mathrm{ht}(\beta)$, it will suffice to show that $\mathrm{ht}(\hat{\pi}, \hat{\beta}, \omega, \delta) \leqslant \mathrm{ht}(\beta)$ for every $\delta > 0$ and for every finite subset ω of $\{\pi(a)\lambda_g \mid a \in A,\ g \in G\}$. Let ν and K be finite subsets of A and, respectively, G, so that $\omega \subseteq \{\pi(a)\lambda_g \mid a \in \nu,\ g \in K\}$. Let $\eta > 0$ and let F be a finite subset of G so that $|F \cap gF| \geqslant (1-\eta)|F|$ for every $g \in K$; (F exists by amenability of G). Let $\nu' = \{\alpha_{t^{-1}}(a) \mid a \in \nu,\ t \in F\}$, let n be a positive integer and let $k = \mathrm{rcp}(\sigma, \nu' \cup \beta(\nu') \cup \cdots \cup \beta^{n-1}(\nu'), \eta)$. Let $\phi : A \to M_k(\boldsymbol{C})$ and $\psi : M_k(\boldsymbol{C}) \to B(\mathcal{H})$ be completely positive contractions such that for every $a \in \nu$ and every $j \in \{0, 1, \ldots, n-1\}$, $\|\psi \circ \phi(\beta^j(a)) - \sigma(\beta^j(a))\| < \eta$. Let $f = |F|^{-1/2} 1_F \in \ell^\infty(G)$ be the normalized characteristic function of F. Let $\varPhi : \hat{A} \to M_{|F|}(\boldsymbol{C}) \otimes M_k(\boldsymbol{C})$ and $\varPsi : M_{|F|}(\boldsymbol{C}) \otimes M_k(\boldsymbol{C}) \to B(\ell^2(G, \mathcal{H}))$ be the completely positive contractions defined in [**3**, Proposition 2.5]. By [**3**, Proposition 2.6] we have, for every $\tilde{a} \in A$ and $g \in G$,

$$\varPsi \circ \varPhi(\pi(\tilde{a})\lambda_g) = \frac{1}{|F|} \sum_{t \in F \cap gF} \pi(\alpha_t \circ \psi \circ \phi \circ \alpha_{t^{-1}}(\tilde{a}))\lambda_g.$$

If $g \in K$ and $\tilde{a} = \beta^j(a)$ for some $0 \leqslant j \leqslant n-1$ and $a \in \nu$, then for every $t \in F \cap gF$ we have

$$\|\alpha_t \circ \psi \circ \phi \circ \alpha_{t^{-1}}(\tilde{a}) - \tilde{a}\| = \|\psi \circ \phi \circ \beta^j \circ \alpha_{t^{-1}}(a) - \beta^j \circ \alpha_{t^{-1}}(a)\| < \eta,$$

because $\alpha_t^{-1}(a) \in \nu'$. Using that $(1-\eta)|F| \leqslant |F \cap gF| \leqslant |F|$ we obtain

$$\|\varPsi \circ \varPhi(\pi(\tilde{a})\lambda_g) - \pi(\tilde{a})\lambda_g\| < \eta(\|a\| + 1).$$

We could have chosen η so small that $\eta(\|a\| + 1) < \delta$ for every $a \in \nu$, which would have given the estimate

$$\mathrm{rcp}(\hat{\pi}, \omega \cup \hat{\beta}(\omega) \cup \cdots \cup \hat{\beta}^{n-1}(\omega), \delta) \leqslant |F|\, \mathrm{rcp}(\sigma, \nu' \cup \beta(\nu') \cup \cdots \cup \beta^{n-1}(\nu'), \eta).$$

Therefore, $\mathrm{ht}(\hat{\pi}, \hat{\beta}, \omega, \delta) \leqslant \mathrm{ht}(\sigma, \beta, \nu', \eta) \leqslant \mathrm{ht}(\beta)$. □

We now turn to the crossed product $A \rtimes_\alpha \mathbb{N}$ of a C^*-algebra A by a single endomorphism α; this construction was introduced by Cuntz [**5**], when he described his algebras $\mathcal{O}_n$ as crossed products of UHF algebras by endomorphisms. Later [**6**, p. 101] he pointed out that this construction applies more generally; see Stacey [**30**] for a more detailed discussion, including the non-unital case (we consider only his multiplicity one crossed product). If A is a C^*-algebra and if α is an injective endomorphism of A, let $\bar{A}$ be the inductive limit of the system $A \xrightarrow{\alpha} A \xrightarrow{\alpha} \ldots$, with corresponding injective homomorphisms $\mu_n : A \to \bar{A}$ ($n \in \mathbb{N}$). Let p denote the element $\mu_0(1)$ of $\bar{A}$ if A is unital, and the corresponding element of the multiplier algebra of $\bar{A}$ if A is non-unital. There is an automorphism $\bar{\alpha}$ of $\bar{A}$ given by $\bar{\alpha}(\mu_n(a)) = \mu_n(\alpha(a))$, with inverse $\mu_n(a) \mapsto \mu_{n+1}(a)$. Then the crossed product $\hat{A} = A \rtimes_\alpha \mathbb{N}$ is defined to be the hereditary C^*-subalgebra $p(\bar{A} \rtimes_{\bar{\alpha}} \mathbb{Z})p$ of the crossed product of $\bar{A}$ by $\bar{\alpha}$. The map μ_0 followed by the embedding of $\bar{A}$ into $\bar{A} \rtimes_{\bar{\alpha}} \mathbb{Z}$ gives an embedding $\pi : A \to \hat{A}$, and the compression by p of the unitary in $\bar{A} \rtimes_{\bar{\alpha}} \mathbb{Z}$

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implementing $\bar{\alpha}$ is an isometry S belonging to $\hat{A}$ if A is unital and to the multiplier algebra of $\hat{A}$ if A is non-unital, and satisfying

$$S\pi(a)S^* = \pi(\alpha(a)), \quad (a \in A). \tag{2.1}$$

If A is unital, then $\hat{A}$ is the universal unital C^*-algebra generated by a copy $\pi(A)$ of A and an isometry S satisfying (2.1); if A is non-unital, then $\hat{A}$ satisfies a similar universal property and is the closed linear span of the set of all elements of the forms $\pi(a)S^k$ and $(S^*)^k\pi(a)$ for $k \geqslant 0$ and $a \in A$ (see [**30**]).

Lemma 2.3. *Let A be an exact C^*-algebra and let α be an injective endomorphism of A. Then the crossed product C^*-algebra $A \rtimes_\alpha \mathbb{N}$ is exact.*

Proof. The C^*-algebra $\bar{A}$, being an inductive limit of exact C^*-algebras, is exact. Now, [**16**, Proposition 7.1] implies that $\bar{A} \rtimes_{\bar{\alpha}} \mathbb{Z}$ is exact, hence that $A \rtimes_\sigma \mathbb{N}$ is exact. □

The following lemma follows easily from the universal property for the crossed product by an endomorphism, but we will exhibit the automorphism $\hat{\beta}$ directly, for use in the next proposition.

