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WEAKLY COMPACT OPERATORS AND THE STRICT TOPOLOGIES

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Let X be a completely regular space. We denote by $C_b(X)$ the Banach space of all real-valued bounded continuous functions on X endowed with the supremum-norm.

In this paper we prove some characterisations of weakly compact operators defined from $C_b(X)$ into a Banach space E which are continuous with respect to β_t, β_τ and β_σ introduced by Sentilles.

We also prove that $(C_b(X), \beta_i)$, $i = t, \tau, \sigma$, has the Dunford-Pettis property.

INTRODUCTION AND NOTATIONS

In this paper, E denotes a Banach space, X a completely regular Hausdorff space, $C_b(X)$ the set of all continuous bounded real-valued functions on X, $\mathcal{F}$ the algebra generated by zero sets, that is, sets of the form $f^{-1}(0)$, $f \in C_b(X)$, and $Ba(X)$ the σ-algebra generated by zero-sets.

On $C_b(X)$ there are three important topologies, the so-called strict topologies, which are denoted by β_t, β_τ, β_σ; the dual of $(C_b(X), \beta_i)$ is $M_i(X)$, for $i = t, \tau, \sigma$ (these topologies and duals are discussed in [**5**]). It is known that $\beta_t \leqslant \beta_\tau \leqslant \beta_\sigma \leqslant \| \, \|_\infty$ and they have the same bounded sets [**5**].

Let $\mathcal{A}$ be an algebra of subsets of X and let $m\colon \mathcal{A} \to E$ a finite additive vector-measure. We shall say that m is strongly additive if the series $\Sigma m(A_n)$ converges for every disjoint sequence $(A_n)_{n \in N}$ of elements of $\mathcal{A}$.

The set-functions of $\mathcal{A}$ into R defined by $v(m)(A) = \sup\{\Sigma \|m(A_i)\| \, A_i \in \mathcal{A}, \, A_i \cap A_j = \emptyset, \, i \neq j, \, A = \cup A_i, \, i = 1 \ldots n, \, n \in N\}$ and by

$$\|m\|(A) = \sup\{\|\Sigma \alpha_i m(A_i)\| \, A_i \in \mathcal{A}, \, A_i \cap A_j = \emptyset, \, i \neq j, \, A = \cup A_i, \, i = 1 \ldots n, \, n \in N, \, |\alpha_i| \leqslant 1\}$$

are called the *variation* and *semi-variation* of m, respectively. If $v(m)(X)$ $(\|m\|(X))$ is finite, we shall say that m is of *bounded* variation (semi-variation). It is known ([**1**]) that $\|m\|(A) = \sup\{|x' \circ m|(A) \mid x' \in E', \|x'\| \leqslant 1\}$. We will denote by $ba(\mathcal{A}, E)$ the space of all bounded semi-variation vector measures m of $\mathcal{A}$ into E. $ba(\mathcal{A}, E)$ is a Banach space with the norm $m \to \|m\|(X)$.

If $S(\mathcal{A})$ denotes the space of all simple functions with the supremum-norm, then for each $m \in ba(\mathcal{A}, E)$, we define $T_m(f) = \Sigma \alpha_i m(A_i)$, where $f = \Sigma \alpha_i \chi_{A_i} \in S(\mathcal{A})$. Hence

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T_m is a continuous linear operator of $S(\mathcal{A})$ into E and $\|T\| = \|m\|(X)$. Conversely, each continuous linear operator T of $S(\mathcal{A})$ into E defines a bounded vector measure of semi-variation m, with $m(A) = T(\chi_A)$.

Let $B(\mathcal{A})$ denote the completion of $S(\mathcal{A})$ in the supremum-norm. It is proved in [**1**] that the mapping $m \to T_m$ is an isomorphism of the space $ba(\mathcal{A}, E)$ with the space $L(B(\mathcal{A}), E)$. $T_m(f)$ is denoted by $\int f\,dm$.

We shall say that an operator T in $L(B(\mathcal{A}), E)$ is *weakly compact* if T maps any bounded subset of $B(\mathcal{A})$ to a relatively weakly compact subset of E. The following theorem is proved in [**1**].

