ANZIAM J. 49(2008), 337–346
doi:10.1017/S1446181108000059

THREE NONNEGATIVE SOLUTIONS FOR SECOND-ORDER IMPULSIVE DIFFERENTIAL EQUATIONS WITH A THREE-POINT BOUNDARY VALUE PROBLEM

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(Received 9 September, 2006; revised 4 January, 2008)

Abstract

In this paper, by using the Leggett–Williams fixed point theorem, we prove the existence of three nonnegative solutions to second-order nonlinear impulsive differential equations with a three-point boundary value problem.

2000 *Mathematics subject classification*: 34B10, 34B37.

Keywords and phrases: differential equation, boundary value problem, nonnegative solution, fixed point theorem.

1. Introduction

Let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$ be given. In this paper we present results which guarantee the existence of three nonnegative solutions to the second-order impulsive equation

$$\begin{cases} y''(t) + h(t) f(y(t)) = 0 & \text{for } t \in (0, 1) \setminus \{t_1, \dots, t_m\}, \\ \Delta y(t_k) = I_k(y(t_k^-)), & k = 1, \dots, m, \\ \Delta y'(t_k) = J_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = 0, \\ \alpha y(\eta) = y(1), \end{cases} \tag{1.1}$$

where $0 < \eta < 1$, $0 < \alpha < 1/\eta$, $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, and $y(t_k^+)$ and $y(t_k^-)$ respectively denote the right limit and left limit of $y(t)$ at $t = t_k$. Also $\Delta y'(t_k) = y'(t_k^+) - y'(t_k^-)$. We define the Banach space ($r = 0$ or 2 in this paper),

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$$PC^r[0, 1] = \{y : [0, 1] \times R : \text{for all } j = 0, \ldots, m \text{ there exists } y_j \in C^r[t_j, t_{j+1}] \\ \text{such that } y = y_j \text{ on } (t_j, t_{j+1}],\ y(0) = y_0(0)\}$$

with the norm

$$\|y\|_{PC^r} = \max\{\|y\|, \|y'\|, \ldots, \|y^{(r)}\|\}.$$

Here

$$\|y\| = \sup\{|y(t)|,\ t \in [0, 1]\}.$$

By a solution to (1.1) we mean a function $y \in PC^2[0, 1]$ which satisfies (1.1). In (1.1), as $\alpha = 0$, the existence of three nonnegative solutions was considered by Agarwal and O'Regan [3], and as $I_k(y(t_k^-)) = J_k(y(t_k^-)) \equiv 0$, $k = 1, \ldots, m$, the positive solution was obtained by Ma [6]. In this paper, motivated by [3] and [6], we shall show the existence of three nonnegative solutions to (1.1) by the Leggett–Williams fixed point theorem [5]. Recently [1–4, 7, 8] this fixed point theorem has been used to establish multiplicity results for differential, integral and difference equations.

Now we present some preliminaries which will be needed in Section 3. First, $E = (E, \|\cdot\|)$ is a Banach space and $P \subset E$ is a cone. By a concave nonnegative continuous functional ψ on P we mean a continuous mapping $\psi : P \to [0, \infty)$ with

$$\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y) \quad \text{for all } x, y \in P \text{ and all } \lambda \in [0, 1].$$

Let $K, L, r > 0$ be constants with P and ψ as defined above. Let

$$P_K = \{y \in P : \|y\| < K\} \quad \text{and} \quad P(\psi, r, L) = \{y \in P : \psi(y) \geq r \text{ and } \|y\| \leq L\}.$$

We now state the Leggett–Williams fixed point theorem [5].