Lemma 2.4. *Let A be a C^*-algebra with an injective endomorphism α and an automorphism β that commutes with α. Then there is a unique automorphism $\hat{\beta}$ of $A \rtimes_\alpha \mathbb{N}$ satisfying $\hat{\beta}(\pi(a)S^k) = \pi(\beta(a))S^k$ for every $a \in A$ and $k \geqslant 0$.*

Proof. Uniqueness is clear. Let $\hat{A} = A \rtimes_\alpha \mathbb{N}$; then $\hat{A} = p(\bar{A} \rtimes_{\bar{\alpha}} \mathbb{Z})p$ as above. The commuting diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \cdots \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \\ A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \cdots \end{array}$$

gives rise to an automorphism $\bar{\beta}$ of $\bar{A}$ that commutes with $\bar{\alpha}$. Let γ be the automorphism of $\bar{A} \rtimes_{\bar{\alpha}} \mathbb{Z}$ arising from $\bar{\beta}$ via Lemma 2.1. Then $\gamma(\hat{A}) = \hat{A}$ and the restriction of γ to $\hat{A}$ is the desired automorphism $\hat{\beta}$. □

Proposition 2.5. *Let A be an exact C^*-algebra with an injective endomorphism α and an automorphism β that commutes with α; let $\hat{A} = A \rtimes_\alpha \mathbb{N}$. Let $\hat{\beta} \in \mathrm{Aut}(\hat{A})$ be the automorphism found in Lemma 2.4. Then $\mathrm{ht}(\hat{\beta}) = \mathrm{ht}(\beta)$.*

Proof. Let us use the notation of the proof of Lemma 2.4. Then we have

$$\mathrm{ht}(\beta) \leqslant \mathrm{ht}(\hat{\beta}) \leqslant \mathrm{ht}(\gamma) = \mathrm{ht}(\bar{\beta}),$$

where the inequalities follow from monotonicity of ht and the equality follows from Proposition 2.2. However, Brown's result [**2**, Proposition 2.14] on inductive limit automorphisms gives $\mathrm{ht}(\bar{\beta}) = \mathrm{ht}(\beta)$. □

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3. Exactness of the Cuntz–Pimsner algebras

In this section we prove that the extended Cuntz–Pimsner algebra $E(H)$ and the Cuntz–Pimsner algebra $O(H)$ of a Hilbert B,B-bimodule H are exact C^*-algebras whenever B is an exact C^*-algebra. We begin by reviewing Pimsner's construction [**23**] of these algebras and some facts about them.

Let B be a C^*-algebra and let H be a Hilbert bimodule over B. By this we mean that H is a right Hilbert B-module with an injective homomorphism $B \to \mathcal{L}(H)$; we further assume that $\{\langle h_1, h_2\rangle_B \mid h_1, h_2 \in H\}$ generates B as a C^*-algebra, where $\langle \cdot, \cdot\rangle_B$ is the B-valued inner product on H. Let $\mathcal{F}(H) = B \oplus \bigoplus_{n\geqslant 1} H^{(\otimes_B)n}$ be the full Fock space over H; here $H^{(\otimes_B)n}$ denotes the n-fold tensor product $H \otimes_B H \otimes_B \cdots \otimes_B H$. Note that $\mathcal{F}(H)$ is a Hilbert B,B-bimodule. For each vector $h \in H$, the operator $l(h) : \mathcal{F}(H) \to \mathcal{F}(H)$ defined by

$$l(h)h_1 \otimes \cdots \otimes h_n = h \otimes h_1 \otimes \cdots \otimes h_n, \quad h, h_1, \dots, h_n \in H,$$
$$l(h)b = hb, \quad h \in H, \quad b \in B,$$

is a bounded adjointable operator on $\mathcal{F}(H)$. These $l(h)$ are called creation operators and satisfy the relations

$$l(h)^* l(g) = \langle h, g\rangle_B, \qquad h, g \in H, \tag{3.1}$$
$$b_1 l(h) b_2 = l(b_1 h b_2), \qquad h \in H, \quad b_1, b_2 \in B. \tag{3.2}$$

Pimsner defined the *extended Cuntz–Pimsner algebra* $E(H) \subset \mathcal{L}(\mathcal{F}(H))$ to be

$$E(H) = C^*(l(h) : h \in H)$$

(where his notation is $\mathcal{T}_H$). Since we assumed that B is generated by the set of inner products $\langle h_1, h_2\rangle_B$, the copy of B acting on the left of $\mathcal{F}(H)$ is contained in $E(H)$. Pimsner showed [**23**, Theorem 3.4] that $E(H)$ is in fact the universal C^*-algebra generated by B and elements $l(h)$, satisfying relations (3.1) and (3.2). The orthogonal projection onto $B \subset \mathcal{F}(H)$ defines a canonical conditional expectation $\mathcal{E}$ from $E(H)$ onto B.

If $K \subset H$ is a Hilbert subbimodule, then $C^*(l(h) : h \in K) \cong E(K)$, so that there is an inclusion $E(K) \subset E(H)$; this inclusion preserves $\mathcal{E}$. Note, furthermore, that if H' is a closed $\boldsymbol{C}$-linear subspace of H and if B' is a C^*-subalgebra of B such that

$$\langle h_1, h_2\rangle_B \in B' \quad (h_1, h_2 \in H'),$$
$$b_1 h b_2 \in H' \quad (b_1, b_2 \in B',\ h \in H'),$$

then H' is a Hilbert bimodule over B' and $E(H') \subset E(H)$.

Theorem 3.1. *Let B be a C^*-algebra and let H be a Hilbert B,B-bimodule such that $\{\langle h_1, h_2\rangle \mid h_1, h_2 \in H\}$ generates B. Then $E(H)$ is exact if and only if B is exact.*

Proof. Since C^*-subalgebras of exact C^*-algebras are exact and since $B \subset E(H)$, if $E(H)$ is exact, then B is exact.

Assume now that B is exact, and let us show that $E(H)$ is exact. There is a net, ordered by inclusions, of pairs (B'_λ, H'_λ), where each B'_λ is a separable C^*-subalgebra of B and where each H'_λ is a separable closed linear subspace of H such that the restriction of the usual operations makes H'_λ a Hilbert bimodule over B'_λ and such that

$$\overline{\bigcup_\lambda B'_\lambda} = B \quad \text{and} \quad \overline{\bigcup_\lambda H'_\lambda} = H.$$

From the inclusions $E(H'_\lambda) \subset E(H)$ mentioned early in this section, we see that $E(H)$ is the direct limit of the $E(H'_\lambda)$. Hence we may and do assume without loss of generality that B and H are separable.

Let $\tilde{H} = H \oplus B$. Since $E(H) \subset E(\tilde{H})$, it will be sufficient to prove that $E(\tilde{H})$ is exact. Denote by $\xi \in \tilde{H}$ the vector $0 \oplus 1_B$, and let $L = l(\xi)$. Then L satisfies the following relations:

$$\begin{aligned} L^*L &= 1, \\ L^*l(h) &= 0, \quad h \in H \\ l(h)^*L &= 0, \quad h \in H. \end{aligned}$$

The C^*-algebra $E(\tilde{H})$ is the closed linear span of the set of all elements of the form

$$W = b_0 l(h_1)^{g(1)} b_1 l(h_2)^{g(2)} b_2 \dots l(h_n)^{g(n)} b_n, \tag{3.3}$$

where $n \geqslant 0$, $b_j \in B$, $g(j) \in \{*, \cdot\}$, $h_1, \dots, h_n \in \tilde{H}$ and where $l(h_j)^{g(j)} = l(h_j)$ if $g(j) = \cdot$.

Consider the unitary $\alpha_t : \tilde{H} \to \tilde{H}$ given by $\alpha_t(h) = \mathrm{e}^{2\pi \mathrm{i} t} h$, $(h \in \tilde{H})$. Denote by β_t the resulting automorphism $E(\alpha_t)$ of $E(\tilde{H})$. Thus $\beta_t(L) = \mathrm{e}^{2\pi\mathrm{i}t} L$, $\beta_t(l(h)) = \mathrm{e}^{2\pi\mathrm{i}t} l(h)$ $(h \in H)$ and $\beta_t(b) = b$ $(b \in B)$. Note that $t \mapsto \beta_t$ is an action of the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ on $E(\tilde{H})$. Let A be the fixed point subalgebra of β, i.e. $A = \{a \in E(H) \mid \forall t \in \mathbb{T},\ \beta_t(a) = a\}$.

Claim 3.2. *A is the closed linear span of the set of operators of the form* (3.3) *for which* $\#\{i : g(i) = *\} = \#\{i : g(i) = \cdot\}$.

