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# THE GENERALIZED GOODWIN–STATON INTEGRAL

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*Abstract* Some properties of the generalized Goodwin–Staton integral are derived. Explicit error bounds for the asymptotic expansion are determined. In addition, results are obtained for the oscillatory case and when logarithmic factors are present.

*Keywords:* Goodwin–Staton integral; asymptotics; Stieltjes transform

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## 1. Introduction

The Goodwin–Staton integral

$$\int_0^\infty \frac{\mathrm{e}^{-t^2}}{t+x}\,\mathrm{d}t$$

has been discussed by Goodwin and Staton [**4**], Ritchie [**7**] and Erdélyi [**3**]. Its asymptotic properties can be found in [**6**]. The aim of the present investigation is to consider its generalization when a factor $t^\mu$ is added to the integrand and $x$ is replaced by a complex variable. The performance of such an integral is of interest because a saddle point, branch point and pole are present and can interact with one another. The interaction is of special relevance to asymptotic approximations.

Representations of the generalized integral by means of series are derived in § 2. Straightforward asymptotic expansions when the pole is in certain regions of the complex plane are obtained in § 3. In § 4 the question of what happens when the pole approaches the positive real axis is examined. As a result a formula for the optimal remainder is determined. That the formula is asymptotic is verified in § 5, which also supplies a bound for the error committed by truncation. Section 6 is devoted to the consequences of the theory when the exponent in the integrand is imaginary. Integrals with logarithmic factors are treated briefly in § 7.

## 2. Alternative representations

The integral to be discussed is

$$I(\mu,z)=\int_0^\infty \frac{t^\mu \mathrm{e}^{-t^2}}{t-z}\,\mathrm{d}t \tag{2.1}$$

so it can be viewed as a Stieltjes transform. It will be assumed that $z$ does not lie on the positive real axis. Also, $\mu$ will be taken to be real to simplify some formulae, though many of the results hold for complex $\mu$. The condition $\mu > -1$ will be imposed to secure convergence of the integral. In general, $\mu$ will be regarded as fixed while $z$ varies. However, it is straightforward to change the value of $\mu$ since

$$I(\mu+1, z) = (\tfrac{1}{2}\mu - \tfrac{1}{2})!\tfrac{1}{2} + zI(\mu, z). \tag{2.2}$$

Note also that

$$\frac{\mathrm{d}}{\mathrm{d}z} I(\mu, z) = \mu I(\mu-1, z) - 2I(\mu+1, z) \tag{2.3}$$

when $\mu > 0$.

Suppose firstly that $\frac{1}{4}\pi < \operatorname{ph} z < \frac{3}{4}\pi$. Then, with $\operatorname{ph}(-z^2) = 2\operatorname{ph} z - \pi$, $|\operatorname{ph}(-z^2)| < \frac{1}{2}\pi$. Hence

$$\begin{aligned} I(\mu, x) &= \int_0^\infty \frac{t^\mu (t+z)}{t^2 - z^2} \mathrm{e}^{-t^2}\,\mathrm{d}t \\ &= \int_0^\infty t^\mu (t+z) \mathrm{e}^{-t^2} \int_0^\infty \mathrm{e}^{-u(t^2 - z^2)}\,\mathrm{d}u\,\mathrm{d}t \\ &= \frac{1}{2} \int_0^\infty \left\{ \frac{(\frac{1}{2}\mu)!}{(u+1)^{\mu/2+1}} + \frac{(\frac{1}{2}\mu - \frac{1}{2})! z}{(u+1)^{\mu/2+1/2}} \right\} \mathrm{e}^{uz^2}\,\mathrm{d}u \end{aligned}$$

after the order of integration is interchanged. Since $|\operatorname{ph}(-z^2)| < \frac{1}{2}\pi$, the integrals can be expressed in terms of the complementary incomplete gamma function $\varGamma(\lambda, z)$, which is given by

$$\varGamma(\lambda, z) = \mathrm{e}^{-z} z^\lambda \int_0^\infty \mathrm{e}^{-zt} (1+t)^{\lambda-1}\,\mathrm{d}t$$

when $|\operatorname{ph} z| < \frac{1}{2}\pi$. Therefore,

$$I(\mu, z) = \tfrac{1}{2}\mathrm{e}^{-z^2} \{ (\tfrac{1}{2}\mu)! (-z^2)^{\mu/2} \varGamma(-\tfrac{1}{2}\mu, -z^2) + (\tfrac{1}{2}\mu - \tfrac{1}{2})! z (-z^2)^{\mu/2 - 1/2} \varGamma(\tfrac{1}{2} - \tfrac{1}{2}\mu, -z^2) \}. \tag{2.4}$$

Another version of (2.4) is obtained by the substitution

$$\varGamma(\lambda, z) = (\lambda - 1)! - \gamma(\lambda, z),$$

where $\gamma(\lambda, z)$ is the incomplete gamma function given by

$$\gamma(\lambda, z) = z^\lambda \sum_{m=0}^\infty \frac{(-z)^m}{m!(\lambda+m)}. \tag{2.5}$$

Thus

$$\begin{aligned} I(\mu, z) = &-\frac{\pi z^\mu \mathrm{e}^{-\mu\pi\mathrm{i} - z^2}}{\sin \mu\pi} \\ &- \tfrac{1}{2}\mathrm{e}^{-z^2} \left\{ (\tfrac{1}{2}\mu)! \sum_{m=0}^\infty \frac{z^{2m}}{m!(m - \frac{1}{2}\mu)} + (\tfrac{1}{2}\mu - \tfrac{1}{2})! \sum_{m=0}^\infty \frac{z^{2m+1}}{m!(m - \frac{1}{2}\mu + \frac{1}{2})} \right\}. \end{aligned} \tag{2.6}$$

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The formula (2.6) has been established subject to $\frac{1}{4}\pi < \operatorname{ph} z < \frac{3}{4}\pi$. However, $I(\mu, z)$ is regular for $0 < \operatorname{ph} z < 2\pi$, and so is the right-hand side of (2.6). Accordingly, by analytic continuation, (2.6) holds for $0 < \operatorname{ph} z < 2\pi$.

The discontinuity in $I(\mu, z)$ as $z$ crosses the positive real axis can be deduced from (2.6). It is

$$I(\mu, x) - I(\mu, x\mathrm{e}^{2\pi \mathrm{i}}) = 2\pi \mathrm{i} x^{\mu} \mathrm{e}^{-x^2}. \tag{2.7}$$

Consequently,

$$I(\mu, z\mathrm{e}^{2\pi \mathrm{i}}) - I(\mu, z) = -2\pi \mathrm{i} z^{\mu} \mathrm{e}^{-z^2}$$

or, more generally,

$$I(\mu, z\mathrm{e}^{2k\pi \mathrm{i}}) - I(\mu, z) = -2\pi \mathrm{i} z^{\mu} \frac{\sin k\mu\pi}{\sin \mu\pi} \exp\{(k-1)\mu\pi \mathrm{i} - z^2\} \tag{2.8}$$

(where $k$ is an integer) can be used to extend (2.6) to other ranges of $\operatorname{ph} z$. Alternatively, one can observe that a change of phase of $2\pi$ in $z$ does not alter the integral in $I(\mu, z)$. So the role of (2.8) is merely to adjust the phase of $z$ in the first term of (2.6) to keep it within the range of $(0, 2\pi)$.

