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GROUPS OF FINITE NORMAL LENGTH

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Abstract

Let k be a nonnegative integer. A subgroup X of a group G has normal length k in G if all chains between X and its normal closure X^G have length at most k, and k is the length of at least one of these chains. The group G is said to have finite normal length if there is a finite upper bound for the normal lengths of its subgroups. The aim of this paper is to study groups of finite normal length. Among other results, it is proved that if all subgroups of a locally (soluble-by-finite) group G have finite normal length in G, then the commutator subgroup G' is finite and so G has finite normal length. Special attention is given to the structure of groups of normal length 2. In particular, it is shown that finite groups with this property admit a Sylow tower.

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1. Introduction

An important theorem of Neumann [5] proves that if every subgroup X of a group G has finite index in its normal closure X^G, then the commutator subgroup G' of G is finite and hence the index $|X^G : X|$ is bounded by the order of G'. This result suggests that the structure of a group all of whose subgroups are not too far from their normal closure must be somehow restricted, and the aim of this paper is to give some further evidence of this phenomenon.

Let $\mathcal{L}$ be any partially ordered set and let $\mathcal{A}(\mathcal{L})$ be the set of all ascending chains of elements of $\mathcal{L}$. The *length* of $\mathcal{L}$ is the ordinal number

$$\sup_{A\in\mathcal{A}(\mathcal{L})} \ell(A),$$

where the symbol $\ell(A)$ denotes the length of the ascending chain $A \in \mathcal{A}(\mathcal{L})$. Notice that a partially ordered set $\mathcal{L}$ has finite length m if and only if $\mathcal{L}$ admits a finite chain of length m and all ascending chains of $\mathcal{L}$ are finite and have length at most m.

Consider now a subgroup X of a group G, and let $[X^G/X]$ be the interval of the subgroup lattice of G consisting of all subgroups Y such that $X \leq Y \leq X^G$. The length

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of the partially ordered set $[X^G/X]$ will be called the *normal length* of X in G and will be denoted by $\ell_\lhd(X,G)$. Then a subgroup X of a group G is normal if and only if it has normal length 0, while a subgroup of normal length 1 is a nonnormal subgroup which is maximal in its normal closure. If G is any group, the ordinal number

$$\ell_\lhd(G) = \sup_{X\leq G} \ell_\lhd(X,G)$$

is the *normal length* of G. In particular, $\ell_\lhd(G) = 0$ if and only if G is a Dedekind group, that is, if all subgroups of G are normal.

It follows from Neumann's theorem that if each subgroup of a group G has finite index in its normal closure, then G has finite normal length. On the other hand, the consideration of Tarski groups (infinite simple groups whose proper nontrivial subgroups have prime order) shows that a group of normal length 1 may have an infinite commutator subgroup. The structure of groups of normal length 1 was described in [3], where the wider class consisting of all groups in which every cyclic subgroup has normal length at most 1 was studied, and it was proved that these groups are soluble of derived length at most 3, provided that they are locally finite.

It turns out that certain groups of generalised Tarski type are the only infinite simple groups of finite normal length, and they represent the main obstacle in the study of groups of finite normal length. In fact, we will prove that if G is a locally (soluble-by-finite) group in which all subgroups have finite normal length, then G' is finite. Moreover, we shall see that any locally soluble group of finite normal length k is soluble and has derived length bounded in terms of k.

In the last section the case of groups of normal length 2 will be investigated in detail. In particular, it will be shown that any group of normal length 2 is soluble and admits a Sylow tower, provided that it is locally graded. Recall here that a group G is *locally graded* if every finitely generated nontrivial subgroup of G has a proper subgroup of finite index, and that locally graded groups form a large class of generalised soluble groups, containing all locally (soluble-by-finite) groups.

Most of our notation is standard and can be found in [8].