Proof. If W is of the form (3.3), then

$$\beta_t(W) = \mathrm{e}^{2\pi\mathrm{i}(\#\{i:g(i)=\cdot\} - \#\{i:g(i)=*\})t} W.$$

Hence, if $\#\{i : g(i) = *\} = \#\{i : g(i) = \cdot\}$, then $W \in A$. The map $\Phi(T) = \int_0^1 \beta_t(T)\,\mathrm{d}t$ is a faithful conditional expectation from $E(\tilde{H})$ onto A; letting W be as above, if $\#\{i : g(i) = *\} = \#\{i : g(i) = \cdot\}$, then $\Phi(W) = W$, while otherwise $\Phi(W) = 0$. If $T \in A$, then T can be approximated by a linear combination of operators of the form (3.3). Since $\Phi(T) = T$ by assumption, it then follows that T can be approximated by a linear span of operators of the form (3.3) for which $\#\{i : g(i) = *\} = \#\{i : g(i) = \cdot\}$. This proves Claim 3.2. □

Claim 3.3. $E(\tilde{H}) = C^*(A, L)$.

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Proof. It is sufficient to show that any operator W of the form (3.3) can be written as $W=(L^*)^kW'$ or $W=W'L^k$ for some $W'\in E(H)$ and $k\in\mathbb{N}$. Let $k=\#\{i:g(i)=\cdot\}-\#\{i:g(i)=*\}$. If $k=0$, then $W\in A$ and we are done. If $k>0$, then since $L^*L=1$ we have $W=(W(L^*)^k)L^k$ and $W'=W(L^*)^k\in A$. If $k<0$, then $W=(L^*)^kL^kW$, and $W'=L^kW\in A$. This proves Claim 3.3. □

Claim 3.4. *$\Psi:a\mapsto LaL^*$ defines an injective endomorphism of A. $E(\tilde{H})$ is isomorphic to the (universal) crossed product of A by this endomorphism, namely $E(\tilde{H})\cong A\rtimes_\Psi\mathbb{N}$.*

Proof. Ψ is an injective endomorphism because $L^*L=1$. Let $C=A\rtimes_\Psi\mathbb{N}$ and let $V\in C$ denote the isometry arising from the crossed product construction and implementing Ψ; thus we have $VaV^*=\Psi(a)$, $a\in A$. As is well known, there exists a continuous family $(\gamma_t)_{t\in\mathbb{T}}$ of automorphisms γ_t of C, such that $\gamma_t(V)=\mathrm{e}^{2\pi\mathrm{i}t}V$, and $\gamma_t(a)=a$ for every $a\in A$ (where we identify the circle $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$). Let $\Gamma:C\to A$ be the conditional expectation $\Gamma(T)=\int_0^1\gamma_t(T)\,\mathrm{d}t$. Then Γ is faithful. By universality of C, there is a surjective map $\rho:C\to E(\tilde{H})$, such that $\rho(a)=a$ for $a\in A$, and $\rho(V)=L$. Let $T\in\ker\rho$. Then $T^*T\in\ker\rho$. Since $\beta\circ\rho=\rho\circ\gamma$, we have $\gamma_t(T^*T)\in\ker\rho$, so that $\Gamma(T^*T)\in\ker\rho$. But $\rho|_A$ is injective, so $\Gamma(T^*T)=0$ and hence $T^*T=0$ and $T=0$. It follows that $C\cong E(\tilde{H})$. This proves Claim 3.4. □

Claim 3.5. *The C^*-algebra A is exact.*

Proof. Denote by A_n the subspace of A that is the closed linear span of the set of words of the form $W=b_0l(h_1)^{g(1)}b_1l(h_2)^{g(2)}b_2\dots l(h_{2m})g(2m)b_{2m}$ with $m\leqslant n$, and for which $\#\{i:g(i)=\cdot\}=\#\{i:g(i)=*\}$. Note that, in light of equations (3.1) and (3.2), we may assume without loss of generality that in W, $g(1)=g(2)=\dots=g(m)=\cdot$ and $g(m+1)=g(m+2)=\dots=g(2m)=*$. Now it is easily seen that A_n is a C^*-subalgebra of A and that $B=A_0\subset\dots\subset A_n\subset A_{n+1}\subset A$ is an increasing sequence of subalgebras with $\bigcup_{n\geqslant1}A_n$ dense in A. Hence it will suffice to prove that each A_n is exact.

Denote by π_n the restriction and compression of the representation of A_n on $\mathcal{F}(\tilde{H})$ to $\mathcal{F}_n(\tilde{H})=B\oplus\bigoplus_{k\leqslant n}\tilde{H}^{(\otimes_B)k}$. Since in the decomposition

$$\begin{aligned}\mathcal{F}(\tilde{H})&=\bigoplus_{k\geqslant0}\bigoplus_{j=0}^{n}\tilde{H}^{\otimes_B(k(n+1)+j)}\\&=\bigoplus_{j=0}^{n}\bigoplus_{k\geqslant0}\tilde{H}^{\otimes_Bj}\otimes(\tilde{H}^{\otimes_B(n+1)})^{\otimes_Bk}\\&=\mathcal{F}_n(\tilde{H})\otimes\mathcal{F}(\tilde{H}^{\otimes_B(n+1)}),\end{aligned}\tag{3.4}$$

A_n acts on $\mathcal{F}(\tilde{H})$ as $\pi_n(A_n)\otimes1$, it follows that π_n is a faithful representation. Let $I_n\subset A_n$ be the closed linear span of the set of all words of the form

$$b_0l(h_1)b_1l(h_2)b_2\dots l(h_n)b_nl(h_{n+1})^*b_{n+1}\dots l(h_{2n})^*b_{2n}.$$

Using the relations (3.1) and (3.2), one easily sees that I_n is a closed two-sided ideal of A_n. Moreover, observing the action of $\pi_n(I_n)$ on $\mathcal{F}_n(\tilde{H})$, one easily sees that, with respect to the decomposition $\mathcal{F}_n(\tilde{H}) = \mathcal{F}_{n-1}(\tilde{H}) \oplus \tilde{H}^{(\otimes_B)n}$, we have $\pi_n(I_n) = 0_{\mathcal{F}_{n-1}(\tilde{H})} + \mathcal{K}(\tilde{H}^{(\otimes_B)n})$, where $\mathcal{K}(\tilde{H}^{(\otimes_B)n})$ denotes the algebra of compact operators on the Hilbert B,B-bimodule $\tilde{H}^{(\otimes_B)n}$. Now the quotient A_n/I_n is canonically identified with the closed linear span of all words $b_0 l(h_1)^{g(1)} b_1 l(h_2)^{g(2)} \dots l(h_{2m})^{g(2m)} b_{2m}$ in A_n for which $m < n$. Thus A_n/I_n is isomorphic to A_{n-1}, and the canonical inclusion $A_{n-1} \hookrightarrow A_n$ provides a splitting for the short exact sequence $0 \to I_n \to A_n \to A_n/I_n \to 0$.