THEOREM 1.1. *Let $T\colon B(\mathcal{A}) \to E$ be a bounded linear operator. Then the following are equivalent:*

(i) *T is weakly compact*

(ii) *m is strongly additive*

The space of all finite, finitely additive, zero set regular, real-valued measures is denoted by $M(X)$ (zero set regular means that for any $\epsilon > 0$ and any $F \in \mathcal{F}$, there exist a zero set Z and a cozero set U, with $Z \subseteq F \subseteq U$, such that $|\mu|(U \setminus Z) < \epsilon$).

Alexandroff's Theorem ([**6**]) establishes that $M(X)$ is the dual of $C_b(X)$ with the supremum-norm.

As in the case when X is a compact Hausdorff space we can identify $B(\mathcal{F})$ with $M(X)'$ by the isometry $f \to \lambda_f$ where $\lambda_f(\mu) = \int_X f\,d\mu$.

LEMMA 1.2. *Each $f \in C_b(X)$ is the uniform limit of simple functions, that is $C_b(X) \subseteq B(\mathcal{A})$.*

PROOF: First note that if $f \in C_b(X)$, $\{x \in X : f(x) \geqslant \alpha\}$ is a zero set. Let $\epsilon > 0$; since $f(X)$ is totally bounded, there is a finite subset $\{x_1, x_2, \ldots x_n\}$ of X such that $f(X) \subseteq \cup f(x_i) + (-\epsilon, \epsilon)$.

Write $U_i = f(x_i) + (-\epsilon, \epsilon)$ and $A_i = f^{-1}(U_i)$. If the family $\{A_i : i = 1, \ldots, n\}$ is not disjoint, then we define $B_1 = A_1$ and $B_i = A_i \setminus B_i$, $i = 2, \ldots, n$; thus, $\{B_i \mid i = 1, \ldots n\}$ is a $\mathcal{F}$-partition of X. If $f_0 = \Sigma f(x_i)\chi_{B_i}$ and if $x \in X$, then there exists $i \in \{1, \ldots, n\}$ such that $x \in A_i \setminus B_i$ and $f(x) = f(x_i) + u$, with $|u| < \epsilon$, which implies that $|f(x) - f(x_i)| < \epsilon$. Thus, $|f(x) - f_0(x)| < \epsilon$ and then $\|f - f_0\| < \epsilon$ since x was arbitrary. ∎

THEOREM 1.3. *Let $T\colon C_b(X) \to E$ be a bounded linear operator. Then, there exists a unique finitely additive vector measure $m\colon \mathcal{F} \to E''$ of bounded semi-variation such that:*

(a) *for any $x' \in E'$, $x' \circ m \in M(X)$;*

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(b) *the mapping of E into $M(X)$ defined by $x' \to x' \circ m$ is $\sigma(E', E)$ – $\sigma(M(X), C_b(X))$ continuous;*
(c) $T(f) = \int_X f dm$, $\forall f \in C_b(X)$;
(d) $\|T\| = \|m\|(X)$.

Conversely, if $m: \mathcal{F} \to E''$ is a finitely additive vector measure of bounded semi-variation satisfying (a) and (b), then (c) defines a bounded linear operator $T: C_b(X) \to E$ such that $\|T\| = \|m\|(X)$.

PROOF: Since $T'': C_b(X)'' \to E''$ is a bounded linear operator and $B(\mathcal{F}) \subset C_b(X)''$, there is a vector measure associated with $\overline{T} = T''_{|B(F)}$. Since $C_b(X) \subseteq B(\mathcal{F})$, we have that, for each $f \in C_b(X)$,

$$T(f) = \overline{T}(f) = \int_X f dm \text{ and } \|T\| \leqslant \|\overline{T}\| = \|m\|(X) \leqslant \|T''\| = \|T\|.$$

Part (a) and (b) follow easily from the continuity of T and T'.

Conversely, since $\{x' \circ m \colon \|x'\| \leqslant 1\}$ is $\sigma(M(X), C_b(X))$ relatively compact, we have that

$$\|m\|(X) = \sup\{|x' \circ m|(X) \colon \|x'\| \leqslant 1\} < \infty$$

and then m is of bounded semi-variation. The result follows from this. ∎

2. Weakly compact operators and strict topologies

In this section we shall study the weakly compact operators of $C_b(X)$ into E which are continuous in the strict topologies β_t, β_τ and β_σ, respectively, and their associated vector measures.