There is a variant of (2.6) which stems from

$$\gamma(\lambda, z) = (\lambda - 1)! z^{\lambda} \mathrm{e}^{-z} \sum_{m=0}^{\infty} \frac{z^m}{(m+\lambda)!}. \tag{2.9}$$

The insertion of (2.9) in (2.4) leads to

$$I(\mu, z) = -\frac{\pi z^{\mu}}{\sin \mu\pi} \mathrm{e}^{-\mu\pi \mathrm{i} - z^2} + \frac{\pi}{2 \sin \frac{1}{2}\mu\pi} \sum_{m=0}^{\infty} \frac{(-)^m z^{2m}}{(m - \frac{1}{2}\mu)!} - \frac{\pi}{2 \cos \frac{1}{2}\mu\pi} \sum_{m=0}^{\infty} \frac{(-)^m z^{2m+1}}{(m + \frac{1}{2} - \frac{1}{2}\mu)!}. \tag{2.10}$$

When $\mu$ is zero or a positive integer neither (2.6) nor (2.10) is very convenient without modification. Let $\mu \to 0$ in (2.6). The first term on the right-hand side combines with the first term of the series to provide a limit and

$$I(0, z) = \mathrm{e}^{-z^2} \left\{ \pi \mathrm{i} - \ln z - \tfrac{1}{2}\gamma - \frac{1}{2} \sum_{m=1}^{\infty} \frac{z^{2m}}{m! m} - \tfrac{1}{2}\pi^{1/2} \sum_{m=0}^{\infty} \frac{z^{2m+1}}{m!(m + \frac{1}{2})} \right\}, \tag{2.11}$$

where $\gamma$ is Euler's constant. Observe that (2.7) and (2.8) remain valid in the limit as $\mu \to 0$.

Expansions when $\mu$ is a positive integer can be inferred from (2.11) and (2.2).

When $z = -x$ with $x > 0$, $I(0, z)$ can be expressed in terms of the extended gamma function defined by [**1**, **2**]

$$\Gamma(\alpha, x; b; \beta) = \int_x^{\infty} t^{\alpha - 1} \mathrm{e}^{-t - b/t^{\beta}} \, \mathrm{d}t.$$

The formula is

$$I(0,-x) = \tfrac{1}{2}\mathrm{e}^{-x^2}\Gamma(0,x^2;-2x;-\tfrac{1}{2}).$$

The expressions (2.6), (2.10) and (2.11) permit the calculation of $I(\mu,z)$ for small to moderate values of $z$ but are not very suitable for large $|z|$. The behaviour when $|z| \gg 1$ is examined in the next section.

## 3. Asymptotic expansions

Repeated application of (2.2) leads to

$$I(\mu,z) = -\frac{1}{2}\sum_{m=0}^{n-1}\frac{(\frac{1}{2}\mu+\frac{1}{2}m-\frac{1}{2})!}{z^{m+1}} + \frac{1}{z^n}I(\mu+n,z), \tag{3.1}$$

which offers the asymptotic expansion

$$I(\mu,z) \sim -\frac{1}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}\mu+\frac{1}{2}m-\frac{1}{2})!}{z^{m+1}} \tag{3.2}$$

when $|z| \gg 1$ provided that the last term in (3.1) can be bounded appropriately. It is assumed that $\mu$ is fixed, independent of $z$, and generally that $|z| \gg \mu$.

Now, when $\frac{1}{2}\pi \leqslant \operatorname{ph} z \leqslant \pi$, $|t-z| \geqslant |z|$ and so

$$\left|\frac{1}{z^n}I(\mu+n,z)\right| \leqslant \frac{(\frac{1}{2}\mu+\frac{1}{2}n-\frac{1}{2})!}{2|z|^{n+1}}. \tag{3.3}$$

This shows that the statement (3.2) is valid for $\frac{1}{2}\pi \leqslant \operatorname{ph} z \leqslant \pi$ and supplies a bound for the error when the series is truncated.

To extend the range of $\operatorname{ph} z$ down to $\frac{1}{6}\pi$, we use the fact that

$$(t^2 - 3^{1/2}t + 1)^{-1/2} - 1 - 2t$$

has a negative derivative and, consequently, does not exceed 0 for $t \geqslant 0$. Then, for $\operatorname{ph} z \geqslant \frac{1}{6}\pi$,

$$\frac{1}{|t-z|} \leqslant \frac{1}{|z|} + \frac{2t}{|z|^2},$$

so

$$\left|\frac{1}{z^n}I(\mu+n,z)\right| \leqslant \frac{(\frac{1}{2}\mu+\frac{1}{2}n-\frac{1}{2})!}{2|z|^{n+1}} + \frac{(\frac{1}{2}\mu+\frac{1}{2}n)!}{|z|^{n+2}}. \tag{3.4}$$

Again (3.2) has been verified and a bound for the error in truncation obtained.

The replacement of $z$ by its complex conjugate does not affect the estimates of $|t-z|$ above. Therefore, $\operatorname{ph} z$ can be changed to $-\operatorname{ph} z$ without altering (3.3) or (3.4), and (3.2) holds for $\frac{1}{6}\pi \leqslant |\operatorname{ph} z| \leqslant \pi$.

The bound in (3.4) is larger than that in (3.3) and it may be expected that the bound will increase with reducing $\operatorname{ph} z$ if the same technique of estimation is followed; indeed a possible bound for $0 < |\operatorname{ph} z| \leqslant \frac{1}{2}\pi$ comes from dividing the right-hand side of (3.3) by $|\sin(\operatorname{ph} z)|$. Therefore, another approach must be adopted for small $\operatorname{ph} z$. This is considered in the next section.

Another way of finding bounds similar to the above is set out in the appendix.

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## 4. The optimal remainder

The remainder in the expansion

$$I(\mu, z) = -\frac{1}{2}\sum_{m=0}^{n-1}(\tfrac{1}{2}\mu + \tfrac{1}{2}m - \tfrac{1}{2})!\frac{1}{z^{m+1}} + \frac{1}{z^n}I(\mu + n, z) \tag{4.1}$$

as $z$ approaches the positive real axis is dictated by the performance of $I(\mu + n, z)$ and this section is concerned with deriving suitable estimates of this integral.