We now show exactness of A_n by induction on n. For $n = 0$, $A_0 \cong B$ is exact by assumption. Having restricted to the separable case, we have that $\tilde{H}^{(\otimes_B)n}$ is separable and hence $\mathcal{K}(\tilde{H}^{(\otimes_B)n})$ is an exact C^*-algebra by the Kasparov stabilization lemma [**14**]. Using the induction hypothesis that A_{n-1} is exact, we find that in the split exact sequence $0 \to I_n \to A_n \to A_n/I_n \to 0$ the algebras $A_n/I_n \cong A_{n-1}$ and $I_n \cong \mathcal{K}(\tilde{H}^{(\otimes_B)n})$ are exact. By [**7**, Proposition 2] (see also [**16**, Proposition 7.1]), A_n is exact. This completes the proof of Claim 3.5. □

Now we can finish the proof of the theorem. By Claims 3.4 and 3.5, $E(\tilde{H})$ is isomorphic to the universal crossed product $A \rtimes_\Psi \mathbb{N}$ of an exact C^*-algebra A by an injective endomorphism Ψ. By Lemma 2.3, $A \rtimes_\Psi \mathbb{N}$ is exact. □

Corollary 3.6. *Let B be a C^*-algebra and let H be a Hilbert B,B-bimodule. Then the Cuntz–Pimsner algebra $\mathcal{O}(H)$ associated to H is exact if and only if B is exact.*

Proof. The Cuntz–Pimsner algebra $\mathcal{O}_H$ is defined as a certain quotient of $E(H)$ (see [**23**] for details). If B is exact, then $E(H)$ is exact, and it was proved by Kirchberg [**16**] that quotients of exact C^*-algebras are exact. Since $\mathcal{O}(H)$ contains a copy of B as a C^*-subalgebra, exactness of $\mathcal{O}(H)$ implies exactness of B. □

Given a C^*-algebra B and a completely positive map $\eta : B \to B \otimes M_{n\times n}(\boldsymbol{C})$, the C^*-algebra $\hat{\Phi}(B,\eta)$ was constructed in [**27**]. This algebra is a subalgebra of $E(H)$, where H is the Hilbert B,B-bimodule associated to η (see [**25**]). Since $B \subset \hat{\Phi}(B,\eta)$, from Theorem 3.1 we immediately have the following.

Corollary 3.7. *Let B be a C^*-algebra and let $\eta : B \to B \otimes M_{n\times n}$ be a completely positive map. Then the C^*-algebra $\hat{\Phi}(B,\eta)$ is exact if and only if B is exact.*

4. Exactness of reduced free product C^*-algebras

Theorem 3.1 can be used to give a new proof of the recent result [**9**] that the class of exact unital C^*-algebras is closed under taking reduced amalgamated free products; this alternative proof will be given in section 5 below. In this section, however, we give an easier argument that makes use of Theorem 3.1 and proves a special case, namely that every reduced free product (with amalgamation over the scalars) of exact C^*-algebras is exact. (See [**11**] for an easier version of the argument of [**9**] in a special case.)

In light of Example 1.4 of [**12**] (see also Question 1 of [**1**] and the answers provided), one must be careful about embeddings of reduced free products of C^*-algebras. Thus we provide a proof of the following lemma.

Lemma 4.1. *Let N be an integer greater than or equal to 2 or ∞ and for every $1 \leqslant k < N+1$ let A_k be a unital C^*-algebra having a state ϕ_k whose Gelfand–Naimark–Segal (GNS) representation is faithful. Let $(A,\phi) = *_{k=1}^N (A_k,\phi_k)$ be the reduced free product of C^*-algebras. Let $B = \bigotimes_{k=1}^N A_k$ be the minimal tensor product of C^*-algebras and let $\rho = \otimes_{k=1}^N \phi_k$ be the tensor product state. Let D_1 be a C^*-algebra with a state ψ_1 whose GNS representation is faithful and having a unitary $u \in D_1$ such that $\psi_1(u^n) = 0$ for every non-zero integer n. Let $(D,\psi) = (D_1,\psi_1) * (B,\rho)$ be the reduced free product of C^*-algebras. Consider the embeddings $\pi_k : A_k \to D$ given by $\pi_k(a) = u^k a u^{-k}$. Then there is an injective homomorphism $\pi : A \to D$ whose restriction to A_k is π_k, for every k and such that $\psi \circ \pi = \phi$.*

Proof. Let us abuse notation by writing A_k for all of the corresponding unital subalgebras $A_k \subseteq A$, $A_k \subseteq B$ and $A_k \subseteq D$ arising from the free product and tensor product constructions, and similarly for $B \subseteq D$. The unitary u generates a copy of $C(\mathbb{T})$, the continuous functions on the circle, on which ψ_1 is given by integration with respect to Haar measure. Thus

$$\left(C(\mathbb{T}), \int \mathrm{d}\lambda\right) \subseteq (D_1,\psi_1),$$

where $\mathrm{d}\lambda$ is Haar measure. By the main result of [**1**], without loss of generality we may and do assume that

$$(D_1,\psi_1) = \left(C(\mathbb{T}), \int \mathrm{d}\lambda\right).$$

It is easily checked that in (D,ψ) the family $(u^k B u^{-k})_{k\in\mathbb{Z}}$ is free; letting $\bar{B}$ be the C^*-subalgebra of D generated by $\bigcup_{k\in\mathbb{Z}} u^k B u^{-k}$, conjugation by u acts as the free shift on $\bar{B}$. As $\bar{B} \cup \{u\}$ generates D and as $\bar{B}u^k \subseteq \ker\psi$ for every non-zero integer k, we see that $D \cong \bar{B} \rtimes \mathbb{Z}$ and the GNS representation of the restriction of ψ to $\bar{B}$ is faithful on $\bar{B}$. Therefore, we may use the uniqueness of the free product construction to see that

$$(\bar{B}, \psi\restriction_{\bar{B}}) \cong *_{k=-\infty}^{\infty} (u^k B u^{-k}, \psi\restriction_{u^k B u^{-k}}).$$

Regarding $A_k \subseteq B$, we have the embeddings $\pi_k : A_k \to u^k A_k u^{-k} \subset u^k B u^{-k}$, and these satisfy $\psi \circ \pi_k = \phi_k$. Hence by the main result of [**1**] there is an injective homomorphism $\pi : A \to \bar{B} \subset D$ extending each π_k and satisfying $\psi \circ \pi = \phi$. □

Proposition 4.2. *Let I be a set having at least two elements and for every $\iota \in I$ let A_ι be a unital C^*-algebra and let ϕ_ι be a state on A_ι whose GNS representation is faithful. Let $(A,\phi) = *_{\iota\in I}(A_\iota,\phi_\iota)$ be the reduced free product of C^*-algebras, let $B = \bigotimes_{\iota\in I} A_\iota$ be the minimal tensor product of C^*-algebras and let $\rho = \otimes_{\iota\in I}\phi_\iota$ be the tensor product state. Then there is a Hilbert B,B-bimodule H and an injective homomorphism $\pi : A \to E(H)$ such that letting $\mathcal{E} : E(H) \to B$ be the canonical vacuum expectation, we have $\rho \circ \mathcal{E} \circ \pi = \phi$.*

Proof. Let H be the Hilbert B,B-bimodule associated to ρ. This means that H is obtained by separation and completion of the algebraic tensor product $B \otimes_{\mathrm{alg}} B$ with

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respect to the norm induced by the B-valued inner product

$$\langle a \otimes b, a' \otimes b' \rangle = b^* \rho(a^* a') b', \quad a, a', b, b' \in B,$$

or, in other notation, $H = L^2(B, \rho) \otimes_{\boldsymbol{C}} B$. Denote by $\xi \in H$ the vector $1 \otimes 1$. Let $\mathcal{E} : E(H) \to B$ be the canonical vacuum expectation given by compression with the projection $\mathcal{F}(H) \to B$. Consider $C = C^*(l(\xi)) \subset E(H)$. By [**26**], the restriction of the conditional expectation $\mathcal{E} : E(H) \to B$ to C is scalar-valued; we denote this restriction by ψ. In fact, as is easily seen, C is isomorphic to the algebra of Toeplitz operators generated by the non-unitary isometry $l(\xi)$ and ψ is the state whose support is $1 - l(\xi)l(\xi)^*$. We have by [**26**, Theorem 2.3] that $E(H)$ is a reduced free product, $(E(H), \rho \circ \mathcal{E}) \cong (C, \psi) * (B, \rho)$, because $l(\xi)$ satisfies $l(\xi)^* b l(\xi) = \rho(b)$ for all $b \in B$. The algebra C contains a unitary u with the property that $\psi(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$; for example, u can be obtained by continuous functional calculus from the semicircular element $l(\xi) + l(\xi)^*$. Then by Lemma 4.1 there is an injective homomorphism $\pi : A \to E(H)$ satisfying $\rho \circ \mathcal{E} \circ \pi = \phi$. □

Corollary 4.3 (see [9]). *Let I be a set having at least two elements and for every $\iota \in I$ let A_ι be a unital C^*-algebra and let ϕ_ι be a state on A_ι whose GNS representation is faithful. Let $(A, \phi) = *_{\iota \in I}(A_\iota, \phi_\iota)$ be the reduced free product of C^*-algebras. Then A is exact if and only if every A_ι is exact.*

Proof. Since each A_ι is canonically embedded as a C^*-subalgebra of A, exactness of A implies exactness of every A_ι.