THEOREM 1.1. *Let $E = (E, \|\cdot\|)$ be a Banach space, $P \subset E$ a cone of E and $R > 0$ a constant. Suppose there exists a concave nonnegative continuous functional ψ on P with $\psi(y) \leq \|y\|$ for all $y \in \bar{P}_R$ and let $A : \bar{P}_R \to \bar{P}_R$ be a continuous compact map. Assume there are numbers r, L and K with $0 < r < L < K \leq R$ such that:*

(H1) *$\{y \in P(\psi, L, K) : \psi(y) > L\} \neq \emptyset$ and $\psi(Ay) > L$ for all $y \in P(\psi, L, K)$;*
(H2) *$\|Ay\| < r$ for all $y \in \bar{P}_r$; and*
(H3) *$\psi(Ay) > L$ for all $y \in P(\psi, L, R)$ with $\|Ay\| > K$.*

Then A has at least three fixed points y_1, y_2 and y_3 in $\bar{P}_R$. Furthermore

$$y_1 \in P_r, \quad y_2 \in \{y \in P(\psi, L, R) : \psi(y) > L\} \quad \text{and} \quad y_3 \in \bar{P}_R \setminus (P(\psi, L, R) \cup \bar{P}_r).$$

2. Some lemmas

Consider the impulsive integral equation

$$y(t) = \int_0^1 H(t, s)h(s)f(y(s))\,ds + \sum_{k=1}^m W_k(t, y), \quad t \in [0, 1], \tag{2.1}$$

https://doi.org/10.1017/S1446181108000059 Published online by Cambridge University Press

where

$$H(t,s)=\frac{1}{1-\alpha\eta}t(1-s)-U(t,s)-\frac{\alpha}{1-\alpha\eta}V(t,s),\quad 0\le t,s\le 1,$$

$$U(t,s)=\begin{cases}t-s, & s\le t,\\ 0, & t\le s,\end{cases}\qquad V(t,s)=\begin{cases}t(\eta-s), & s\le\eta,\\ 0, & \eta\le s,\end{cases}$$

and for $k=1,\ldots,m$

$$W_k(t,y)=\begin{cases}\dfrac{1-t-\alpha\eta+\alpha t}{1-\alpha\eta}[I_k(y(t_k^-))-t_kJ_k(y(t_k^-))], & 0<t_k<\min\{t,\eta\},\\[2mm] \dfrac{t}{1-\alpha\eta}\{-I_k(y)-(1-t_k)J_k(y)+\alpha[I_k(y)+(\eta-t_k)J_k(y)]\}, & \\ & t\le t_k<\max\{t,\eta\},\\[2mm] \dfrac{1}{1-\alpha\eta}\{(1-\alpha\eta)(I_k(y)-t_kJ_k(y))-t[I_k(y)-(t_k-\alpha\eta)J_k(y)]\}, & \\ & \eta\le t_k<\max\{t,\eta\},\\[2mm] \dfrac{t}{1-\alpha\eta}[-I_k(y(t_k^-))-(1-t_k)J_k(y(t_k^-))], & \max\{t,\eta\}\le t_k<1.\end{cases}$$

LEMMA 2.1. *We have that $y\in PC[0,1]\cap PC^2[0,1]$ is a solution of (1.1) if and only if $y\in PC[0,1]$ is a solution of the integral equation (2.1).*

PROOF. Suppose that $y\in PC[0,1]$ is a solution of (2.1). Then for $t\ne t_k$,

$$\begin{aligned}y'(t)&=\frac{1}{1-\alpha\eta}\int_0^1(1-s)h(s)f(y(s))\,ds-\frac{\alpha}{1-\alpha\eta}\int_0^\eta(\eta-s)h(s)f(y(s))\,ds\\&\quad-\frac{1}{1-\alpha\eta}\sum_{0<t_k<1}[J_k(y)(1-t_k)+I_k(y)]\\&\quad+\frac{\alpha}{1-\alpha\eta}\sum_{0<t_k<\eta}[J_k(y)(\eta-t_k)+I_k(y)]\\&\quad-\int_0^t h(s)f(y(s))\,ds+\sum_{0<t_k<t}J_k(y),\\ y''(t)&=-h(t)f(y(t)),\end{aligned}$$

and for $t=t_k$,

$$\begin{aligned}\Delta y(t_k)&=y(t_k^+)-y(t_k^-)=I_k(y(t_k^-)),\\ \Delta y'(t_k)&=y'(t_k^+)-y'(t_k^-)=J_k(y(t_k^-)),\end{aligned}$$

and

$$y(0)=0,\quad \alpha y(\eta)=y(1).$$

So y is a solution of (1.1).