2. Locally soluble groups

Let G be a group and let X be a subgroup of G. Clearly,

$$\ell_\lhd(X,H) \leq \ell_\lhd(X,G)$$

for each subgroup H of G containing X and

$$\ell_\lhd(X^\sigma,G^\sigma) \leq \ell_\lhd(X,G)$$

whenever σ is an epimorphism of G onto a group G^σ. It follows that, for each positive integer k, the class consisting of all groups of normal length at most k is closed with respect to forming subgroups and homomorphic images.

Our first lemma describes the behaviour of cyclic subgroups of finite normal length in a generalised soluble group, and shows in particular that if G is a locally (soluble-by-finite) group and $\langle x\rangle$ is a cyclic subgroup of G of finite normal length, then the element x has only finitely many conjugates in G.

Lemma 2.1. *Let G be a locally (soluble-by-finite) group and let x be an element of G such that $\langle x\rangle$ has finite normal length in G. Then the index $|\langle x\rangle^G : \langle x\rangle|$ is finite.*

Proof. As the interval $[\langle x\rangle^G/\langle x\rangle]$ has finite length, the normal closure $\langle x\rangle^G$ is finitely generated and so it is soluble-by-finite. Then $\langle x\rangle^G$ contains a largest soluble normal subgroup S and the index $|\langle x\rangle^G : S|$ is finite. Assume for a contradiction that the statement is false and choose a counterexample such that the soluble radical S of $\langle x\rangle^G$ has smallest derived length. Obviously $S \neq \{1\}$ and the smallest nontrivial term A of the derived series of S is an abelian normal subgroup of G. It follows from our minimal assumption that the product $\langle x\rangle A$ is a subgroup of finite index in $\langle x\rangle^G$, so that $\langle x\rangle A$ is finitely generated and hence the metabelian group $\langle x\rangle A/\langle x\rangle \cap A$ is residually finite (see [8, Part 2, Theorem 9.51]). Let k be the normal length of $\langle x\rangle$ in G. As the index

$$|A : \langle x\rangle \cap A| = |\langle x\rangle A : \langle x\rangle|$$

is infinite, $A/\langle x\rangle \cap A$ is an infinite normal subgroup of the residually finite group $\langle x\rangle A/\langle x\rangle \cap A$ and so A contains $\langle x\rangle$-invariant subgroups

$$A_0, A_1, \ldots, A_k, A_{k+1}$$

such that

$$\langle x\rangle \cap A = A_0 < A_1 < \cdots < A_k < A_{k+1} = A.$$

Thus, $\langle x\rangle A_i \neq \langle x\rangle A_{i+1}$ for each $i = 0, 1, \ldots, k$ by the Dedekind modular law and so

$$\langle x\rangle = \langle x\rangle A_0 < \langle x\rangle A_1 < \cdots < \langle x\rangle A_k < \langle x\rangle A_{k+1} = \langle x\rangle A$$

is a chain of length $k + 1$ in the interval $[\langle x\rangle^G/\langle x\rangle]$. This contradiction proves the lemma. □

Recall that a group G is an *FC-group* if each of its elements has only finitely many conjugates or equivalently if the centraliser of any element of G has finite index. The class of FC-groups plays a major role in many important problems of the theory of infinite groups, and we refer to the monographs [1] and [10] for the main properties of groups with the FC-property. Here we just mention that any FC-group is locally finite over the centre, so that in particular it has a locally finite commutator subgroup, and that any maximal subgroup of an FC-group has finite index.

Our first main result is an easy consequence of Lemma 2.1.