We already know that if T is weakly compact, then T' from E' into $M(X)$ is also weakly compact; thus, $T'(B_{E'}) = \{x' \circ m : \|x'\| \leqslant 1\}$ is relatively $\sigma\big(M(X), C_b(X)''\big)$-compact and then $\{|x' \circ m| : \|x'\| \leqslant 1\}$ is relatively $\sigma\big(M(X), C_b(X)''\big)$-compact ([**1**]) which implies that $\{|x' \circ m| : \|x'\| \leqslant 1\}$ is relatively $\sigma(M(X), C_b(X))$-compact.

The following theorems characterise the β_i-continuous, weakly compact linear operators, where $i = t, \tau, \sigma$.

From now on, we will assume that $T: C_b(X) \to E$ is a weakly compact operator.

THEOREM 2.1. *If m is the associated vector measure of T, then the following are equivalent:*

(i) *m is σ-additive vector measure;*
(ii) *if $\{f_n\}_{n \in N}$ is any decreasing sequence in $C_b(X)$, with*

$$f_n(x) \to 0 \text{ for each } x \in X, \text{ then } \|Tf_n\| \to 0;$$

(iii) *T is β_σ-continuous.*

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PROOF: (i) $\Rightarrow$ (ii) Let $\{f_n\}_{n\in N}$ be a sequence in $C_b(X)$ such that $f_n \downarrow 0$ pointwise and $\epsilon > 0$. By the Caratheodory-Hahn Extension Theorem ([**1**]), there exists a nonnegative real-valued σ-additive measure μ on $Ba(X)$ such that $m << \mu$. Thus, for the given ϵ, there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $|x' \circ m|(A) < \epsilon$ uniformly in $x' \in B_{E'}$.

On the other hand, Egoroff's Theorem gives us a subset $F \in Ba(X)$ such that $f_n \to 0$ uniformly on $X \setminus F$ and $\mu(F) < \delta$. Therefore, there exists $n_0 \in N$ such that $n \geqslant n_0$ implies $\|f_n\| < \epsilon/2M$ on $X \setminus F$, where $M = \|m\|(X)$.

Now, if $n \geqslant n_0$, then

$$\begin{aligned}|x' \circ T(f_n)| &\leqslant \int_X |f_n|\, d\,|x' \circ m| \\ &= \int_{X\setminus F} |f_n|\, d\,|x' \circ m| + \int_F |f_n|\, d\,|x' \circ m| \\ &< (\epsilon/2m)\,|x' \circ m|\,(X \setminus F) + |x' \circ m|\,(F) \\ &< \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

uniformly in $x' \in B_{E'}$.

The conclusion follows from the fact that

$$\|Tf\| = \sup\{|x' \circ T(f)| : \|x'\| \leqslant 1\}.$$

(ii) $\Rightarrow$ (iii) If $\|Tf_n\| \to 0$ for any sequence $\{f_n\}_{n\in N}$ in $C_b(X)$, then $|x' \circ m| \in M_\sigma(X)$ for all $x' \in B_{E'}$. Since $\{|x' \circ m| : \|x'\| \leqslant 1\}$ is relatively $\sigma(M(X), C_b(X))$-compact, we have $\{|x' \circ m| : \|x'\| \leqslant 1\}$ is relativley compact and then β_σ-equicontinuous [**5**, Theorem 5.2, p.322].

Let $\{f_\alpha\}_{\alpha\in I}$ be a net in $C_b(X)$ such that $f_\alpha \to 0$ with respect to β_σ. Hence $|x' \circ T|(f_\alpha) \to 0$ uniformly in $x' \in B_{E'}$; since $|x' \circ T(f_\alpha)| \leqslant |x' \circ T|(f_\alpha)$, we get $|x' \circ T(f_\alpha)| \to 0$ uniformly in $x' \in B_{E'}$ and then $\|Tf_\alpha\| = \sup\{|x' \circ T(f_\alpha)| : \|x'\| \leqslant 1\} \to 0$. Consequently, T is continuous with respect to β_σ.

(iii) $\Rightarrow$ (i) Since $|x' \circ T(f)| \leqslant \|Tf\|\,\|x'\|$ for all $x' \in B_{E'}$ and T is continuous with respect to β_σ we have that $x' \circ T$ is continuous with respect to β_σ for any $x' \in B_{E'}$; hence $|x' \circ m|$ is a real-valued σ-additive measure for any $x' \in B_{E'}$. The conclusion follows from [**1**, Theorem 2 p.27]. ∎

The next theorem characterises the weakly compact operators which are continuous with respect to β_t. By Sentilles [**5**], β_t is the finest locally convex topology on $C_b(X)$ agreeing with the compact-open topology on the norm-bounded subsets of $C_b(X)$.