Suppose that every A_ι is exact and let $B = \bigotimes_{\iota \in I} A_\iota$ be the minimal tensor product of C^*-algebras. Then B is exact, as is easily seen from the definition of exactness and by taking inductive limits if necessary. By Proposition 4.2, A is isomorphic to a C^*-subalgebra of $E(H)$, for some Hilbert B, B-bimodule H. By Theorem 3.1 $E(H)$ is exact, and it thus follows that A is exact. □

5. Exactness of reduced amalgamated free product C^*-algebras

In this section, we give an alternative proof, using Theorem 3.1, of the result [**9**] that the class of exact unital C^*-algebras is closed under taking reduced amalgamated free products.

Proposition 5.1. *Let B be a unital C^*-algebra and let A_1 and A_2 be unital C^*-algebras each containing a copy of B as a unital C^*-subalgebra and having a conditional expectation $\phi_\iota : A_\iota \to B$ whose GNS representation is faithful. Let*

$$(\tilde{A}, \tilde{\phi}) = (A_1, \phi_1) *_B (A_2, \phi_2)$$

be the reduced amalgamated free product of C^*-algebras and let $A = A_1 \oplus A_2$. Then there is a Hilbert A, A-bimodule H such that $\tilde{A}$ is isomorphic to a C^*-quotient of a C^*-subalgebra of $E(H)$.

Proof. Let $D = B \oplus B = \{(b_1, b_2) \in A \mid b_1, b_2 \in B\}$ and consider the conditional expectation $\phi = \phi_1 \oplus \phi_2 : A \to D$ and the completely positive map $\eta : A \to A$ given by $\eta((a_1, a_2)) = (\phi_2(a_2), \phi_1(a_1))$. Note that η takes values in D and that $\eta \circ \phi = \eta$. We now consider the η-creation operator L; this construction was introduced by Speicher [**29**] and Pimsner [**23**], and proceeds as follows. We take the Hilbert A, A-bimodule H which is obtained by separation and completion of the algebraic tensor product $A \otimes_{\text{alg}} A$ equipped with the natural left and right actions of A and with the inner product

$$\langle a_1 \otimes a_2, a_1' \otimes a_2' \rangle = a_2^* \eta(a_1^* a_1') a_2'.$$

We let $\xi \in H$ be the element corresponding to $1 \otimes 1 \in A \otimes A$; taking the Fock space $\mathcal{F}(H)$ we let $L = l(\xi) \in E(H) \in \mathfrak{L}(\mathcal{F}(H))$ be the corresponding creation operator. As is well known and is easily seen using equations (3.1) and (3.2), $L^* a L = \eta(a)$ for every $a \in A$.

Claim 5.2. *Let $\mathcal{E} : E(H) \to A$ be the conditional expectation defined by compression with the orthogonal projection $\mathcal{F}(H) \to A$ and let $\psi = \phi \circ \mathcal{E} : E(H) \to D$. Then $\{L, L^*\}$ and A are free with respect to ψ.*

Proof. This claim is proved by applying [**26**, Theorem 2.3]. In fact, from its proof, we see that statements (i), (ii) and (b) of that theorem imply statement (a) of that theorem. Hence, in order to apply [**26**, Theorem 2.3] to show freeness of A and $\{L, L^*\}$, we need only show

(α) $\psi(a_1 L a_2 \dots L a_k L a_1' L^* a_2' \dots L^* a_{\ell+1}') = 0$ whenever $k, \ell \geqslant 0$, $k + \ell > 0$ and $a_j, a_j' \in A$; and

(β) $L^* a L = \eta(\psi(a))$ for every $a \in A$.

Indeed, (α) and (β) together show that L is distributed with respect to ψ as an $(\eta\restriction_D)$-creation operator, showing part (c) of [**26**, Theorem 2.3] holds, while (α) is part (i) and (β) is part (ii) of [**26**, Theorem 2.3]. But (α) and (β) follow from the facts that L is distributed as an η-creation operator and $\eta \circ \psi(a) = \eta \circ \phi(a) = \eta(a)$ for $a \in A$. This finishes the proof of Claim 5.2. □

Now let $P = 1 - L^2(L^*)^2$ and $W = P(L + L^*)P$.

Claim 5.3. *P is a projection, W is a partial isometry, $W = W^*$ and $W^2 = P$.*

Proof. The creation operator L is easily seen to be an isometry; thus P is a projection. Clearly $W = W^*$. Straightforward computations reveal that

$$W = L + L^* - L(L^*)^2 - L^2 L^*, \tag{5.1}$$

and then that $W^2 = P$. So Claim 5.3 is proved. □

Claim 5.4. *Let $\mathfrak{A} = C^*(\{W\} \cup A) \subseteq E(H)$ and let π be the GNS representation of $\mathfrak{A}$ arising from the conditional expectation ψ. Then $\pi(1 - P) = 0$; hence $\pi(W)$ is unitary.*

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Proof. We must show that

$$\psi(a_1 W^{q_1} \dots a_k W^{q_k} a_{k+1}(1-P)a'_{\ell+1} W^{q'_\ell} a'_\ell \dots W^{q'_1} a'_1) = 0 \tag{5.2}$$

for all integers $k, \ell \geqslant 0$ and $q_j, q'_j \geqslant 1$ and all $a_j a'_j \in A$. We will show (5.2) by induction on $k+\ell$. For $k+\ell=0$, clearly $a_1(1-P)a'_1 = a_1 l^2 (L^*)^2 a'_1$ is in the kernel of $\mathcal{E}$, hence also of ψ. Suppose now $k+\ell>0$. Writing $a_j = (a_j - \phi(a_j)) + \phi(a_j)$ and similarly for a'_j and then distributing, we may assume that each a_j and each a'_j lies either in $\ker\phi$ or in D. For every $d=(b_1,b_2)\in D$ we have $Ld = \alpha(d)L$, where α is the automorphism of D given by $\alpha((b_1,b_2)) = (b_2,b_1)$. Indeed, we may take $d=d^*$ and then

$$\begin{aligned}(Ld-\alpha(d)L)^*(Ld-\alpha(d)L) &= dL^*Ld - dL^*\alpha(d)L - L^*\alpha(d)Ld + L^*\alpha(d)\alpha(d)L \\ &= d^2 - d^2 - d^2 + d^2 = 0,\end{aligned}$$

because $L^*dL = \eta(d) = \alpha(d)$ for $d\in D$. From $Ld=\alpha(d)L$ it follows that

$$L^*d = \alpha(d)L^*, \quad Pd = dP \quad \text{and} \quad Wd = \alpha(d)W.$$

Note also that $W(1-P)=0$. If $k\geqslant 1$ and $a_{k+1}\in D$, then

$$W^{q_k} a_{k+1}(1-P) = W^{q_k}(1-P)a_{k+1} = 0;$$

similarly, if $\ell \geqslant 1$ and $a'_{\ell+1}\in D$, then $(1-P)a'_{\ell+1}W^{q'_\ell} = 0$. If $a_j \in D$ for some $1 < j \leqslant k$, then $W^{q_{j-1}} a_j W^{q_j} = W^{q_{j-1}+q_j}\alpha^{q_j}(a_j)$ and we may use the induction hypothesis to conclude that (5.2) holds; similarly, if $a'_j\in D$ for some $1<j\leqslant \ell$, then (5.2) holds. Hence we may assume that $a_j \in \ker\phi$ for every $1<j\leqslant k+1$ and $a'_j\in\ker\phi$ for every $1<j\leqslant \ell+1$. Writing $W^{q_j} = (W^{q_j} - \psi(W^{q_j})) + \psi(W^{q_j})$ and similarly for $W^{q'_j}$, distributing and letting