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On the other hand, if y is a solution of (2.1), then

$$y'(t) = y'(0) - \int_0^t h(s) f(y(s))\, ds + \sum_{0<t_k<t} J_k(y),$$

$$y(t) = y'(0)t - \int_0^t (t-s) h(s) f(y(s))\, ds + \sum_{0<t_k<t} [J_k(y)(t-t_k) + I_k(y)].$$

This and the boundary value condition $y(0) = 0$ and $\alpha y(\eta) = y(1)$ imply that

$$\begin{aligned} y'(0) = &\frac{1}{1-\alpha\eta} \int_0^1 (1-s) h(s) f(y(s))\, ds - \frac{\alpha}{1-\alpha\eta} \int_0^\eta (\eta - s) h(s) f(y(s))\, ds \\ &- \frac{1}{1-\alpha\eta} \sum_{0<t_k<1} [J_k(y)(1-t_k) + I_k(y)] \\ &+ \frac{\alpha}{1-\alpha\eta} \sum_{0<t_k<\eta} [J_k(y)(\eta - t_k) + I_k(y)]. \end{aligned}$$

Therefore

$$\begin{aligned} y(t) = &\frac{t}{1-\alpha\eta} \int_0^1 (1-s) h(s) f(y(s))\, ds - \frac{\alpha t}{1-\alpha\eta} \int_0^\eta (\eta - s) h(s) f(y(s))\, ds \\ &- \frac{t}{1-\alpha\eta} \sum_{0<t_k<1} [J_k(y)(1-t_k) + I_k(y)] \\ &+ \frac{\alpha t}{1-\alpha\eta} \sum_{0<t_k<\eta} [J_k(y)(\eta - t_k) + I_k(y)] \\ &- \int_0^t (t-s) h(s) f(y(s))\, ds + \sum_{0<t_k<t} [J_k(y)(t-t_k) + I_k(y)]. \end{aligned}$$

For $0 \le t \le \eta$,

$$\begin{aligned} y(t) = &\int_0^1 H(t,s) h(s) f(y(s))\, ds + \frac{1 - t - \alpha\eta + \alpha t}{1-\alpha\eta} \sum_{0<t_k<t} [I_k(y) - t_k J_k(y)] \\ &+ \frac{t}{1-\alpha\eta} \sum_{t\le t_k<\eta} \{-I_k(y) - (1-t_k) J_k(y) + \alpha [I_k(y) + (\eta - t_k) J_k(y)]\} \\ &+ \frac{t}{1-\alpha\eta} \sum_{\eta \le t_k < 1} [-I_k(y) - (1-t_k) J_k(y)]. \end{aligned}$$

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For $\eta \le t \le 1$,

$$
\begin{aligned}
y(t) = & \int_0^1 H(t, s)h(s) f(y(s))\, ds + \frac{1 - t - \alpha\eta + \alpha t}{1 - \alpha\eta} \sum_{0 < t_k < \eta} [I_k(y) - t_k J_k(y)] \\
& + \frac{1}{1 - \alpha\eta} \sum_{\eta \le t_k < t} \{(1 - \alpha\eta)(I_k(y) - t_k J_k(y)) - t[I_k(y) - (t_k - \alpha\eta) J_k(y)]\} \\
& + \frac{t}{1 - \alpha\eta} \sum_{\eta \le t_k < 1} [-I_k(y) - (1 - t_k) J_k(y)].
\end{aligned}
$$

So

$$
y(t) = \int_0^1 H(t, s)h(s) f(y(s))\, ds + \sum_{k=1}^{m} W_k(t, y), \quad t \in [0, 1]. \qquad \square
$$