Theorem 2.2. *Let G be a locally (soluble-by-finite) group in which all subgroups have finite normal length. Then the commutator subgroup G' of G is finite and so G has finite normal length.*

Proof. It follows from Lemma 2.1 that every cyclic subgroup of G has finite index in its normal closure, so that G is an FC-group. In particular, all maximal subgroups of G have finite index and so $|X^G : X|$ is finite for each subgroup X of G, because the interval $[X^G/X]$ has finite length. Therefore, G' is finite by Neumann's theorem and hence also the normal length of G is finite. □

Lemma 2.3. *Let G be a group and let X be an ascendant subgroup of G. If X has finite normal length k in G, then it is subnormal in G of defect at most $k + 1$.*

Proof. As the interval $[X^G/X]$ has finite length k, the ascendant subgroup X is subnormal in X^G with defect at most k. Then X is subnormal also in G and has defect at most $k + 1$. □

It is well known that a group all of whose subgroups are subnormal need not be nilpotent. However, an important theorem of Roseblade shows that if all subgroups of a group G are subnormal of defect at most k (for some fixed positive integer k), then G is nilpotent with nilpotency class at most $\xi(k)$ for a suitable function ξ (see [8, Part 2, Theorem 7.42]). Clearly, $\xi(1) = 2$ and Mahdavianary [4] proved that $\xi(2) = 3$. Thus, Lemma 2.3 has the following consequence.

Corollary 2.4. *Let G be a locally nilpotent group of finite normal length k. Then G is nilpotent and its nilpotency class is at most $\xi(k + 1)$.*

Proof. Let E be any finitely generated subgroup of G. Then E is nilpotent and so all of its subgroups have subnormal defect at most $k + 1$ by Lemma 2.3. Thus, it follows from Roseblade's theorem that the nilpotency class of E is bounded by $\xi(k + 1)$ and hence G itself is nilpotent of class at most $\xi(k + 1)$. □

As we mentioned in the introduction, it follows from results in [3] that any (locally) finite group of normal length 1 is soluble of derived length at most 3, and it is actually easy to give a direct proof of this fact. On the other hand, the consideration of the special linear group $SL(2, 3)$ shows that such groups need not be metabelian. It follows from Lemma 2.3 that in a group with normal length at most 2 every subnormal subgroup has defect at most 3, but it is known that every finite soluble group can be embedded in a finite soluble group whose subnormal subgroups have defect at most 3 (see [2]), so that the derived length of such groups cannot be bounded. On the other hand, our next result shows that the derived length of any soluble group of finite normal length k can be bounded in terms of k.

Theorem 2.5. *Let G be a locally soluble group of finite normal length. Then G is soluble. Moreover, there exists a function ψ such that $\psi(k)$ bounds the derived length of any locally soluble group of normal length at most k.*

Proof. Assume first that G is a finite soluble group of normal length at most k. Consider a minimal normal subgroup N of G and let x be a nontrivial element of N. Clearly, N is abelian of prime exponent p and $\langle x\rangle^G = N$, so that the interval $[N/\langle x\rangle]$ has length at most k and hence $|N| \leq p^{k+1}$. Then $G/C_G(N)$ is isomorphic to a subgroup of the general linear group $GL(k + 1, p)$ and so its derived length is bounded by $\theta(k + 1)$ for a suitable function θ (see for instance [8, Part 1, Theorem 3.23]). Since all homomorphic images of G have normal length at most k, it follows that the factor group $G/C_G(H/K)$ has derived length at most $\theta(k + 1)$ for each chief factor H/K of G. On the other hand, the Fitting subgroup F of G is the intersection of the centralisers of all chief factors of G, so that G/F has derived length at most $\theta(k + 1)$. Moreover, F

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has nilpotency class at most $\xi(k+1)$ by Corollary 2.4 and hence the derived length of G is bounded by

$$\psi(k) = \theta(k+1) + [\log_2(\xi(k+1))].$$

Suppose now that G is any locally soluble group of finite normal length $\leq k$ and let E be a finitely generated subgroup of G. As G' is finite by Theorem 2.2, the subgroup E is polycyclic and so also residually finite. Moreover, each finite homomorphic image has derived length at most $\psi(k)$ by the first part of the proof and hence also the derived length of E is bounded by $\psi(k)$. Therefore, the group G is soluble and its derived length is at most $\psi(k)$. □

3. Simple groups

This section contains some remarks on the behaviour of simple groups of finite normal length. The first lemma shows that a finite group of normal length 2 cannot be simple.