$$y_j = W^{q_j} - \psi(W^{q_j}) \quad \text{and} \quad y'_j = W^{q'_j} - \psi(W^{q'_j}),$$

we find that $a_1 W^{q_1}\dots a_k W^{q_k} a_{k+1}(1-P)a'_{\ell+1}W^{q'_\ell}a'_\ell \dots W^{q'_1}a'_1$ is equal to a sum of $2^{k+\ell}$ terms which are obtained by replacing each W^{q_j} variously with y_j and with $\psi(W^{q_j})$, and each $W^{q'_j}$ variously with y'_j and with $\psi(W^{q'_j})$. If z is one of these terms where at least one W^{q_j} or $W^{q'_j}$ has been replaced by its expectation under ψ, then we can see that $\psi(z)=0$ by using the induction hypothesis; indeed, we write $y_j = W^{q_j} - \psi(W^{q_j})$ for each y_j appearing in z and similarly for each y'_j and then we distribute; this expresses z as a sum of terms to each of which the induction hypothesis applies to show it has expectation zero under ψ. We are left to show only that

$$\Psi(a_1 y_1 \dots a_k y_k a_{k+1}(1-P)a'_{\ell+1} y'_\ell a'_\ell \dots y'_1 a'_1) = 0.$$

But this holds by the freeness proved in Claim 5.2. Thus the proof of Claim 5.4 is finished. □

Claim 5.5. *The restriction of π to A is faithful.*

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Proof. This follows from the fact that ϕ_1 and ϕ_2 have faithful GNS representations. □

The inner product arising from the GNS construction for ψ gives a map $\tilde{\psi} : \pi(\mathfrak{A}) \to D$, which upon identifying D with $\pi(D)$ becomes a conditional expectation $\tilde{\psi} : \pi(\mathfrak{A}) \to \pi(D)$; moreover, we have $\tilde{\psi} \circ \pi = \pi \circ \psi\restriction_{\mathfrak{A}}$.

Claim 5.6. *Let $q = \pi((1,0)) \in \pi(D)$ and let $V = \pi(W)$. Then*

(a) *V is a unitary satisfying $V = V^*$ and $VqV = 1 - q$.*

Consider the subalgebras $M_1 = q\pi(A)q$, $M_2 = qV\pi(A)Vq = V(1-q)\pi(A)(1-q)V$ and $N = q\pi(D)q$. Then

(b) *$N \subseteq M_\iota$ ($\iota = 1, 2$);*

(c) *M_ι is isomorphic to A_ι via an isomorphism that sends N to B and conjugates $\tilde{\psi}\restriction_{M_\iota}$ to ϕ_ι ($\iota = 1, 2$); and*

(d) *M_1 and M_2 are free with respect to $\tilde{\psi}$.*

Proof. (a) follows from Claim 5.3, Claim 5.4 and the fact that $W(b_1, b_2) = (b_2, b_1)W$ for all $(b_1, b_2) \in D$; note that (b) holds for the same reason.

For (c), the isomorphisms are

$$M_1 \ni \pi((a_1, 0)) \mapsto a_1 \in A_1,$$
$$M_2 \ni V\pi((0, a_2))V \mapsto a_2 \in A_2,$$

which we denote σ_1 and σ_2, respectively. That σ_1 sends N to B and conjugates $\tilde{\psi}\restriction_{M_\iota}$ to ϕ_1 is straightforward to see. We have $N \subseteq M_2$ because $\pi((b, 0)) = V\pi((0, b))V$ for every $b \in B$, and this also shows that N is mapped by σ_2 onto B. We must show that

$$\sigma_2 \circ \tilde{\psi}(V\pi((0, a_2))V) = \phi_2(a_2) \tag{5.3}$$

for every $a_2 \in A_2$. But

$$\tilde{\psi}(V\pi((0, a_2))V) = \tilde{\psi} \circ \pi(W(0, a_2)W) = \pi \circ \psi(W(0, a_2)W).$$

Write $a_2 = (a_2 - \phi_2(a_2)) + \phi_2(a_2)$. It is easily seen using (5.1) that $\psi(W) = 0$. This and the freeness result proved in Claim 5.2 show that $\psi(W(0, a_2 - \phi_2(a_2))W) = 0$, while since $\phi_2(a_2) \in B$ we have

$$\begin{aligned}\pi \circ \psi(W(0, \phi_2(a_2))W) &= \pi \circ \psi((\phi_2(a_2), 0)) \\ &= \pi((\phi_2(a_2), 0)) = V\pi((0, \phi_2(a_2)))V \overset{\sigma_2}{\longmapsto} \phi_2(a_2).\end{aligned}$$

Hence (5.3) holds and (c) is proved.

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To prove (d) it will suffice to show that $\tilde{\psi}(x) = 0$ whenever $x = x_1 x_2 \dots x_n$, where $n \geqslant 1$, $x_j \in M_{\iota_j} \cap \ker \tilde{\psi}$ and $\iota_1 \neq \iota_2, \iota_2 \neq \iota_3, \dots, \iota_{n-1} \neq \iota_n$. Writing

$$x_j = \begin{cases} \pi((a_j, 0)) \text{ for } a_j \in A_1 \cap \ker \phi_1, & \text{if } \iota_j = 1, \\ \pi(W(0, a_j)W) \text{ for } a_j \in A_2 \cap \ker \phi_2, & \text{if } \iota_j = 2, \end{cases}$$

we find that $\tilde{\psi}(x) = \tilde{\psi} \circ \pi(y) = \pi \circ \tilde{\psi}(y)$, where $y = y_1 y_2 \dots y_n$ and

$$y_j = \begin{cases} (a_j, 0), & \text{if } \iota_j = 1, \\ (0, a_j), & \text{if } \iota_j = 2. \end{cases}$$

Rewriting y as a product of Ws alternating with $(0, a_j)$s and $(a_j, 0)$s, using that each $(a_j, 0)$ and $(0, a_j)$ lies in $\ker \psi$, that $\psi(W) = 0$ and the freeness proved in Claim 5.2, we find that $\psi(y) = 0$. This finishes the proof of Claim 5.6. □

We now finish the proof of Proposition 5.1. By properties of the free product construction, $\tilde{A}$ is isomorphic to the image of $C^*(M_1 \cup M_2)$ under the GNS representation of $\tilde{\psi}\restriction_{C^*(M_1 \cup M_2)}$, while $C^*(M_1 \cup M_2)$ is itself a subalgebra of a quotient of a subalgebra of $E(H)$. Finally, the property of being a quotient of a subalgebra of a given C^*-algebra is preserved under taking quotients and subalgebras. □

Corollary 5.7 (see [9]). *Let B be a unital C^*-algebra and let A_1 and A_2 be unital C^*-algebras each containing a copy of B as a unital C^*-subalgebra and having a conditional expectation $\phi_\iota : A_\iota \to B$ whose GNS representation is faithful. Let*

$$(\tilde{A}, \tilde{\phi}) = (A_1, \phi_1) *_B (A_2, \phi_2)$$

be the reduced amalgamated free product of C^*-algebras. Then $\tilde{A}$ is an exact C^*-algebra if and only if A_1 and A_2 are exact.

Proof. Since A_1 and A_2 are C^*-subalgebras of $\tilde{A}$, exactness of $\tilde{A}$ implies that of A_1 and A_2.