THEOREM 2.2. *If m is the associated vector measure of T, then the following are equivalent:*

(i) $(\forall\epsilon > 0)(\exists\, K \subset X,\ K$ *compact*$)(\|m\|\,(X \setminus K) < \epsilon)$;

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(ii) *T is continuous with respect to β_t;*

(iii) *T is continuous on the unit ball with respect to the compact-open topology.*

PROOF: (i) $\Rightarrow$ (ii) If $\epsilon > 0$ is given, then there exists a compact subset K of X such that $|x' \circ m|(X \setminus K) < \epsilon$ uniformly in $x' \in B_{E'}$. Therefore $\{|x' \circ m| : \|x'\| \leqslant 1\}$ and then $\{x' \circ m : \|x'\| \leqslant 1\}$ is β_t-equicontinuous ([**5**]). Thus, T is continuous with respect to β_t.

(ii) $\Rightarrow$ (iii) Follows from the definition of β_t.

(iii) $\Rightarrow$ (i) From the fact that T is continuous with respect to β_t, we have that $\{|x' \circ m| : \|x'\| \leqslant 1\}$ is β_t-equicontinuous. The result follows from [**5**, Theorem 5.1]. ∎

THEOREM 2.3. *If m is the associated vector measure of T, then the following are equivalent:*

(i) *T is continuous with respect to β_τ;*

(ii) *for any decreasing net $\{f_\alpha\}_{\alpha \in I}$ in $C_b(X)$ with $f_\alpha(x) \to 0$ for each $x \in X$, $\|Tf_\alpha\| \to 0$;*

(iii) *for any net of zero sets Z_α decreasing to the null set, $\|m\|(Z_\alpha) \to 0$.*

PROOF: (i) $\Rightarrow$ (ii) If $\{f_\alpha\}_{\alpha \in I}$ is a decreasing net in $C_b(X)$ such that $f(x)_\alpha \to 0$, for each $x \in X$, then $f_\alpha \to 0$ in the topology β_σ ([**6**]). Thus, $\|Tf_\alpha\| \to 0$.

(ii) $\Rightarrow$ (i) Since $|x' \circ T(f)| \leqslant \|Tf\| \, \|x'\|$, for any $x' \in E'$, we have that $x' \circ T$ is τ-additive and then $\{|x' \circ m| : \|x'\| \leqslant 1\}$ is relatively $\sigma(M_\tau(X), C_b(X))$-compact. Therefore, $\{|x' \circ m| : \|x'\| \leqslant 1\}$ is β_τ-equicontinuous ([**5**]). The statement follows easily from this.

(i) $\Rightarrow$ (iii) Let $\{Z_\alpha\}_{\alpha \in I}$ be a net of zero sets decreasing to the null set. Consider $D = \{f \in C_b(X) : 0 \leqslant f(x) \leqslant 1 \ \& \ (\exists \alpha)(f \equiv 1 \text{ in } Z_\alpha)$. We index the elements of D as follows: $D = \{f_\lambda\}_{\lambda \in A}$ so that $\lambda > \mu$ if and only if $f_\lambda \leqslant f_\mu$. Thus $\{f_\lambda\}_{\lambda \in A}$ is a net in $C_b(X)$; further $f_\lambda \downarrow 0$. Hence $\|Tf_\alpha\| \to 0$.

Since $|x' \circ T(f)| \leqslant \|Tf\| \, \|x'\|$, we have that $x' \circ m \in M_\tau(X)$ and $\{|x' \circ m| : \|x'\| \leqslant 1\}$ is β_τ-equicontinuous. From this we get that $|x' \circ T|(f_\lambda) \to 0$ uniformly in $x' \in B_{E'}$.