Instead of dealing with the remainder for general $n$, our attention will be concentrated on that value of $n$ for which the remainder is optimal. A bound for general $n$ which covers part of the region $\operatorname{ph} z \leqslant \frac{1}{4}\pi$ including a portion of the upper side of the positive real axis is in Equation (A 7) of the appendix.

The ratio of the moduli of successive terms in the series of Equation (4.1) is roughly $(\mu + m)^{1/2}/(2|z|)$ as $m$ increases. Hence the optimal remainder occurs when $\mu + n$ is approximately $2|z|^2$. Choose $n$ such that $\mu + n - 1 < 2|z|^2$ and $\mu + n \geqslant 2|z|^2$. Then $\mu + n = 2|z|^2 + \nu$, where $0 \leqslant \nu < 1$.

Let $z = |z|\mathrm{e}^{\mathrm{i}\theta}$ with $0 < \delta \leqslant |\theta| \leqslant \pi$. Then

$$I(\mu + n, z) = |z|^{\mu+n}J(\nu, z), \tag{4.2}$$

where

$$J(\nu, z) = \int_0^\infty \frac{t^{r+\nu}\mathrm{e}^{-rt^2/2}}{t - \mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{d}t \tag{4.3}$$

and $r = 2|z|^2$.

The integrand of $J(\nu, z)$ has a saddle point at $t = 1$ on account of the largeness of $r$. The standard method of dealing with the saddle point [**6**] is to make the change of variable

$$s = \tfrac{1}{2}t^2 - \tfrac{1}{2} - \ln t \tag{4.4}$$

with $s$ non-negative. Then, for small $s$, $t - 1 \approx s^{1/2}$ when $t \geqslant 1$ and $t - 1 \approx -s^{1/2}$ when $t \leqslant 1$. The asymptotic series is obtained by expansion about $t = 1$ of that part of the integrand which does not involve the saddle point. In terms of $s$ the form of expansion is

$$\frac{t^{\nu+1}}{(t - \mathrm{e}^{\mathrm{i}\theta})(t^2 - 1)} = \begin{cases} \displaystyle\sum_{m=0}^\infty a_m s^{m/2-1/2}, & t \geqslant 1, \\ \displaystyle\sum_{m=0}^\infty a_m (-)^{m+1} s^{m/2-1/2}, & t \leqslant 1. \end{cases} \tag{4.5}$$

As a result,

$$J(\nu, z) \sim \frac{2}{r^{1/2}}\mathrm{e}^{-r/2}\sum_{m=0}^\infty (m - \tfrac{1}{2})!\frac{a_{2m}}{r^m} \tag{4.6}$$

for $|z| \gg 1$.

It is evident from the mode of derivation of (4.5) that the coefficients $a_m$ become infinite as $\mathrm{e}^{\mathrm{i}\theta} \to 1$. Therefore, a restriction such as $\pi \geqslant |\theta| \geqslant \delta > 0$ must be applied to (4.6). Note that a change in $\theta$ by $2\pi$ does not affect (4.6).

To allow $\theta$ to approach 0, the influence of the pole has to be separated off. Put $s = u^2$ in (4.4). Take $u$ to be negative when $t < 1$ and positive when $t > 1$. In other words

$$t - 1 \approx u \tag{4.7}$$

when $t$ is in the neighbourhood of 1. The change of variable gives

$$J(\nu, z) = \mathrm{e}^{-r/2} \int_{-\infty}^{\infty} \frac{2ut^{\nu+1}\mathrm{e}^{-ru^2}}{(t - \mathrm{e}^{\mathrm{i}\theta})(t^2 - 1)}\,\mathrm{d}u.$$

Extract the pole by writing

$$\frac{2ut^{\nu+1}}{(t - \mathrm{e}^{\mathrm{i}\theta})(t^2 - 1)} = \frac{\mathrm{e}^{\mathrm{i}\nu\theta}}{u - \alpha} + f(u), \tag{4.8}$$

where

$$\alpha^2 = \tfrac{1}{2}\mathrm{e}^{2\mathrm{i}\theta} - \tfrac{1}{2} - \mathrm{i}\theta. \tag{4.9}$$

As $\theta \to 0$, $\alpha \approx \mathrm{i}\theta$ to comply with (4.7). Since $\operatorname{Im}\alpha$ cannot change sign according to (4.9) when $\pi \geqslant \theta \geqslant 0$, it follows that $\alpha$ is in the upper half of the $u$-plane for $\pi > \theta > 0$. Correspondingly, $\alpha$ is in the lower half of the $u$-plane when $-\pi < \theta < 0$.

For $\operatorname{Im}\alpha > 0$,

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-ru^2}}{u - \alpha}\,\mathrm{d}u = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{e}^{-ru^2} \int_0^{\infty} \mathrm{e}^{-\mathrm{i}y(u-\alpha)}\,\mathrm{d}y\,\mathrm{d}u = \mathrm{i}\left(\frac{\pi}{r}\right)^{1/2} \int_0^{\infty} \mathrm{e}^{\mathrm{i}y\alpha - y^2/4r}\,\mathrm{d}y$$

after interchanging the order of integration. The final integral can be expressed in terms of the complementary error function

$$\operatorname{erfc}(w) = \frac{2}{\pi^{1/2}} \int_w^{\infty} \mathrm{e}^{-y^2}\,\mathrm{d}y$$

and

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-ru^2}}{u - \alpha}\,\mathrm{d}u = \pi\mathrm{i}\mathrm{e}^{-r\alpha^2} \operatorname{erfc}(-\mathrm{i}\alpha r^{1/2})$$

when $\operatorname{Im}\alpha > 0$. For $\operatorname{Im}\alpha < 0$, change i to $-$i on the right-hand side. Then, since

$$\operatorname{erfc}(-w) = 2 - \operatorname{erfc}(w), \tag{4.10}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-ru^2}}{u - \alpha}\,\mathrm{d}u = \pi\mathrm{i}\mathrm{e}^{-r\alpha^2}\{2H(\operatorname{Im}\alpha) - \operatorname{erfc}(\mathrm{i}\alpha r^{1/2})\}, \tag{4.11}$$

where $H(x)$ is the Heaviside step function which is 1 for $x > 0$ and 0 for $x < 0$.

