8

Gauge theories in two dimensions – basics

As an introduction to the cast of characters of two-dimensional gauge theories, we briefly summarize here the basics of pure Maxwell theory, QED, pure YM theory and QCD. This includes the corresponding actions, symmetries, equations of motion and their solutions.

The basics of gauge theories in two dimensions is "standard material" which appears in many books and review articles, for instance [66], [178], [1] and [2]. For treatment in non-covariant gauges see [28].

8.1 Pure Maxwell theory

The simplest theory of gauge fields in two dimensions is obviously the abelian Maxwell theory defined by the classical action,

$$S = \int d^2x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \tag{8.1}$$

where the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{8.2}$$

has, in two dimensions, only one non-trivial component $E_1 \equiv F_{10} = -F_{01} = \partial_1 A_0 - \partial_0 A_1$. The action is invariant under the full global two-dimensional conformal symmetry $SO(2,2)$, discussed in Section 2.1, which includes in particular the $ISO(1,1)$, where I stands for inhomogeneous, namely adding the momenta, thus going over to the Poincare group from the Lorentz group. The action is by construction also invariant under the gauge transformation,

$$A_\mu(x,t) \to A_\mu(x,t) + \partial_\mu \Lambda(x,t). \tag{8.3}$$

The canonical dimension of A_μ is clearly zero. The corresponding equation of motion reads,

$$\partial^\mu F_{\mu\nu} = 0 \quad \partial_0 E_1 = \partial_1 E_1 = 0 \quad \to \quad E_1 = \text{constant}. \tag{8.4}$$

Thus we conclude that the two-dimensional Maxwell theory is an empty theory on an $R^{1,1}$ manifold. On such a space-time requiring finite energy implies that $E_1 = 0$. This is of course not surprising. In d-dimensional space-time the number of degrees of freedom of an abelian gauge field is $d-2$ and hence there are no degrees of freedom in two dimensions.

8.2 QED_2 – Schwinger's model

Next we couple the two-dimensional abelian gauge fields to a Dirac fermion. The Lagrangian density of this model is given by,[1]

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\partial\!\!\!/ - e A\!\!\!/ \ - m)\Psi \\ &= \frac{1}{2}(\bar{\partial}A - \partial\bar{A})^2 + \psi^\dagger\bar{\partial}\psi + \tilde{\psi}^\dagger\partial\tilde{\psi} + e\psi^\dagger\psi\bar{A} + e\tilde{\psi}^\dagger\tilde{\psi}A - m(\psi^\dagger\tilde{\psi} + \tilde{\psi}^\dagger\psi),\end{aligned} \tag{8.5}$$

where in the second line the action is expressed in terms of the light-cone derivatives and components of the gauge fields, and the Dirac fermion is decomposed into its left and right chiral fermions, as discussed in Section 3.8. It is evident that with the gauge field having a vanishing dimension, the gauge coupling e has a dimension of mass. Thus the action is not invariant any more under the two-dimensional global conformal symmetry, but rather only under the $ISO(1,1)$ Poincare group. For the massless case, the action is classically invariant under the global transformations,

$$\begin{aligned}&\Psi \to e^{i\alpha}\Psi \quad \Psi \to e^{i\tilde{\alpha}\gamma_5}\Psi \\ &\psi \to e^{i(\alpha+\tilde{\alpha})}\psi \quad \tilde{\psi} \to e^{i(\alpha-\tilde{\alpha})}\tilde{\psi}.\end{aligned} \tag{8.6}$$

In fact the left and right chiral transformations, for the massless case, can be lifted also into holomorphic and anti-holomorphic transformations, as was discussed in Section 3.7.1. The corresponding vector and axial currents

$$\begin{aligned}J^\mu &= \bar{\Psi}\gamma^\mu\Psi \quad J_5^\mu = \bar{\Psi}\gamma^\mu\gamma_5\Psi \\ J &= \psi^\dagger\psi \quad \bar{J} = \tilde{\psi}^\dagger\tilde{\psi}.\end{aligned} \tag{8.7}$$

Again by construction the action is also invariant under the gauge transformation,

$$\Psi \to e^{-i\Lambda(x,t)}\Psi \quad A_\mu(x,t) \to A_\mu(x,t) + \frac{1}{e}\partial_\mu\Lambda(x,t). \tag{8.8}$$

Quantum mechanically the axial current is not conserved even for the massless case due to an anomaly,

$$\partial_\mu J_5^\mu = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}. \tag{8.9}$$

We will derive this result using the bosonized version, see Section 9.1. Unlike Maxwell's theory, this theory has non-trivial degrees of freedom. However, once again the gauge field is not dynamical. This phenomenon can be easily demonstrated in the axial gauge $A_1 = 0$, where the other component A_0 can be solved

[1] The Schwinger model was introduced in [190] and further analyzed in [68] and [64].

as a function of the electric current. The resulting electric field is,

$$E_1 = -F_{01} = -e\partial_1^{-1} J_0 - \frac{e\theta}{2\pi}, \quad J_0 =: \Psi^\dagger \Psi :, \tag{8.10}$$

θ is a new parameter in the theory, the vacuum angle.[2] In Part 3 of the book we will describe its four-dimensional analog which is the vacuum angle due to QCD_4 instanton tunneling.

The massless Schwinger model can easily be solved using the anomaly equation combined with the equation of motion of the system. This will be done in Section 9.1 using the bosonized version where we also address the massive case. In Chapter 15 we determine the spectrum of the massless case using a BRST quantization approach. In Chapter 14 we analyze the nature of the system and determine when it confines and when it admits a screening behavior.