LEMMA 3.1. *Let G be a finite simple nonabelian group. Then $\ell_{\triangleleft}(G) \geq 3$.*

PROOF. Since G is not soluble, it contains a maximal subgroup M which is not supersoluble. Then M cannot have square-free order and so for some prime number p there exists a Sylow p-subgroup P of M such that $|P| > p$. Let X be a subgroup of P of order p. Then

$$X < P < M < G = X^G$$

is a chain of length 3 in the interval $[X^G/X]$, so that $\ell_{\triangleleft}(X, G) \geq 3$ and hence the group G has normal length at least 3. □

COROLLARY 3.2. *Let G be a finite group of normal length at most 2. Then G is soluble.*

Notice here that the alternating group Alt(5) has normal length 3 and that

$$\ell_{\triangleleft}(\mathrm{Alt}(n)) < \ell_{\triangleleft}(\mathrm{Alt}(n+1))$$

for all $n \geq 5$, so that the normal length of finite simple groups cannot be bounded.

Let n be a positive integer. We shall say that an infinite simple group G satisfying both the minimal and the maximal conditions on subgroups is a *Tarski n-monster* if every proper subgroup of G can be generated by at most n elements, and n is the smallest positive integer with such property. Thus, Tarski 1-monsters are precisely the ordinary Tarski groups, whose existence was proved by Ol'shanskiĭ [6]. Observe also that any Tarski n-monster is finitely generated and has finite rank, either n or $n+1$, and that all soluble subgroups of a Tarski n-monster are finite.

Our next result proves in particular that an infinite simple group of normal length 1 is a Tarski group (see also [3, Theorem 32]).

THEOREM 3.3. *Let G be an infinite simple group of finite normal length k. Then G is a Tarski n-monster for some positive integer $n \leq k$.*

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Proof. Let X be any proper subgroup of G and let x be a nontrivial element of X. As $\langle x\rangle^G = G$, the interval $[G/\langle x\rangle]$ has length at most k and so G can be generated by at most $k+1$ elements. Moreover, the interval $[X/\langle x\rangle]$ has length at most $k-1$ and hence X is k-generated. In particular, the group G satisfies the maximal condition on subgroups. It follows from the assumption that all chains of subgroups of G are finite, so that G satisfies the minimal condition on subgroups. Therefore, G is a Tarski n-monster for a suitable positive integer $n \le k$. □

Corollary 3.4. *Let G be a simple nonperiodic group. Then G has infinite normal length.*

It follows from Ol'shanskiĭ's methods that for each prime number $p > 2 \times 10^{77}$ and for each positive integer k there exists a simple two-generator infinite p-group G_k of normal length k and G_k is a Tarski k-monster. In order to prove this, denote by G_1 a Tarski p-group and suppose that a simple p-group G_k of normal length k has been constructed for some $k \ge 1$. Consider now an abelian group A_k of exponent p and order p^{k+1} and apply [7, Theorem 35.1] to the triple (G_k, A_k, p) in order to obtain a simple two-generator group G_{k+1} in which every proper noncyclic subgroup is contained either in a conjugate of G_k or in a conjugate of A_k. In particular, G_k and A_k are maximal subgroups of G_{k+1}. It is easy to prove that G_{k+1} has normal length $k+1$, and it is a $(k+1)$-Tarski monster, because the subgroup A_k is generated by $k+1$ elements.

We point out here that a simple group satisfying both the minimal and the maximal conditions on subgroups may have infinite rank. To see this, take for each positive integer k a finite group E_k of rank strictly larger than k and apply [7, Theorem 35.1] to the collection $\{E_k \mid k > 0\}$. In this way we obtain a periodic infinite simple two-generator group G in which every proper noncyclic subgroup is contained in a conjugate of some E_k. In particular, all proper subgroups of G are finite and so G satisfies both the minimal and the maximal conditions on subgroups. On the other hand, it is clear that the group G has infinite rank.