The function $f(u)$ has no singularity at $u=\alpha$ and is regular in that part of the $u$-plane mapped by the $t$-plane. Hence $f(u)$ can be expanded in a power series about the origin in the form

$$f(u)=\sum_{m=0}^{\infty} f_m(\theta)u^m. \tag{4.12}$$

In fact, it is clear from (4.8) and (4.5) with $s=u^2$ that

$$f_m(\theta)=2a_m+\frac{\mathrm{e}^{\mathrm{i}\nu\theta}}{\alpha^{m+1}}. \tag{4.13}$$

The first few $a_m$ are given by

$$\begin{aligned}
2a_0 &= \frac{1}{1-\mathrm{e}^{\mathrm{i}\theta}},\\
2a_1 &= \frac{-1}{(\mathrm{e}^{\mathrm{i}\theta}-1)^2}-\frac{3\nu+1}{3(\mathrm{e}^{\mathrm{i}\theta}-1)},\\
2a_2 &= -\frac{1}{(\mathrm{e}^{\mathrm{i}\theta}-1)^3}-\frac{2\nu+1}{2(\mathrm{e}^{\mathrm{i}\theta}-1)^2}+\frac{1-3\nu^2}{6(\mathrm{e}^{\mathrm{i}\theta}-1)},\\
2a_4 &= -\frac{1}{(\mathrm{e}^{\mathrm{i}\theta}-1)^5}-\frac{5+6\nu}{6(\mathrm{e}^{\mathrm{i}\theta}-1)^4}+\frac{5-12\nu-18\nu^2}{36(\mathrm{e}^{\mathrm{i}\theta}-1)^3}\\
&\quad -\frac{1-16\nu-6\nu^2+12\nu^3}{72(\mathrm{e}^{\mathrm{i}\theta}-1)^2}-\frac{1+24\nu-6\nu^2-24\nu^3+9\nu^4}{216(\mathrm{e}^{\mathrm{i}\theta}-1)}.
\end{aligned}$$

In the limit as $\theta\to 0$,

$$\begin{aligned}
f_0 &\to \nu+\tfrac{1}{6},\\
f_1 &\to \tfrac{1}{2}\nu^2-\tfrac{1}{6}\nu-\tfrac{5}{36},\\
f_2 &\to \tfrac{1}{6}\nu^3-\tfrac{1}{4}\nu^2-\tfrac{1}{12}\nu+\tfrac{37}{540},\\
f_4 &\to \frac{378\nu^5-2205\nu^4+2730\nu^3+1890\nu^2-2583\nu-215}{45\,360}.
\end{aligned}$$

The substitution (4.8) now gives

$$J(\nu,z)\sim \mathrm{e}^{-r/2}\biggl[\pi\mathrm{i}\mathrm{e}^{-r\alpha^2+\mathrm{i}\nu\theta}\{2H(\operatorname{Im}\alpha)-\operatorname{erfc}(\mathrm{i}\alpha r^{1/2})\}+\sum_{m=0}^{\infty}(m-\tfrac{1}{2})!\frac{f_{2m}(\theta)}{r^{m+1/2}}\biggr]. \tag{4.14}$$

If $\operatorname{Im}\alpha<0$ and $|\alpha r^{1/2}|\gg 1$, the asymptotic expansion

$$\operatorname{erfc} w\sim\frac{\mathrm{e}^{-w^2}}{\pi w}\sum_{m=0}^{\infty}(m-\tfrac{1}{2})!\frac{(-)^m}{w^{2m}}\quad(|\operatorname{ph} w|<\tfrac{1}{2}\pi) \tag{4.15}$$

together with (4.13) shows that (4.14) reproduces (4.6) for $-\pi\leqslant\theta\leqslant-\delta$. Likewise, invocation of (4.10) and (4.15) reveals that (4.14) coincides with (4.6) for $\delta\leqslant\theta\leqslant\pi$.

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On the other hand, (4.14) continues to hold as $\theta \to 0$, so, if $x > 0$,

$$J(\nu, x \pm \mathrm{i}0) \sim \mathrm{e}^{-x^2}\left[\pm\pi\mathrm{i} + \sum_{m=0}^{\infty}(m-\tfrac{1}{2})!\frac{f_{2m}(0)}{2^{m+1/2}x^{2m+1}}\right] \quad \text{as } x \to \infty. \tag{4.16}$$

It can be shown that the expansion (4.14) is asymptotic for $|\theta| \leqslant \pi$. However, it is most useful for the smaller values of $\theta$, since larger values have been dealt with in § 3. Also, for numerical purposes, it is desirable to have some idea of the error committed when the series in (4.14) is truncated and, usually, the estimates increase as the range of $\theta$ widens. For these reasons the discussion of the asymptotic properties of (4.14) in the next section is limited to smaller values of $\theta$.

As regards the last term of (4.1) it can be inferred from (4.2) that

$$\begin{aligned} I(\mu+n, z) \sim \pi\mathrm{i}z^{\mu+n}\mathrm{e}^{-z^2}\{2H(\operatorname{Im}\alpha) - \operatorname{erfc}(\mathrm{i}\alpha 2^{1/2}|z|)\} \\ + |z|^{\mu+n}\mathrm{e}^{-|z|^2}\sum_{m=0}^{\infty}(m-\tfrac{1}{2})!\frac{f_{2m}(\theta)}{2^{m+1/2}|z|^{2m+1}} \end{aligned} \tag{4.17}$$

and, for $x > 0$,

$$I(\mu+n, x \pm \mathrm{i}0) \sim x^{\mu+n}\mathrm{e}^{-x^2}\left\{\pm\pi\mathrm{i} + \sum_{m=0}^{\infty}(m-\tfrac{1}{2})!\frac{f_{2m}(0)}{2^{m+1/2}x^{2m+1}}\right\}. \tag{4.18}$$

Formula (4.17) not only provides an estimate of the error in optimal truncation of the asymptotic series for $I(\mu, z)$ but also offers an asymptotic expansion for the generalized Goodwin–Staton integral when both parameters are large.

## 5. Error bound for the remainder

To establish that (4.14) is asymptotic, put

$$f(u) = \sum_{m=0}^{2p-1} f_m(\theta)u^m + g_p(u)u^{2p}, \tag{5.1}$$

where $p$ is a non-negative integer and the series is absent if $p = 0$.

Then (4.14) is replaced by

$$J(\nu, z) = \mathrm{e}^{-1/2r}\left[\pi\mathrm{i}\mathrm{e}^{-r\alpha^2+\mathrm{i}\nu\theta}\{2H(\operatorname{Im}\alpha) - \operatorname{erfc}(\mathrm{i}\alpha r^{1/2})\} + \sum_{m=0}^{p-1}(m-\tfrac{1}{2})!\frac{f_{2m}(\theta)}{r^{m+1/2}} + G_p(\theta, r)\right] \tag{5.2}$$

with

$$G_p(\theta, r) = \int_{-\infty}^{\infty} g_p(u)u^{2p}\mathrm{e}^{-ru^2/2}\,\mathrm{d}u. \tag{5.3}$$

The main interest in estimating $G_p$ occurs for the smaller values of $\theta$ on account of the formulae of § 3. So consider what happens when $\mathrm{e}^{\mathrm{i}\theta}$ lies in a circle with centre $t = 1$ and radius $\rho$ ($\rho < 1$). In the $u$-plane the point corresponding to the point $t = 1 + \rho e^{i\phi}$ on the circumference of the circle is given by

$$u^2 = \rho e^{i\phi} + \tfrac{1}{2}\rho^2 e^{2i\phi} - \ln(1 + \rho e^{i\phi}). \tag{5.4}$$

There is no difficulty in checking that, as $\phi$ moves round the circle with $\rho = \frac{3}{4}$, $|u|^2$ has a minimum of 0.471 634 and a maximum of 0.917 544. Therefore, when $\rho = \frac{3}{4}$, the boundary in the $u$-plane of the map of the circle lies between the circles with centre the origin and radii 0.686 753, 0.957 885. Similarly, when $\rho = \frac{1}{2}$, the bounding circles in the $u$-plane are of radii 0.468 546 and 0.564 045.