LEMMA 2.2. *We have that:*

(1) $H : [0, 1] \times [0, 1] \to [0, +\infty)$ *is continuous; and*

(2) $H(t, s) \le M_1 H(s, s)$ *for all* $t, s \in [0, 1]$,
$H(t, s) \ge M_2 H(s, s)$ *for* $s \in [0, 1]$, $t \in [a_k, b_k]$,
where

$$
\begin{aligned}
M_1 &= \max \left\{ \frac{1 - \alpha\eta}{\alpha(1 - \eta)}, \frac{1 + \alpha\eta}{\eta} \right\}, \\
M_2 &= \min \left\{ \frac{t_1}{4}, \frac{(1 - \alpha\eta)(1 - t_m)}{4}, \frac{(1 - \alpha\eta)(1 - t_m)}{4\alpha(1 - \eta)} \right\}, \\
a_k &= \frac{3t_k + t_{k+1}}{4}, \quad b_k = \frac{t_k + 3t_{k+1}}{4} \quad \textit{for } k \in \{0, 1, \ldots, m\}.
\end{aligned}
$$

PROOF. Part (1) is part (1) of [9, Lemma 3.1]. Now we prove part (2). We divide the proof into the following six cases.

(i) If $0 \le s \le t \le \eta$, then

$$
\begin{aligned}
\frac{H(t, s)}{H(s, s)} &= \frac{1 - \alpha\eta + t(\alpha - 1)}{1 - \alpha\eta + s(\alpha - 1)} \le \begin{cases} 1, & \alpha \ge 1, \\ \dfrac{1 - \alpha\eta}{\alpha(1 - \eta)}, & \alpha < 1, \end{cases} \\
\frac{H(t, s)}{H(s, s)} &= \frac{1 - \alpha\eta + t(\alpha - 1)}{1 - \alpha\eta + s(\alpha - 1)} \ge (1 - \alpha\eta)(1 - t) \ge (1 - \alpha\eta)\frac{1 - t_m}{4}, \\
& \qquad t \in [a_k, b_k].
\end{aligned}
$$

(ii) If $0 \le t \le s \le \eta$, then

$$
\begin{aligned}
\frac{H(t, s)}{H(s, s)} &= \frac{t}{s} \le 1, \\
\frac{H(t, s)}{H(s, s)} &= \frac{t}{s} \ge \frac{t_1}{4}, \quad t \in [a_k, b_k].
\end{aligned}
$$

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(iii) If $0 \le s \le \eta \le t \le 1$, then

$$\frac{H(t,s)}{H(s,s)} = \frac{1-\alpha\eta + t(\alpha-1)}{1-\alpha\eta + s(\alpha-1)} \le \begin{cases} 1, & \alpha \ge 1, \\ \dfrac{1-\alpha\eta}{\alpha(1-\eta)}, & \alpha < 1, \end{cases}$$

$$\frac{H(t,s)}{H(s,s)} \ge \begin{cases} 1-t \ge \dfrac{1-t_m}{4}, & \alpha \le 1,\ t \in [a_k, b_k], \\ \dfrac{(1-\alpha\eta)(1-t)}{\alpha(1-\eta)} \ge \dfrac{(1-\alpha\eta)(1-t_m)}{4\alpha(1-\eta)}, & \alpha > 1,\ t \in [a_k, b_k]. \end{cases}$$

(iv) If $0 \le t \le \eta \le s \le 1$, then

$$\frac{H(t,s)}{H(s,s)} = \frac{t}{s} \le 1,$$
$$\frac{H(t,s)}{H(s,s)} = \frac{t}{s} \ge \frac{t_1}{4}, \quad t \in [a_k, b_k].$$

(v) If $\eta \le s \le t \le 1$, then

$$\frac{H(t,s)}{H(s,s)} = \frac{s(1-t)+\alpha\eta(t-s)}{s(1-s)} \le \frac{s(1-s)+\alpha\eta(1-s)}{s(1-s)} = \frac{s+\alpha\eta}{s} \le \frac{1+\alpha\eta}{\eta},$$
$$\frac{H(t,s)}{H(s,s)} = \frac{s(1-t)+\alpha\eta(t-s)}{s(1-s)} \ge \frac{1-t}{1-s} \ge \frac{1}{1-\eta}\,\frac{1-t_m}{4}, \quad t \in [a_k, b_k].$$