Thus, if $\epsilon > 0$ is given, then there exists $\lambda_0 \in A$ such that $\lambda > \lambda_0$ implies $|x' \circ T|(f_\lambda) = \int f_\lambda d|x' \circ m| < \epsilon$ uniformly in $x' \in B_{E'}$. For this λ_0 there exists a $\alpha_0 \in I$ so that $f \equiv 1$ on Z_{α_0} and

$$|x' \circ m|(Z_\alpha) = \int \chi_{Z_{\lambda_0}} d|x' \circ m| \leqslant \int f_{\lambda_0} d|x' \circ m| < \epsilon$$

uniformly in $x' \in B_{E'}$. Therefore, if $\alpha > \alpha_0$, $|x' \circ m|(Z_\alpha) < |x' \circ m|(Z_{\alpha_0}) < \epsilon$ uniformly in $x' \in B_{E'}$. The statement follows from the fact that $\|m\|(A) = \sup\{|x' \circ m|(A) : \|x'\| \leqslant 1\}$.

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(iii) $\Rightarrow$ (ii) Let $\{f_\lambda\}_{\lambda\in A}$ be a decreasing net in $C_b(X)$ such that $\|f_\alpha\| \leqslant 1$ and $f_\alpha(x) \to 0$ for each $x \in X$, and $\epsilon > 0$. Define $Z_\alpha = \{x \in X : f_\alpha(x) \geqslant \epsilon/2M \;\&\; M = \|m\|(X)\}$. $\{Z_\alpha\}_{\alpha\in I}$ is a decreasing net of zero sets such that $Z_\alpha \downarrow \emptyset$. Then there exists $\alpha_0 \in I$ so that $\alpha > \alpha_0$ implies $\|m\|(Z_\alpha) < \epsilon/2$.

Take $x' \in E'$, $\|x'\| \leqslant 1$, and $\alpha > \alpha_0$. Then

$$\begin{aligned} |x' \circ T(f_\alpha)| &\leqslant \int_X f_\alpha d\,|x' \circ m| = \int_{Z_\alpha} f_\alpha d\,|x' \circ m| + \int_{X\setminus Z_\alpha} f_\alpha d\,|x' \circ m| \\ &\leqslant |x' \circ m|(Z_\alpha) + (\epsilon/2M)\,|x' \circ m|(X \setminus Z_\alpha) < \epsilon \end{aligned}$$

uniformly in $x' \in B_{E'}$. Thus $Tf_\alpha \to 0$. ∎

We shall say that a topological vector space E has the strict Dunford-Pettis property if for any Banach space F and every linear continuous weakly compact operator T from E into F transforms weakly Cauchy sequences into convergent sequences [**1**].

The following theorem applies the previous result to prove that $(C_b(X), \beta_i)$, $i = t, \tau, \sigma$, possess the strict Dunford–Pettis property. This result was proved by Khurana [**4**].

THEOREM 2.4. *$(C_b(X), \beta_i)$, $i = t, \tau, \sigma$, possess the strict Dunford–Pettis property.*

PROOF: Since $\beta_t \leqslant \beta_\tau \leqslant \beta_\sigma$, it is enough to show the statement for the case $(C_b(X), \beta_\sigma)$.

Let T be a linear β_σ-continuous operator of $C_b(X)$ to E which is weakly compact. Thus, its associated vector measure m is σ-additive and it admits a control measure μ. So if $\epsilon > 0$ is given, there exists $\delta > 0$ such that $\mu(F) < \delta$ implies $\|m\|(F) < \epsilon$.

Let $\{f_n\}_{n\in N}$ be a weakly Cauchy sequence in $C_b(X)$. Then $\{f_n(x)\}_{n\in N}$ is Cauchy in $\mathbb{R}$ for each $x \in X$. By Egoroff's Theorem, there exists $F_\delta \in Ba(X)$ such that $\{f_n\}_{n\in N}$ is Cauchy on $X \setminus F_\delta$ and $\mu(F_\delta) < \delta$.

Let $n_0 \in \mathbb{N}$ such that for n, $m \geqslant n_0$ we have $\sup\{\|f_n(x) - f_m(x)\| : x \in X\setminus F_\delta\} < \epsilon/2M$, where $M = \|m\|(X)$. Thus

$$\begin{aligned} \|Tf_n - Tf_m\| &\leqslant \left\| \int_{X\setminus F_\delta} (f_n - f_m)dm \right\| + \left\| \int_{F_\delta} (f_n - f_m)dm \right\| \\ &\leqslant \sup\{\|f_n(x) - f_m(x)\| : x \in X\}\,\|m\|(X) + L\,\|m\|(F_\delta) < \epsilon, \end{aligned}$$

where $\|f_n\| \leqslant L$ for all $n \in \mathbb{N}$. ∎

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