The value of $|\alpha|$ increases with $\theta$. In particular, $|\alpha| = 0.614\,629$ when $\theta = \frac{1}{5}\pi$ and $|\alpha| = 0.515\,66$ when $\theta = \frac{1}{6}\pi$. Hence $\alpha$ lies in the inner of the two bounding circles in the $u$-plane for $|\theta| \leqslant \frac{1}{5}\pi$ when $\rho = \frac{3}{4}$. However, the $\alpha$ corresponding to $\theta = \frac{1}{6}\pi$ lies outside the inner circle when $\rho = \frac{1}{2}$. Accordingly, to have a region which includes $\theta = \frac{1}{6}\pi$ (so that there is an overlap with the region covered by (3.1) and (3.4)) and which keeps $\alpha$ within the inner circle, the choice $\rho = \frac{3}{4}$ will be made. Also the restriction $|\theta| \leqslant \frac{1}{5}\pi$ is enforced.

Let $b = 0.686\,753$. By virtue of (4.8)

$$|f(u)| \leqslant \frac{3.43}{0.75 - 2\sin(\frac{1}{2}|\theta|)} + \frac{1}{b - |\alpha|} \tag{5.5}$$

on the boundary in the $u$-plane. Denote the right-hand side of (5.5) by $A(\theta)$ so that

$$A(0) = 6.02. \tag{5.6}$$

By Cauchy's inequality and (5.5)

$$|f_m(\theta)| \leqslant \frac{A(\theta)}{b^m}. \tag{5.7}$$

A bound for $G_p$ is derived by splitting the interval of integration into three pieces. In the first piece $|u| \leqslant \frac{1}{2}b$ and here

$$|g_p(u)| = \left|\sum_{m=2p}^{\infty} f_m(\theta)u^{m-2p}\right| \leqslant \frac{2A(\theta)}{b^{2p}}$$

on account of (5.7). Therefore,

$$\begin{aligned}\left|\int_{-b/2}^{b/2} g_p(u)u^{2p}e^{-ru^2/2}\,du\right| &\leqslant \frac{2A(\theta)}{b^{2p}}\int_{-b/2}^{b/2} u^{2p}e^{-ru^2/2}\,du \\ &\leqslant \frac{2A(\theta)}{b^{2p}}\gamma(p + \tfrac{1}{2}, \tfrac{1}{8}rb^2)\left(\frac{2}{r}\right)^{p+1/2}.\end{aligned} \tag{5.8}$$

Outside the interval just discussed, some information about $dt/du$ is required. Now

$$\frac{d^2t}{du^2} = \frac{4t}{(t^2-1)^3}\{(t^2+1)\ln t - t^2 + 1\}.$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\ln t-\frac{t^2-1}{t^2+1}\right\}=\frac{(t^2-1)^2}{t(t^2+1)^2}$$

which is non-negative. Consequently, $(t^2+1)\ln t-t^2+1$ is an increasing function of $t$ which is negative for $t<1$ and positive for $t>1$. Hence $\mathrm{d}^2t/\mathrm{d}u^2$ is always non-negative. It follows that $\mathrm{d}t/\mathrm{d}u$ is an increasing function of $t$ which starts at 0 when $t=0$ and approaches $2^{1/2}$ as $t\to\infty$, passing through 1 at $t=1$. In particular $\mathrm{d}t/\mathrm{d}u<2^{1/2}$ throughout the interval $t\geqslant 0$.

For $u\geqslant\frac{1}{2}b$, the properties of $\mathrm{d}t/\mathrm{d}u$ just established show that $t-1>u$. Since $t=1.36$ when $u=\frac{1}{2}b$ it is clear that $t\leqslant 4u$. Hence,

$$\left|\frac{t^\nu}{t-\mathrm{e}^{\mathrm{i}\theta}}\frac{\mathrm{d}t}{\mathrm{d}u}\right|<2^{2\nu+1/2}u^{\nu-1}.$$

The points $\alpha$ lie close to the imaginary axis in the $u$-plane for $|\theta|\leqslant\frac{1}{5}\pi$. Hence, from Equation (4.8),

$$|f(u)|<2^{2\nu+1/2}u^{\nu-1}+1/u$$

on $u\geqslant\frac{1}{2}b$. It follows from (5.1) and (5.7) that, for $u\geqslant\frac{1}{2}b$,

$$|g_p(u)|<B_p, \tag{5.9}$$

where

$$B_p=\left(\frac{2}{b}\right)^{2p+1}\{2^{\nu+1/2}b^\nu+1+A(\theta)b\}, \tag{5.10}$$

the term involving $A(\theta)$ being omitted when $p=0$. Hence

$$\left|\int_{b/2}^{\infty}g_p(u)u^{2p}\mathrm{e}^{-ru^2/2}\,\mathrm{d}u\right|<\tfrac{1}{2}B_p\left(\frac{2}{r}\right)^{p+1/2}\Gamma(p+\tfrac{1}{2},\tfrac{1}{8}rb^2). \tag{5.11}$$

On the interval $u\leqslant-\frac{1}{2}b$, $t\leqslant 0.68$, so $0\leqslant\mathrm{d}t/\mathrm{d}u<1$. Consequently,

$$|f(u)|<3+\frac{1}{|u|} \tag{5.12}$$

for $u\leqslant-\frac{1}{2}b$ and

$$|g_p(u)|<\left\{3+\frac{2}{b}+2A(\theta)\right\}\left(\frac{2}{b}\right)^{2p},$$

the term containing $A(\theta)$ being absent when $p=0$. Therefore,

$$\left|\int_{-\infty}^{-b/2}g_p(u)u^{2p}\mathrm{e}^{-ru^2/2}\,\mathrm{d}u\right|<\frac{1}{2}\left(\frac{2}{b}\right)^{2p}\left(\frac{2}{r}\right)^{p+1/2}\left\{3+\frac{2}{b}+2A(\theta)\right\}\Gamma(p+\tfrac{1}{2},\tfrac{1}{8}rb^2). \tag{5.13}$$

The combination of (5.3), (5.8), (5.11) and (5.13) gives

$$\begin{aligned}|G_p(\theta,r)|<\left(\frac{2}{r}\right)^{p+1/2}\bigg[&\frac{2A(\theta)}{b^{2p}}\gamma(p+\tfrac{1}{2},\tfrac{1}{8}rb^2)\\&+\left(\frac{2}{b}\right)^{2p+1}\{2^{\nu-1/2}b^\nu+\tfrac{3}{4}b+1+A(\theta)\}\Gamma(p+\tfrac{1}{2},\tfrac{1}{8}rb^2)\bigg].\end{aligned} \tag{5.14}$$

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The bound in (5.14) shows that the error in truncating the series in $J(\nu, z)$ is $O(r^{-p-1/2})$. It confirms that an asymptotic expansion as $r \to \infty$ has been obtained.