(vi) If $\eta \le t \le s \le 1$, then

$$\frac{H(t,s)}{H(s,s)} = \frac{t}{s} \le 1,$$
$$\frac{H(t,s)}{H(s,s)} = \frac{t}{s} \ge \frac{t_1}{4}, \quad t \in [a_k, b_k].$$

Thus

$$H(t,s) \le M_1 H(s,s) \quad \text{for } t, s \in [0,1],$$
$$H(t,s) \ge M_2 H(s,s) \quad \text{for } s \in [0,1],\ t \in [a_k, b_k]. \qquad \square$$

REMARK 2.1. Note that $M_1 > 1$.

3. Existence

We will use Theorem 1.1 to establish the existence of three nonnegative solutions to (1.1). The following conditions will be assumed:

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$$h \in C(0,1) \text{ with } h > 0 \text{ on } (0,1) \text{ and } h \in L^1[0,1], \tag{3.1}$$

$$f : [0,\infty) \to [0,\infty) \text{ is continuous and nondecreasing}, \tag{3.2}$$

$$I_k,\ J_k : [0,\infty) \to R \text{ are continuous for } k = 1, \dots, m, \tag{3.3}$$

$$t_k J_k(v) \le I_k(v) \le (t_k - 1) J_k(v) \text{ for } v \ge 0 \text{ and } k = 1, \dots, m, \tag{3.4}$$

$$\begin{cases} I_k(v) \ge (t_k - \eta) J_k(v) & \text{for } v \ge 0,\ t_k < \eta \text{ and } k \in \{1, \dots, m\}, \\ I_k(v) \le (t_k - \alpha\eta) J_k(v) & \text{for } v \ge 0,\ t_k \ge \eta \text{ and } k \in \{1, \dots, m\}, \end{cases} \tag{3.5}$$

$$\begin{cases} W_k(t,u) \le \Omega_k(u(t_k)) \text{ for } t \in [0,1] \text{ and } u \in C[0,1] \text{ with } u \ge 0, \\ \text{and with } \Omega_k \ge 0 \text{ continuous and nondecreasing on } [0,\infty), \end{cases} \tag{3.6}$$

$$\exists\, r > 0 \text{ with } f(r) \sup_{t\in[0,1]} \int_0^1 H(t,s)h(s)\,ds + \sum_{i=1}^m \Omega_i(r) < r, \tag{3.7}$$

$$\exists\, L > r \text{ with } f(L) \min_{k\in\{0,1,\dots,m\}} \min_{t\in[a_k,b_k]} \int_{a_k}^{b_k} H(t,s)h(s)\,ds > L, \tag{3.8}$$

$$\begin{cases} \exists\, c_0,\ 0 < c_0 < 1 \text{ with, for each } j \in \{1,2,\dots,m\},\ W_j(t,y) \ge c_0 \Omega_j(y(t_j)), \\ \text{for each } t \in [a_k,b_k],\ k \in \{0,1,\dots,m\} \text{ and } y \in C[0,1] \text{ with } y \ge 0, \end{cases} \tag{3.9}$$

and

$$\exists\, R \ge L M^{-1} M_1 \text{ with } f(R) \sup_{t\in[0,1]} \int_0^1 H(t,s)h(s)\,ds + \sum_{j=1}^m \Omega_j(R) \le R, \tag{3.10}$$

where

$$M = \min\{c_0, M_2\}. \tag{3.11}$$

THEOREM 3.1. *Suppose that (3.1)–(3.10) hold. Then (1.1) has at least three nonnegative solutions y_1, y_2 and y_3 in $PC^2[0,1]$ such that*

$$\|y_1\| < r, \quad y_2(t) > L \quad \text{for } t \in [a_k,b_k],\ k \in \{0,1,\dots,m\},$$

and

$$\|y_3\| > r \quad \text{with} \quad \min_{k\in\{0,\dots,m\}} \min_{t\in[a_k,b_k]} y_3(t) < L.$$