4. Groups of normal length 2

The first result of this section shows that all locally graded groups of normal length at most 2 are soluble.

Theorem 4.1. *Let G be a locally graded group of normal length at most* 2*. Then G is soluble.*

Proof. Let E be any finitely generated subgroup of G and let J be the finite residual of E. It follows from Corollary 3.2 that every finite homomorphic image $\overline{E}$ of E is soluble, and $\overline{E}$ has derived length at most $\psi(2)$ by Theorem 2.5. Thus, also E/J is soluble. An application of Theorem 2.2 yields now that E/J has finite commutator subgroup and in particular it is polycyclic. Thus, there exists a finitely generated subgroup X such that $J = X^E$. Assume for a contradiction that $J \neq \{1\}$. Since X has

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finite normal length in E, the subgroup J is finitely generated and so it admits a finite nontrivial homomorphic image. Then J' is properly contained in J and E/J' is soluble. It follows that the commutator subgroup of E/J' is finite, so that E/J' is polycyclic and hence also residually finite. This contradiction shows that $J = \{1\}$, so that E is soluble and G is locally soluble. Therefore, the group G is soluble by Theorem 2.5. □

We shall say that a group G has a *Sylow tower* if it admits an ascending normal series

$$\{1\} = G_0 < G_1 < \cdots < G_\alpha < \cdots < G_\mu \leq G_{\mu+1} = G$$

such that $G_{\alpha+1}/G_\alpha$ is a Sylow subgroup of G/G_α for each ordinal $\alpha < \mu$ and G/G_μ is torsion-free. In particular, a periodic group G has a Sylow tower if and only if every nontrivial homomorphic image of G contains a nontrivial normal Sylow subgroup. It is obvious that any finite group with a Sylow tower is soluble and p-nilpotent for some prime number p dividing its order, and a classical result of Zappa [11] states that any finite supersoluble group has a Sylow tower.

It is easy to show that the direct product $\mathrm{Alt}(4) \times \mathrm{Sym}(3)$ has normal length 3, but of course it does not have a Sylow tower. The aim of this section is to prove that all locally graded groups of normal length at most 2 admit a Sylow tower.

The following elementary lemma is probably well known and shows that every group of order p^2q^2 (p, q prime numbers) has a Sylow tower.

Lemma 4.2. *Let p and q be prime numbers and let G be a finite group of order p^2q^2. Then G contains a nontrivial Sylow subgroup which is normal.*

Proof. Assume for a contradiction that the statement is false and let N be a minimal normal subgroup of G. Since G is soluble by the celebrated Burnside theorem, it follows from the assumption that N has prime order, p say. Obviously, G is not supersoluble and so a minimal normal subgroup M/N of G/N must be elementary abelian of order q^2. On the other hand, the group $M/C_M(N)$ is cyclic, so that $C_M(N) = M$ and N is contained in $Z(M)$. It follows that M is abelian and hence $M = N \times Q$, where Q is a characteristic subgroup of order q^2. This contradiction shows that G has a normal nontrivial Sylow subgroup. □

Lemma 4.3. *Let G be a finite group of normal length at most 2 and let N be a minimal normal subgroup of G. Then N has order p^k, where p is a prime number and $k \leq 3$.*

Proof. As the group G is soluble by Theorem 4.1, its minimal normal subgroup N is abelian and has prime exponent, p say. Consider an element x of N of order p. Then $\langle x\rangle^G = N$, so that $N/\langle x\rangle$ has order at most p^2 and hence $|N| \leq p^3$. □

As the class of groups of normal length at most 2 is subgroup closed, our next theorem actually proves that any finite group of normal length at most 2 has a Sylow tower.