For the converse, suppose A_1 and A_2 are exact and let $A = A_1 \oplus A_2$. Then A is an exact C^*-algebra. By Theorem 3.1, $E(H)$ is an exact C^*-algebra whenever H is a Hilbert A, A-bimodule. Using Proposition 5.1, the well-known fact that C^*-subalgebras of exact C^*-algebras are exact and Kirchberg's result [**16**] that exactness passes to quotients, we have that $\tilde{A}$ is exact. □

6. Bogljubov automorphisms

In this section we consider the analogues of Bogljubov automorphisms on the extended Cuntz–Pimsner algebras $E(H)$ when H is a B, B-bimodule and we prove that if B is finite dimensional, then the topological entropy of every Bogljubov automorphism is zero. We begin, however, by showing that if B is a finite-dimensional C^*-algebra and if H is a Hilbert B-module, then every countably generated Hilbert B-submodule of H is complemented in H. We are convinced this is well known, but we do not know of a reference.

Proposition 6.1 (Gram–Schmidt procedure). *Let B be a finite-dimensional C^*-algebra and let H be a right Hilbert B-module. Let X be a finite or countably infinite subset of H. Then there is a finite or countably infinite subset V of H such that*

(i) *the submodule of H generated by V equals the submodule of H generated by X;*

(ii) *if $v, w \in V$ and $v \neq w$, then $\langle v, w\rangle = 0$; and*

(iii) *if $v \in V$, then $\langle v, v\rangle$ is a minimal projection in B.*

Proof. Let $e_1, \dots, e_n$ be minimal projections in B such that $e_1 + e_2 + \dots + e_n = 1$. We may without loss of generality assume that for every $x \in X$, $x = xe_\ell$ for some $\ell \in \{1, \dots, n\}$. Let $X = \{x_1, x_2, \dots\}$ be an enumeration of X and let ℓ_j be such that $x_j = x_j e_{\ell_j}$. Note that $\langle x_1, x_1\rangle = \|x_1\|^2 e_{\ell_1}$ and let

$$v_1 = \begin{cases} 0, & \text{if } x_1 = 0, \\ \|x_1\|^{-1} x_1, & \text{if } x_1 \neq 0. \end{cases}$$

Then either $v_1 = 0$ or $\langle v_1, v_1\rangle = e_{\ell_1}$. We now recursively define elements $v_2, v_3, \dots,$ so that for all j we have $v_j e_{\ell_j} = v_j$, $\langle v_j, v_j\rangle \in \{0, e_{\ell_j}\}$, and, if $i < j$, then $\langle v_i, v_j\rangle = 0$. For the recursive step, if $n \geqslant 2$, if $v_1, \dots, v_{n-1}$ have been defined and if X has at least n elements, then let $w_n = x_n - \sum_{j=1}^{n-1} v_j \langle v_j, x_n\rangle$. Then for every $i \in \{1, \dots, n-1\}$ we have

$$\langle v_i, w_n\rangle = \langle v_i, x_n\rangle - \langle v_i, v_i\langle v_i, x_n\rangle\rangle = \langle v_i, x_n\rangle - \langle v_i, v_i\rangle\langle v_i, x_n\rangle = 0.$$

Note that $w_n = w_n e_{\ell_n}$; let

$$v_n = \begin{cases} 0, & \text{if } w_n = 0, \\ \|w_n\|^{-1} w_n, & \text{if } w_n \neq 0. \end{cases}$$

Letting $V = \{v_j \mid v_j \neq 0\}$ does the job. □

Although we will only apply the following proposition when the submodule K is assumed to be finitely generated, for the sake of completeness we would like to give the more general result.

Proposition 6.2. *Let B be a finite-dimensional C^*-algebra, let H be a right Hilbert B-module and let K be a closed B-submodule of H that is finitely or countably generated (meaning that K has a dense submodule that is finitely or countably generated). Then K is a complemented submodule of H.*

Proof. Let X be a finite or countable set such that the submodule of H generated by X is dense in K. Let V be the set obtained from X using the Gram–Schmidt procedure of Proposition 6.1. We shall define $P : H \to H$ by

$$Ph = \sum_{v \in V} v\langle v, h\rangle,$$

where, when V is infinite, we shall show that the sum converges in H. Suppose first that V is finite. Then easy calculations show that $P \in \mathcal{L}(H)$, $P^2 = P = P^*$ and consequently $\|P\| \leqslant 1$.

Now suppose that V is infinite and enumerate it by $V = \{v_1, v_2, \dots\}$. For every positive integer n and $h \in H$ let $P_n h = \sum_{j=1}^{n} v_j \langle v_j, h\rangle$. Then by the result for the case of V finite we have

$$\langle h, h\rangle \geqslant \langle P_n h, P_n h\rangle = \sum_{j=1}^{n} \langle h, v_j\rangle\langle v_j, h\rangle.$$

For every state ϕ on B, the sequence $(\phi(\langle P_n h, P_n h\rangle))_{n=1}^{\infty}$ is a bounded and increasing sequence of positive numbers, hence converges. Therefore the sequence $(\langle P_n h, P_n h\rangle)_{n=1}^{\infty}$ converges in B. Then for $m < n$ we have that

$$\langle P_n h - P_m h, P_n h - P_m h\rangle = \sum_{j=m+1}^{n} \langle h, v_j\rangle\langle v_j, h\rangle \leqslant \sum_{j=m+1}^{\infty} \langle h, v_j\rangle\langle v_j, h\rangle,$$

and the right-hand side tends to zero as $m \to \infty$. Therefore the sequence $(P_n h)_{n=1}^{\infty}$ is Cauchy in H, hence converges in H. We may thus define $Ph = \sum_{j=1}^{\infty} v_j \langle v_j, h\rangle$. From the corresponding facts for the finite-dimensional case we obtain that $P \in \mathcal{L}(H)$, $P^2 = P = P^*$ and $\|P\| \leqslant 1$.

It remains to show that $PH = K$. If V is finite, then this is clear, so suppose V is infinite. Given $h \in H$ we have $Ph = \lim_{n\to\infty} P_n h$ and $P_n h \in K$, so $Ph \in K$. Given $k \in K$ then for all $\epsilon > 0$ there are $n \geqslant 1$ and $k_\epsilon \in \operatorname{span}\{v_1, \dots, v_n\}$ such that $\|k - k_\epsilon\| < \epsilon$. But $Pk_\epsilon = k_\epsilon$ and, hence, $\|Pk - k\| \leqslant 2\epsilon + \|Pk_\epsilon - k_\epsilon\| = 2\epsilon$. Therefore $Pk = k$. □

Definition 6.3. Let B be a C^*-algebra and let H be a Hilbert B,B-bimodule such that $\{\langle h_1, h_2\rangle \mid h_1, h_2 \in H\}$ generates B; suppose that $U : H \to H$ is a $\boldsymbol{C}$-linear map such that for some automorphism β of B we have

$$\langle U(h_1), U(h_2)\rangle = \beta(\langle h_1, h_2\rangle), \quad h_1, h_2 \in H,$$
$$U(b_1 h b_2) = \beta(b_1) U(h) \beta(b_2), \quad h \in H, \quad b_1, b_2 \in B.$$

(Note that β is uniquely determined by U and the first of the above equations.) Then there is an automorphism $E(U)$ of $E(H)$, given by $E(U)(l(h)) = l(Uh)$ $(h \in H)$ and $E(U)(b) = \beta(b)$ $(b \in B)$. We call $E(U)$ the *Bogljubov automorphism* of $E(H)$ associated to U.

Theorem 6.4. *Let B be a finite-dimensional C^*-algebra and let H be a Hilbert B,B-bimodule such that $\{\langle h_1, h_2\rangle \mid h_1, h_2 \in H\}$ generates B. Then the topological entropy of every Bogljubov automorphism $E(U)$ of $E(H)$ is zero.*

Proof. The Bogljubov automorphism $E(U)$ arises from a $\boldsymbol{C}$-linear map $U : H \to H$ and an affiliated automorphism $\beta \in \operatorname{Aut}(B)$ as in Definition 6.3. Let $\tilde{H} = H \oplus B$ and $\xi = 0 \oplus 1 \in \tilde{H}$. Let $\tilde{U} : \tilde{H} \to \tilde{H}$ be defined by $\tilde{U}(h \oplus b) = U(h) \oplus \beta(b)$ $(h \in H,\ b \in B)$; note that $\tilde{U}$ and β together satisfy the conditions in Definition 6.3, so that we have the Bogljubov automorphism $E(\tilde{U})$ of $E(\tilde{H})$. Now $E(H)$ is canonically embedded in $E(\tilde{H})$

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and the restriction of $E(\tilde{U})$ to $E(H)$ is $E(U)$; by the monotonicity of ht, which was proved in [**2**, Proposition 2.1], it will therefore suffice to show that $\operatorname{ht}(E(\tilde{U})) = 0$.