PROOF. Let

$$E = (PC[0,1], \|\cdot\|) \quad \text{and} \quad P = \{u \in PC[0,1],\ u(t) \ge 0 \text{ for } t \in [0,1]\}.$$

Now let $A : PC[0,1] \to PC[0,1]$ be defined by

$$Ay(t) = \int_0^1 H(t,s)h(s)f(y(s))\,ds + \sum_{k=1}^m W_k(t,y) \quad \text{for } t \in [0,1]. \tag{3.12}$$

For $y \ge 0$ the conditions (3.1), (3.2), (3.4) and (3.5) imply that $Ay(t) \ge 0$ for $t \in [0,1]$. So $A(P) \subset P$. It is easy to show that $A : P \to P$ is continuous and completely continuous [3].

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For $y \in P$, let

$$\psi(y) = \min_{k\in\{0,1,\ldots,m\}} \min_{t\in[a_k,b_k]} y(t).$$

Then ψ is a nonnegative continuous concave functional on P with $\psi(y) \le \|y\|$ for $y \in P$. Next choose and fix K so that

$$LM_1M^{-1} \le K \le R. \tag{3.13}$$

First, we prove that condition (H2) of Theorem 1.1 holds. To do this, let $y \in \bar{P}_r$, then $0 \le y \le r$. Conditions (3.2), (3.6) and (3.7) imply for $t \in [0, 1]$ that

$$Ay(t) \le f(r) \sup_{t\in[0,1]} \int_0^1 H(t, s)h(s)\, ds + \sum_{k=1}^{m} \Omega_k(r) < r.$$

So

$$\|Ay\| < r.$$

This shows that condition (H2) of Theorem 1.1 follows. Also $A : \bar{P}_R \to \bar{P}_R$ since, if $y \in \bar{P}_R$, then

$$\|Ay\| \le f(R) \sup_{t\in[0,1]} \int_0^1 H(t, s)h(s) + \sum_{k=1}^{m} \Omega_k(R) \le R.$$

Next, we show that $\{y \in P(\psi, L, K) : \psi(y) > L\} \ne \emptyset$ and $\psi(Ay) > L$ for all $y \in P(\psi, L, K)$. In fact, take $u(t) \equiv (L + K)/2$ for $t \in [0, 1]$, then

$$u \in \{y \in P(\psi, L, K) : \psi(y) > L\}.$$

Moreover, for $y \in P(\psi, L, K)$, then $\psi(y) = \min_{k\in\{0,1,\ldots,m\}} \min_{t\in[a_k,b_k]} y(t) \ge L$ and $\|y\| \le K$, so for each $k \in \{0, 1, \ldots, m\}$, we have

$$y(t) \in [L, K] \quad \text{for } t \in [a_k, b_k].$$

This together with (3.8) yields

$$\begin{aligned}
\psi(Ay) &= \min_{k\in\{0,1,\ldots,m\}} \min_{t\in[a_k,b_k]} \left(\int_0^1 H(t, s)h(s) f(y(s))\, ds + \sum_{j=1}^{m} W_j(t, y) \right) \\
&\ge \min_{k\in\{0,1,\ldots,m\}} \min_{t\in[a_k,b_k]} \int_{a_k}^{b_k} H(t, s)h(s) f(y(s))\, ds \\
&\ge f(L) \min_{k\in\{0,1,\ldots,m\}} \min_{t\in[a_k,b_k]} \int_{a_k}^{b_k} H(t, s)h(s)\, ds > L.
\end{aligned}$$

So condition (H1) of Theorem 1.1 is satisfied.