Theorem 4.4. *Let G be a finite group of normal length at most 2. Then G is p-nilpotent for some prime divisor p of the order of G.*

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Proof. Let $p_1, \ldots, p_m$ be the prime divisors of the order of G and suppose first that $m \geq 3$. Since G is soluble by Theorem 4.1, it has a Hall p_i'-subgroup Q_i for each $i = 1, \ldots, m$. Consider the Sylow system

$$\mathcal{S} = \{Q_1, \ldots, Q_m\}$$

and let

$$W = \bigcap_{i=1}^{m} N_G(Q_i)$$

be the system normaliser of G arising from $\mathcal{S}$. Then $W^G = G$ (see for instance [9, 9.2.8]). Assume for a contradiction that G is not p_i-nilpotent for any i, so that $N_G(Q_i)$ is a proper subgroup of G for all i. It follows that Q_h is not contained in $N_G(Q_k)$ whenever $h \neq k$ and so

$$W \leq N_G(Q_h) \cap N_G(Q_k) < N_G(Q_h) < G.$$

On the other hand $\ell_{\lhd}(W, G) = 2$, so that

$$W = N_G(Q_h) \cap N_G(Q_k)$$

and hence W contains a Sylow p_j-subgroup P_j of G for some $j \leq m$. Then Q_j is a normal subgroup of $G = P_j Q_j$ and this contradiction completes the proof when $m \geq 3$.

Suppose now that the set $\pi(G)$ consists of only two prime numbers p and q, and assume for a contradiction that the statement is false. Choose a counterexample G of smallest possible order and let N be a minimal normal subgroup of G. Then N has prime exponent, p say, and $|N| \leq p^3$ by Lemma 4.3. Clearly, the statement holds for the factor group G/N and so QN/N is normal in G/N, where Q is any Sylow q-subgroup of G. An application of the Frattini–Capelli argument yields $G = KN$, where $K = N_G(Q)$. As the subgroup Q is not normal in G, the intersection $K \cap N$ is a proper G-invariant subgroup of N and hence $K \cap N = \{1\}$. Notice also that p divides the order of K, because N cannot be a Sylow p-subgroup of G. Assume that the largest normal p-subgroup $O_p(K)$ of K is not trivial. Since the product $O_p(K)N$ is a normal p-subgroup of G,

$$[N, O_p(K)] = [N, NO_p(K)]$$

is a proper G-invariant subgroup of N and so $[N, O_p(K)] = \{1\}$. It follows that $O_p(K)$ is normal in G, so that $QO_p(K)$ is a normal subgroup by the minimal choice of G and hence

$$Q = QN \cap QO_p(K)$$

is likewise normal in G. This contradiction shows that $O_p(K) = \{1\}$. Let K^* be a maximal subgroup of K containing Q, so that K^* is normal in K since K/Q is a p-group. The product K^*N is a proper subgroup of G and hence it has a nontrivial normal Sylow subgroup. On the other hand, Q is not normal in K^*N, so that K^*N has a unique Sylow p-subgroup, and this latter is the product K_p^*N, where K_p^* is the unique Sylow p-subgroup of K^*. Clearly, the subgroup K_p^* is normal in K, so

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that $K_p^* = \{1\}$ since K has no nontrivial normal p-subgroups. It follows that K^* is a q-subgroup and hence $Q = K^*$ is a maximal subgroup of K. Therefore, $|K : Q| = p$, so that $K = Q\langle x\rangle$, where $\langle x\rangle$ has order p and $P = \langle x\rangle N$ is a Sylow p-subgroup of G. In particular, $N = O_p(G)$.

Assume that $O_q(G) = \{1\}$, so that N is the unique minimal normal subgroup of G. As the centraliser $C_K(N)$ is a normal subgroup of G, it follows that

$$O_p(C_K(N)) = O_q(C_K(N)) = \{1\}$$

and hence $C_K(N) = \{1\}$. Thus, K is isomorphic to a group of automorphisms of N and so $|N| > p$ since p divides the order of K. On the other hand,