Insertion of the inequalities

$$\gamma(p + \tfrac{1}{2}, \tfrac{1}{8}rb^2) < (p - \tfrac{1}{2})!, \qquad \Gamma(p + \tfrac{1}{2}, \tfrac{1}{8}rb^2) < (p - \tfrac{1}{2})!$$

into (5.14) leads to a bound which is simpler to evaluate. Its disadvantage is that it is generally quite a bit larger than (5.14) because finite limits of integration are replaced by infinite ones.

On the other hand, a bound which is lower than (5.14) can be derived by retaining the dependence on $u$ in the inequalities for longer. An illustration of the technique is provided by the case $p = 0$. Instead of calling on (5.9) and (5.10), we employ

$$|g_0(u)| < 2^{2\nu+1/2}u^{\nu-1} + \frac{1}{u}.$$

Then

$$\left|\int_{b/2}^{\infty} g_0(u)\mathrm{e}^{-ru^2/2}\,\mathrm{d}u\right| < 2^{2\nu+1/2}\left(\frac{2}{r}\right)^{\nu/2}\Gamma(\tfrac{1}{2}\nu, \tfrac{1}{8}rb^2) + \tfrac{1}{2}\Gamma(0, \tfrac{1}{8}rb^2). \tag{5.15}$$

The formula (5.11) with $p = 0$ is recovered from (5.15) by using the inequality

$$\Gamma(\alpha, x) < \frac{\Gamma(\alpha + \beta, x)}{x^{\beta}}, \tag{5.16}$$

which is valid for $\beta > 0$ and $x > 0$. Take $\beta = \frac{1}{2} - \frac{1}{2}\nu$ in one $\Gamma$ and $\beta = \frac{1}{2}$ in the other. While this method does offer a lower error bound, it is obvious from (5.15) that the bound becomes much more complicated than (5.14) as $p$ increases.

While the above discussion has been confined to $|\theta| < \frac{1}{5}\pi$, the same path may be traced to verify that (4.14) is asymptotic for $|\theta| < \pi$. Bounds similar to (5.7) can be based on Cauchy's theorem, but the contours are less simple than those used for $|\theta| < \frac{1}{5}\pi$. This is because, although the contour in the $t$-plane need only enclose the relevant $\mathrm{e}^{\mathrm{i}\theta}$, the contour in the $u$-plane must circumvent the branch lines going from $\pi^{1/2}\mathrm{e}^{\pm 3\pi\mathrm{i}/4}$ (corresponding to $t = \mathrm{e}^{\pm\pi\mathrm{i}}$) to negative infinity. Moreover, it can be expected that the error bounds will normally be appreciably larger than that of (5.14).

## 6. The oscillatory case

The preceding theory can be applied in order to supply information about the integral

$$K(\mu, z) = \int_0^{\infty} \frac{t^{\mu}\mathrm{e}^{\mathrm{i}t^2}}{t - z}\,\mathrm{d}t. \tag{6.1}$$

The integral converges at infinity for $\mu < 2$ and it will be supposed from now on that this condition holds for $K(\mu, z)$.

If $\frac{1}{4}\pi < \operatorname{ph} z < 2\pi$, deformation of the contour gives

$$K(\mu, z) = \mathrm{e}^{\mathrm{i}\mu\pi/4}I(\mu, z\mathrm{e}^{-\mathrm{i}\pi/4}).$$

By virtue of (2.6),

$$K(\mu,z) = -\frac{\pi z^{\mu}\mathrm{e}^{-\mu\pi\mathrm{i}+\mathrm{i}z^2}}{\sin\mu\pi} - \tfrac{1}{2}\mathrm{e}^{\mathrm{i}z^2+\mathrm{i}\mu\pi/4}\bigg\{(\tfrac{1}{2}\mu)!\sum_{m=0}^{\infty}\frac{(-\mathrm{i})^m z^{2m}}{m!(m-\frac{1}{2}\mu)} + (\tfrac{1}{2}\mu-\tfrac{1}{2})!\,\mathrm{e}^{-\mathrm{i}\pi/4}\sum_{m=0}^{\infty}\frac{(-\mathrm{i})^m z^{2m+1}}{m!(m-\frac{1}{2}\mu+\frac{1}{2})}\bigg\} \tag{6.2}$$

and, from (2.11),

$$K(0,z) = \mathrm{e}^{\mathrm{i}z^2}\bigg\{\frac{5\pi\mathrm{i}}{4} - \ln z - \tfrac{1}{2}\gamma - \frac{1}{2}\sum_{m=1}^{\infty}\frac{(-\mathrm{i})^m z^{2m}}{m!m} - \frac{\pi^{1/2}}{2}\mathrm{e}^{-\mathrm{i}\pi/4}\sum_{m=0}^{\infty}\frac{(-\mathrm{i})^m z^{2m+1}}{m!(m+\frac{1}{2})}\bigg\}. \tag{6.3}$$

The formulae (6.2) and (6.3) have been derived for $\frac{1}{4}\pi < \operatorname{ph} z < 2\pi$. They hold for $0 < \operatorname{ph} z < 2\pi$ by analytic continuation. For this range of $\operatorname{ph} z$,

$$K(\mu,z) = \mathrm{e}^{\mathrm{i}\mu\pi/4} I(\mu, z\mathrm{e}^{-\mathrm{i}\pi/4}) + 2\pi\mathrm{i}z^{\mu}\mathrm{e}^{\mathrm{i}z^2}\{H(\operatorname{ph} z) - H(\operatorname{ph} z - \tfrac{1}{4}\pi)\}. \tag{6.4}$$

The discontinuity in $K$ when $z$ crosses the real axis satisfies

$$K(\mu, z\mathrm{e}^{2\pi\mathrm{i}}) - K(\mu,z) = -2\pi\mathrm{i}z^{\mu}\mathrm{e}^{\mathrm{i}z^2}. \tag{6.5}$$