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Finally, we assert that if $y \in P(\psi, L, R)$ and $\|Ay\| > K$, then $\psi(Ay) > L$. To see this, let $y \in P(\psi, L, R)$ and $\|Ay\| > K$. Now (3.6) and Lemma 2.2 imply that

$$\begin{aligned} \|Ay\| &\le M_1 \int_0^1 H(s, s)h(s)f(y(s))\, ds + \sum_{j=1}^{m} \Omega_j(y(t_j)) \\ &< M_1\Big(\int_0^1 H(s, s)h(s)f(y(s))\, ds + \sum_{j=1}^{m} \Omega_j(y(t_j))\Big). \end{aligned} \tag{3.14}$$

Fix $k \in \{0, 1, \ldots, m\}$ and notice that (3.9), (3.12), (3.14) and Lemma 2.2 yield

$$\begin{aligned} \min_{t\in[a_k,b_k]} Ay(t) &= \min_{t\in[a_k,b_k]} \Big(\int_0^1 H(t, s)h(s)f(y(s))\, ds + \sum_{j=1}^{m} W_j(t, y)\Big) \\ &\ge M_2 \int_0^1 H(s, s)h(s)f(y(s))\, ds + c_0 \sum_{j=1}^{m} \Omega_j(y(t_j)) \\ &\ge M\Big(\int_0^1 H(s, s)h(s)f(y(s))\, ds + \sum_{j=1}^{m} \Omega_j(y(t_j))\Big) \\ &\ge \frac{M}{M_1}\|Ay\| > \frac{M}{M_1}K \ge L. \end{aligned}$$

So we get for each $k \in \{0, 1, \ldots, m\}$ that

$$\psi(Ay) = \min_{k\in\{0,1,\ldots,m\}} \min_{t\in[a_k,b_k]} Ay(t) > L.$$

Thus condition (H3) of Theorem 1.1 holds. By Theorem 1.1, A has at least three fixed points, that is, (1.1) has at least three nonnegative solutions y_1, y_2 and y_3 such that

$$\|y_1\| < r, \quad y_2(t) > L \quad \text{for } t \in [a_k, b_k],\ k \in \{0, 1, \ldots, m\},$$

and

$$\|y_3\| > r \quad \text{with} \quad \min_{k\in\{0,\ldots,m\}} \min_{t\in[a_k,b_k]} y_3(t) < L.$$

The proof is complete. □

We work through an example to illustrate our results.

EXAMPLE 3.1. Consider the following impulsive boundary value problem:

$$\begin{cases} y''(t) + [(y(t) - 1)^{1/3} + 1] = 0, & t \in (0, 1), t \neq \frac{1}{2}, \\ \Delta y(t_1) = \frac{1}{3} y(t_1^-), & t_1 = \frac{1}{2}, \\ \Delta y'(t_1) = -\frac{2}{3} y(t_1^-), & t_1 = \frac{1}{2}, \\ y(0) = 0, \quad y(\frac{2}{3}) = y(1), \end{cases} \tag{3.15}$$

https://doi.org/10.1017/S1446181108000059 Published online by Cambridge University Press

where $h(t) \equiv 1$, $f(y) = (y-1)^{1/3} + 1$, $\alpha = 1$, $\eta = \frac{2}{3}$. It is easy to see that conditions (3.1)–(3.5) hold. Let $\Omega_1(u) = 2u/3$, $c_0 = \frac{1}{8}$; it follows that (3.6) and (3.9) hold. Since

$$\sup_{t\in[0,1]} \int_0^1 H(t,s)h(s)\,ds = \frac{21}{64}, \quad \min_{k\in 0,1} \min_{t\in[a_k,b_k]} \int_{a_k}^{b_k} H(t,s)h(s)\,ds = \frac{1}{32},$$

taking $r = 1$, $L = 2$ and $R = 91 > LM^{-1}M_1 = 90$, then (3.7), (3.8) and (3.10) hold. So all the conditions of Theorem 3.1 hold. By Theorem 3.1, (3.15) has at least three nonnegative solutions.

Acknowledgements

This work is supported by the NNSF of China (No. 10571050 and No. 60671066), a project supported by the Scientific Research Fund of Hunan Provincial Education Department (07B041) and the Program for Young Excellent Talents in Hunan Normal University.

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