$$\langle x\rangle < \langle x\rangle N < \langle x\rangle^G,$$

so that $\langle x\rangle$ must be a maximal subgroup of $\langle x\rangle N$ and hence $|\langle x\rangle N : \langle x\rangle| = p$. This contradiction shows that $O_q(G) \neq \{1\}$ and so G contains a minimal normal subgroup M of exponent q. The replacement of N by M in the above arguments yields $G = LM$ and $L \cap M = \{1\}$, where $L = N_G(P)$ is the normaliser of the Sylow p-subgroup P of G. Moreover, the product PM is a normal subgroup of G and $|Q : M| = q$, so that in particular $M = O_q(G)$ and $Q = \langle y\rangle M$, where $\langle y\rangle$ is a subgroup of order q of L. Clearly,

$$PM \cap QN = M(P \cap QN) = MN(P \cap Q) = MN$$

and so

$$G/MN = (PM/MN) \times (QN/MN)$$

is a cyclic group of order pq. In particular, the commutator subgroup G' is contained in $MN = M \times N$, so that

$$G' = (M \cap G') \times (N \cap G')$$

and hence $G' = MN$, because G' cannot have prime-power order. Notice also that $K \cap L$ is a system normaliser of G, so that it is nilpotent and $(K \cap L)^G = G$. Then

$$K \cap L = \langle x\rangle Q \cap \langle y\rangle P = \langle x, y\rangle = \langle x\rangle \times \langle y\rangle$$

is a cyclic group of order pq.

The group G cannot have order p^2q^2 by Lemma 4.2, so that it can be assumed without loss of generality that $|P| > p^2$. As $\langle y\rangle N$ is a subnormal subgroup of G and $O_q(L) = \{1\}$, the element y cannot centralise N and so there exists $a \in N$ such that $[a, y] \neq 1$. Put $z = xa$ and suppose first that P is abelian. Then z has order p and

$$[z, y] = [xa, y] = [x, y]^a[a, y] = [a, y]$$

is a nontrivial element of N. It follows that p divides the order of the normal subgroup $[z, G]$ of G and so

$$N \leq [z, G] \leq G'.$$

On the other hand, if $[z, G] = N$, then

$$P = \langle x\rangle N = \langle z\rangle N = \langle z\rangle^G$$

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is normal in G, which is not the case. Thus, $[z, G] = G'$ and hence

$$\langle z\rangle < P < PM = \langle z\rangle NM = \langle z\rangle G' = \langle z\rangle^G,$$

which is a contradiction since $\langle z\rangle$ cannot be maximal in P. Therefore, the subgroup P is not abelian. The intersection $X = Z(P) \cap N$ is a nontrivial normal subgroup of $N_G(P)$ and

$$\langle xy\rangle < \langle xy\rangle X < N_G(P) < \langle xy\rangle^G = G.$$

This last contradiction completes the proof of the statement. □

We finally extend Theorem 4.4 to the case of locally graded groups.

Theorem 4.5. *Let G be a locally graded group of normal length at most 2. Then G has a Sylow tower.*

Proof. The group G is soluble by Theorem 4.1 and so it follows from Theorem 2.2 that its commutator subgroup G' is finite. In particular, the set T of all elements of finite order of G is a characteristic subgroup and G/T is torsion-free. Thus, it is enough to prove that T has a Sylow tower and hence we may suppose without loss of generality that G is locally finite.

Assume for a contradiction that G has no nontrivial normal Sylow subgroups and put $C = C_G(G')$. Then C is a nilpotent normal subgroup of G and the factor group G/C is finite. Clearly, each Sylow subgroup of C is normal in G, so that the set of primes $\pi(C)$ is contained in $\pi(G/C)$. It follows that $\pi(G)$ is finite. For each prime number $p \in \pi(G)$, there exist elements x_p and y_p of G whose orders are powers of p, such that the order of the product $x_p y_p$ is not a power of p. Then

$$E = \langle x_p, y_p \mid p \in \pi(G)\rangle$$

is a finite subgroup of G which has no nontrivial normal Sylow subgroups, contrary to the statement of Theorem 4.4. This contradiction proves the theorem. □

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