The asymptotic behaviour of $K$ can be deduced from that of $I$ and (6.4). It is of the form

$$K(\mu,z) = -\tfrac{1}{2}\mathrm{e}^{\mathrm{i}(\mu+1)\pi/4}\sum_{m=0}^{n-1}(\tfrac{1}{2}\mu+\tfrac{1}{2}m-\tfrac{1}{2})!\frac{\mathrm{e}^{\mathrm{i}m\pi/4}}{z^{m+1}} + R_n(\mu,z) + 2\pi\mathrm{i}z^{\mu}\mathrm{e}^{\mathrm{i}z^2}\{H(\operatorname{ph} z) - H(\operatorname{ph} z - \tfrac{1}{4}\pi)\} \tag{6.6}$$

as $|z| \to \infty$. Various bounds for $R_n$ are available from § 3. In particular

$$|R_n(\mu,z)| \leqslant \frac{(\frac{1}{2}\mu+\frac{1}{2}n-\frac{1}{2})!}{2|z|^{n+1}} + \frac{(\frac{1}{2}\mu+\frac{1}{2}n)!}{|z|^{n+2}} \tag{6.7}$$

for $\frac{5}{12}\pi \leqslant \theta \leqslant \pi$ and $-\frac{3}{4}\pi \leqslant \theta \leqslant \frac{1}{12}\pi$, which includes the critical range around $\theta = 0$.

Sections 4 and 5 are pertinent when $\theta$ is in the vicinity of $\frac{1}{4}\pi$. To take advantage of (4.17) in evaluating $R_n$, allowance must be made for the change of phase from $\theta$ to $\theta - \frac{1}{4}\pi$. Let $\phi = \theta - \frac{1}{4}\pi$ and define $\alpha_1$ by

$$\alpha_1^2 = \tfrac{1}{2}\mathrm{e}^{2\mathrm{i}\phi} - \tfrac{1}{2} - \mathrm{i}\phi,$$

so $\alpha_1 \approx \mathrm{i}(\theta - \frac{1}{4}\pi)$ as $\theta \to \frac{1}{4}\pi$. Then

$$R_n(\mu,z) + 2\pi\mathrm{i}z^{\mu}\mathrm{e}^{\mathrm{i}z^2}\{H(\operatorname{ph} z) - H(\operatorname{ph} z - \tfrac{1}{4}\pi\} = \pi\mathrm{i}\mathrm{e}^{\mathrm{i}z^2}z^{\mu}\{2 - \operatorname{erfc}(\mathrm{i}\alpha_1 2^{1/2}|z|\} + \mathrm{e}^{-|z|^2+\mathrm{i}\mu\pi/4-\mathrm{i}n\phi}|z|^{\mu}\sum_{m=0}^{\infty}(m-\tfrac{1}{2})!\frac{f_{2m}(\phi)}{2^{m+1/2}|z|^{2m+1}}, \tag{6.8}$$

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which shows that $K(\mu, z)$ is continuous as $\operatorname{ph} z$ passes through $\frac{1}{4}\pi$. This behaviour is consistent with (6.1). Indeed, when $z = x\mathrm{e}^{\mathrm{i}\pi/4}$, the right-hand side of (6.8) reduces to

$$\mathrm{e}^{-x^2}(x\mathrm{e}^{\mathrm{i}\pi/4})^{\mu}\biggl\{\pi\mathrm{i} + \sum_{m=0}^{\infty}(m-\tfrac{1}{2})!\frac{f_{2m}(0)}{2^{m+1/2}x^{2m+1}}\biggr\}. \tag{6.9}$$

A bound for the error in truncating the series in (6.8) is available from $G_p(\phi, r)$, as given by (5.14) and the subsequent discussion. $A(\phi)$ is obtainable from (5.5) with $\alpha$ replaced by $\alpha_1$.

The somewhat more general integral

$$\int_0^{\infty}\frac{t^{\mu}\mathrm{e}^{wt^2}}{t-z}\,\mathrm{d}t = \frac{\mathrm{e}^{\mathrm{i}\mu\pi/2}}{w^{\mu/2}}I(\mu, -\mathrm{i}w^{1/2}z) + 2\pi\mathrm{i}z^{\mu}\mathrm{e}^{wz^2}\{H(\operatorname{ph} z) - H(\operatorname{ph} z + \tfrac{1}{2}\operatorname{ph} w - \tfrac{1}{2}\pi)\}$$

for $\frac{1}{2}\pi \leqslant \operatorname{ph} w \leqslant \pi$ can be dealt with in a similar manner.

## 7. Other integrals

Logarithmic terms can be introduced by taking derivatives with respect to $\mu$. For example,

$$\begin{aligned}\int_0^{\infty}\frac{t^{\mu}\ln t}{t-z}\mathrm{e}^{-t^2}\,\mathrm{d}t &= \frac{\pi z^{\mu}\mathrm{e}^{-\mu\pi\mathrm{i}-z^2}}{\sin\mu\pi}(\pi\mathrm{i} + \pi\cot\mu\pi - \ln z)\\ &\quad - \tfrac{1}{4}\mathrm{e}^{-z^2}\biggl[(\tfrac{1}{2}\mu)!\sum_{m=0}^{\infty}\biggl\{\frac{\psi(\frac{1}{2}\mu)}{m-\frac{1}{2}\mu} + \frac{1}{(m-\frac{1}{2}\mu)^2}\biggr\}\frac{z^{2m}}{m!}\\ &\qquad + (\tfrac{1}{2}\mu-\tfrac{1}{2})!\sum_{m=0}^{\infty}\frac{z^{2m+1}}{m!}\biggl\{\frac{\psi(\frac{1}{2}\mu-\frac{1}{2})}{m-\frac{1}{2}\mu+\frac{1}{2}} + \frac{1}{(m-\frac{1}{2}\mu+\frac{1}{2})^2}\biggr\}\biggr]\end{aligned} \tag{7.1}$$

where $\psi(z) = z!'/z!$. Allow $\mu \to 0$ in (7.1) and then

$$\begin{aligned}\int_0^{\infty}\frac{\ln t}{t-z}\mathrm{e}^{-t^2}\,\mathrm{d}t &= \mathrm{e}^{-z^2}\{\tfrac{1}{8}\gamma^2 - \tfrac{7}{48}\pi^2 - \tfrac{1}{2}(\ln z - \pi\mathrm{i})^2\}\\ &\quad - \tfrac{1}{4}\mathrm{e}^{-z^2}\biggl[\sum_{m=1}^{\infty}\frac{z^{2m}}{m!}\biggl\{\frac{1}{m^2} - \frac{\gamma}{m}\biggr\}\\ &\qquad + \pi^{1/2}\sum_{m=0}^{\infty}\frac{z^{2m+1}}{m!}\biggl\{\frac{1}{(m+\frac{1}{2})^2} - \frac{\gamma + 2\ln 2}{m+\frac{1}{2}}\biggr\}\biggr].\end{aligned} \tag{7.2}$$