As in the proof of Theorem 3.1, $E(\tilde{H})$ is isomorphic to the crossed product C^*-algebra $A \rtimes_\Psi \mathbb{N}$, where $A = \overline{\operatorname{span}}\,\Omega$ with

$$\Omega = B \cup \{l(h_1)\dots l(h_m)l(h_{m+1})^*\dots l(h_{2m})^* \mid m \geqslant 1,\ h_j \in \tilde{H}\},$$

and where Ψ is the endomorphism of A given by $\Psi(x) = LxL^*$ with $L = l(\xi)$. Since $\tilde{U}(\xi) = \xi$, we have $E(\tilde{U})L = L$ and, hence, the restriction of $E(\tilde{U})$ to A is an automorphism of A that commutes with the endomorphism Ψ. It thus follows from Proposition 2.5 that $\operatorname{ht}(E(\tilde{U})) = \operatorname{ht}(E(\tilde{U})\restriction_A)$. Let γ denote the automorphism $E(\tilde{U})\restriction_A$ of A.

Let $\tau : B \to \mathcal{L}(\mathcal{V})$ be a faithful unital representation of B on a Hilbert space $\mathcal{V}$ and let π denote the representation of A which is the inclusion $A \hookrightarrow \mathcal{L}(\mathcal{F}(\tilde{H}))$ followed by the representation $x \mapsto x \otimes 1$ of $\mathcal{L}(\mathcal{F}(\tilde{H}))$ on the Hilbert space $\mathcal{F}(\tilde{H}) \otimes_\tau \mathcal{V}$. We must show $\operatorname{ht}(\gamma) = 0$ and to do so it will suffice to show that $\operatorname{ht}(\pi, \gamma, \omega, \delta) = 0$ for every $\delta > 0$ and every finite subset ω of Ω. Given a finite subset $\omega \subseteq \Omega$ there are $n \in \mathbb{N}$ and a B,B-subbimodule K of $\tilde{H}$ that is finite dimensional as a $\boldsymbol{C}$-vector space and such that $\omega \subseteq \Omega(n, K)$, where

$$\Omega(n, K) \overset{\text{def}}{=} B \cup \{l(h_1)\dots l(h_m)l(h_{m+1})^*\dots l(h_{2m})^* \mid 1 \leqslant m \leqslant n,\ h_j \in K\}.$$

Recall the definition of $\mathcal{F}_n(\tilde{H})$ from the proof of Claim 3.5 and let $P_n \in \mathcal{L}(\mathcal{F}(\tilde{H}))$ denote the projection onto $\mathcal{F}_n(\tilde{H})$. Consider the completely positive contractions

$$\begin{aligned} \Phi_n :\ & \mathcal{L}(\mathcal{F}(\tilde{H})) \to \mathcal{L}(\mathcal{F}_n(\tilde{H})), & \Phi_n(x) &= P_n x P_n, \\ \Psi_n :\ & \mathcal{L}(\mathcal{F}_n(\tilde{H})) \to \mathcal{L}(\mathcal{F}(\tilde{H})), & \Psi_n(y) &= W_n^*(y \otimes 1)W_n, \end{aligned}$$

where $W_n : \mathcal{F}(\tilde{H}) \to \mathcal{F}_n(\tilde{H}) \otimes_B \mathcal{F}(\tilde{H}^{\otimes_B (n+1)})$ is the unitary operator canonically defined by the decomposition in equation (3.4) in the proof of Claim 3.5. Note that $\Psi_n \circ \Phi_n(x) = x$ for every $x \in \Omega(n, \tilde{H})$.

For every integer $p \geqslant 1$ let $K_p = K + \tilde{U}(K) + \tilde{U}^2(K) + \dots + \tilde{U}^{p-1}(K)$; then

$$\omega \cup \gamma(\omega) \cup \dots \cup \gamma^{p-1}(\omega) \subseteq \Omega(n, K_p).$$

Moreover K_p is a B,B-subbimodule of $\tilde{H}$ whose $\boldsymbol{C}$-linear dimension satisfies $\dim_{\boldsymbol{C}}(K_p) \leqslant p\dim_{\boldsymbol{C}}(K)$. Let

$$\mathcal{F}_n(K_p) = B \oplus \bigoplus_{k=1}^{n} K_p^{(\otimes_B)k} \subseteq \mathcal{F}_n(\tilde{H}) \subseteq \mathcal{F}(\tilde{H}).$$

Clearly, $\mathcal{F}_n(K_p)$ is a finite-dimensional B,B-subbimodule of $\mathcal{F}_n(\tilde{H})$. By Proposition 6.2, there is $Q_{n,p} \in \mathcal{L}(\mathcal{F}_n(\tilde{H}))$, which is the projection onto $\mathcal{F}_n(K_p)$; note that $Q_{n,p}$ commutes with the left action of B on $\mathcal{F}_n(\tilde{H})$. Consider the completely positive contractions

$$\begin{aligned} \Theta_{n,p} :\ & \mathcal{L}(\mathcal{F}_n(\tilde{H})) \to \mathcal{L}(\mathcal{F}_n(K_p)), & \Theta_{n,p}(x) &= Q_{n,p} x Q_{n,p}, \\ \Upsilon_{n,p} :\ & \mathcal{L}(\mathcal{F}_n(K_p)) \to \mathcal{L}(\mathcal{F}_n(\tilde{H})), & \Upsilon_{n,p}(y) &= Q_{n,p} y Q_{n,p} + V_{B,n}^* y V_{B,n}(1 - Q_{n,p}), \end{aligned}$$

where $V_{B,n} \in \mathcal{L}(B, \mathcal{F}_n(\tilde{H}))$ maps B to the submodule $B \oplus 0$ of $\mathcal{F}_n(\tilde{H})$ via $b \mapsto b \oplus 0$. Since $l(h)^*(1 - Q_{n,p}) = 0$ whenever $h \in K_p$, we see that $\Psi_n \circ \Upsilon_{n,p} \circ \Theta_{n,p} \circ \Phi_n(x) = x$ for every $x \in \Omega(n, K_p)$. As $\mathcal{L}(\mathcal{F}_n(K_p))$ is a finite-dimensional C^*-algebra, we have that

$$\mathrm{rcp}(\pi, \omega \cup \gamma(\omega) \cup \cdots \cup \gamma^{p-1}(\omega), \delta) \leqslant \mathrm{rank}\, \mathcal{L}(\mathcal{F}_n(K_p)).$$

Because the C^*-algebra $\mathcal{L}(\mathcal{F}_n(K_p))$ can be faithfully represented on the Hilbert space $\mathcal{F}_n(K_p) \otimes_B \mathcal{V}$, making a crude estimate we get

$$\begin{aligned}\mathrm{rank}\, \mathcal{L}(\mathcal{F}_n(K_p)) &\leqslant \dim(\mathcal{F}_n(K_p) \otimes_B \mathcal{V}) \leqslant \dim(\mathcal{V}) \biggl(\sum_{k=0}^{n} \dim_{\boldsymbol{C}}(K_p)^k \biggr) \\ &\leqslant n \dim(\mathcal{V}) \dim_{\boldsymbol{C}}(K_p)^n \leqslant np^n \dim(\mathcal{V}) \dim_{\boldsymbol{C}}(K)^n.\end{aligned}$$

Because this upper bound grows subexponentially as $p \to \infty$, we conclude that

$$\mathrm{ht}(\pi, \gamma, \omega, \delta) = 0.$$

□

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