The analogue of (3.2) is

$$\int_0^{\infty}\frac{t^{\mu}\ln t}{t-z}\mathrm{e}^{-t^2}\,\mathrm{d}t \sim -\frac{1}{4}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}\mu+\frac{1}{2}m-\frac{1}{2})!\psi(\frac{1}{2}\mu+\frac{1}{2}m-\frac{1}{2})}{z^{m+1}}, \tag{7.3}$$

whereas the analogue of (4.1) is

$$\int_0^\infty \frac{t^\mu \ln t}{t-z}\mathrm{e}^{-t^2}\,\mathrm{d}t \sim -\frac{1}{4}\sum_{m=0}^{n-1}\frac{(\frac{1}{2}\mu+\frac{1}{2}m-\frac{1}{2})!\psi(\frac{1}{2}\mu+\frac{1}{2}m-\frac{1}{2})}{z^{m-1}} + \pi\mathrm{i}\mathrm{e}^{-z^2}z^\mu\ln z\{2H(\operatorname{Im}\alpha)-\operatorname{erfc}(\mathrm{i}\alpha 2^{1/2}|z|)\} + \mathrm{e}^{-|z|^2-\mathrm{i}n\theta}|z|^\mu\sum_{m=0}^{\infty}\frac{(m-\frac{1}{2})!}{2^{m+1/2}|z|^{2m+1}}\biggl\{f_{2m}(\theta)\ln|z|+\frac{\partial f_{2m}(\theta)}{\partial\mu}\biggr\}. \tag{7.4}$$

The derivative of $f_{2m}$ occurs in (7.4) because it depends on $\nu=\mu+n-2|z|^2$.

The expansions in (7.3) and (7.4) must be regarded as purely formal until suitable error bounds have been determined. A quick bound can be found for (7.3) by using

$$|\ln t| < t-1+\frac{1}{t}, \quad t>0. \tag{7.5}$$

This follows immediately from

$$\ln t < t-1, \quad t>1, \tag{7.6}$$

when $t>1$. It is verified for $t<1$ by replacing $t$ by $1/t$ in (7.6). With the aid of (7.5), an error bound for (7.3) on truncation when $\frac{1}{2}\pi \leqslant |\operatorname{ph} z| \leqslant \pi$ is

$$\frac{\{(\frac{1}{2}\mu+\frac{1}{2}n)!+(\frac{1}{2}\mu+\frac{1}{2}n-1)!-(\frac{1}{2}\mu+\frac{1}{2}n-\frac{1}{2})!\}}{2|z|^{n+1}}.$$

Clearly, the methods employed earlier can now be adapted to this case but, as the formulae become increasingly complex, details will be omitted.

## Appendix A.

In this appendix we describe the determination of error bounds by taking advantage of the representations of § 2. It is known that (2.6) holds for $0<\operatorname{ph} z<2\pi$. By reversing the steps which led to (2.6), it can be confirmed that (2.4) is also valid for $0<\operatorname{ph} z<2\pi$ on the understanding that $\operatorname{ph}(-z^2)=2\operatorname{ph} z-\pi$. Therefore, asymptotic results can be deduced from those of the complementary incomplete gamma function.

The pertinent formula is

$$(-\lambda)!w^{-\lambda}\mathrm{e}^w\Gamma(\lambda,w) = -\sum_{m=0}^{q-1}\frac{(m-\lambda)!}{(-w)^{m+1}}+\varepsilon_q \quad (|\omega|<\tfrac{3}{2}\pi), \tag{A 1}$$

where $\omega=\operatorname{ph} w$. Various bounds are available for $\varepsilon_q$ [**5**, **6**]. For instance, if $\lambda<1$,

$$|\varepsilon_q| \leqslant \frac{(q-\lambda)!}{|w|^{q+1}} \qquad (|\omega|\leqslant\tfrac{1}{2}\pi) \tag{A 2}$$

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and

$$|\varepsilon_q| \leqslant \frac{(q-\lambda)!}{|w|^{q+1}|\sin\omega|} \quad (\tfrac{1}{2}\pi \leqslant |\omega| < \pi). \tag{A 3}$$

An alternative to (A 3) is

$$|\varepsilon_q| \leqslant \frac{(q-\lambda)!}{|w|^q\{|w|\cos(|\omega| - \frac{3}{4}\pi) - 2^{1/2}(q-\lambda+1)\ln 2\}} \quad (\tfrac{1}{2}\pi \leqslant |\omega| < \pi), \tag{A 4}$$

so long as the denominator is positive. Although (A 4) has a restriction on $|w|$, in regions where it is valid it is a more satisfactory bound than (A 3) as $|\omega| \to \pi$.

The insertion of (A 1) in (2.4) gives

$$I(\mu, z) = -\frac{1}{2}\sum_{m=1}^{n-1} \frac{(\frac{1}{2}\mu + \frac{1}{2}m - \frac{1}{2})!}{z^{m+1}} + \eta_n, \tag{A 5}$$

provided that an appropriate value of $q$ is chosen for each of the $\Gamma$ which occur.

From (A 2) it can be concluded that

$$|\eta_n| \leqslant \frac{(\frac{1}{2}n + \frac{1}{2}\mu - \frac{1}{2})!}{2|z|^{n+1}} + \frac{(\frac{1}{2}n + \frac{1}{2}\mu)!}{2|z|^{n+2}} \tag{A 6}$$

for $\frac{1}{4}\pi \leqslant \operatorname{ph} z \leqslant \frac{3}{4}\pi$.

This bound is similar to those in (3.3) and (3.4). It is not quite the same because they hold in different regions of the complex $z$-plane.

For $\frac{1}{4}\pi \geqslant \operatorname{ph} z > 0$, (A 3) and (A 4) can be called on. As far as (A 3) is concerned it adds the factor $\sin 2\theta$, $\theta = \operatorname{ph} z$, to the denominators of (A 6). Then the bound becomes very large as $z$ approaches the positive real axis. On the other hand, (A 4) gives

$$|\eta_n| \leqslant \frac{(\frac{1}{2}n + \frac{1}{2}\mu - \frac{1}{2})!}{|z|^{n-1}\{2|z|^2\cos(2\theta - \frac{1}{4}\pi) - (n+\mu+1)2^{1/2}\ln 2\}} + \frac{(\frac{1}{2}n + \frac{1}{2}\mu)!}{|z|^{n}\{2|z|^2\cos(2\theta - \frac{1}{4}\pi) - (n+\mu+2)2^{1/2}\ln 2\}} \tag{A 7}$$

subject to $|z|$ being large enough for both denominators to be positive. Allow $\theta \to 0$. Then (A 7) supplies an error bound on the upper side of the positive real axis in that part where $x^2 > (n+\mu+2)\ln 2$. The larger $x$ is, the more useful is the bound and, generally, (A 7) is more useful the larger $|z|$.

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