8.3 Yang–Mills theory

It is straightforward to generalize the action of the Maxwell theory (8.1) to the non-abelian case.[3] The gauge fields are now in the adjoint representation of a non-abelian gauge group $\mathcal{G}$. We will mainly be interested in the groups $SO(N_c)$, $U(N_c)$ and $SU(N_c)$. Thus A_μ is an $N_c \times N_t$ either orthogonal, or hermitian or traceless hermitian matrix of the form $A_\mu = t^B A_\mu^B$ where t^B are the generators of the group, $B = 1, ..., \dim\mathcal{G}$ and $\dim\mathcal{G}$ is the dimension of the corresponding algebra $[\frac{1}{2}N_c(N_c-1), N_c^2$ and $N_c^2 - 1$, respectively]. The field strength is now,

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \\ F_{\bar{z}z} &= \bar{\partial} A - \partial \bar{A} + i[A, \bar{A}], \end{aligned} \tag{8.11}$$

where again we write it out in light-cone coordinates. The action of two-dimensional Yang–Mills theory reads,

$$S_{YM2} = \int \mathrm{d}^2x \left[-\frac{1}{2e_c^2} \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \right] = \int \mathrm{d}^2x \left[-\frac{1}{4e_c^2} F^a{}_{\mu\nu} F^{a\,\mu\nu} \right], \tag{8.12}$$

and the corresponding equations of motions are,

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + i[A_\mu, F^{\mu\nu}] = 0. \tag{8.13}$$

Note that we have rescaled the fields A by a factor of the gauge coupling, as compared with the abelian case. Note, however, that this does not affect the dynamical dimensions, namely the space-time behavior of Green's functions.

In this formulation A_μ has dimension one and so is the dimension of the color gauge coupling e_c. Again since the coupling constant has a dimension of mass the classical theory is not invariant under the full global conformal symmetry,

[2] The θ angle was introduced by Lowenstein and Swieca [152] and also by Coleman [64].
[3] The Yang–Mills non-abelian gauge theory was introduced in the seminal paper [229].

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but only with respect to the $ISO(1,1)$ Poincare transformations. The action is invariant under a non-abelian gauge transformation, which in infinitesimal form is,

$$A_\mu \to A_\mu + D_\mu \Lambda = A_\mu + \partial_\mu \Lambda + i[A_\mu, \Lambda], \tag{8.14}$$

where $\Lambda = t^A \Lambda^A$. A priori this is not a free theory, but rather an interacting one. However, in a similar manner to Maxwell's theory, this model too on an $R^{1,1}$ manifold has no dynamical degrees of freedom just as the abelian model. This can easily be seen by fixing a gauge, for instance $A_0 = 0$. In this gauge the equations of motion read,

$$\partial_0 F^{01} = 0 \quad \partial_1 F^{10} + i[A_1, F^{01}] = 0. \tag{8.15}$$

From the first we get that $\partial_0^2 A_1 = 0$, and thus $A_1 = f_1(x_1) + x_0 f_2(x_1)$. Using the residual gauge invariance, of gauge transformations that depend only on x_1, we can go to $f_1 = 0$, and then the second equation implies that f_2 is a constant C, which yields $F_{01} = C$, and then again the requirement of finite energy results in $C = 0$. This will also be shown in a complicated way using a BRST approach in Chapter 15.

When the underlying manifold has a non-trivial topology like that of a torus then the theory is not totaly empty but instead has topological degrees of freedom. This will be described in Section 16.

Finally, the non-abelian case is different from the abelian in higher dimensions, as the former is not free there. While the abelian case represents free photons, the non-abelian case represents interacting gluons, which turn to interacting glue balls in the physical space.

8.4 Quantum chromodynamics

The theory of non-abelian gauge fields coupled to Dirac quarks in the fundamental representation of the gauge group, QCD_2, is described by the action,

$$S_{QCD_2} = \int \mathrm{d}^2 x \left\{ -\frac{1}{2e_c^2} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) - \bar{\Psi}^{ai} [(i\partial\!\!\!/ - A\!\!\!/ + m)\Psi_i]_a \right\}. \tag{8.16}$$

The action is invariant under two-dimensional Poincare transformation and the non-abelian generalization of the gauge transformation of (8.8), which in infinitesimal form is,

$$\delta \Psi_a = -i[\Lambda(x,t)]_a^b \Psi_b \quad \delta A_\mu(x,t) = \partial_\mu \Lambda(x,t) + i[A_\mu \Lambda], \tag{8.17}$$

with the non-abelian $\Lambda = \Lambda^A T_A$. Ψ is in the fundamental representation of the gauge group which we take to be $SU(N_c)$ where $a = 1, \ldots, N_c$ denote the color indices. As was discussed in Section 6.3.4 flavor degrees of freedom have been included by assigning a flavor index to the Dirac fermion Ψ_i, $i = 1, \ldots, N_f$. For this case the theory is obviously invariant classically under a global

$U_{\mathrm{L}}(N_f) \times U_{\mathrm{R}}(N_f)$ symmetry. Here there is no anomaly, as in 2d the anomaly occurs via the abelian gauge field only.

In a similar manner to the transition from the empty Maxwell theory to the dynamically viable Schwinger model, so is the transition from the two-dimensional pure Yang–Mills theory to QCD_2. The difference, however, is that in the non-abelian case, even for the massless case there is no simple way to solve the theory. Instead we will need to implement various different techniques developed in the first part of the book. In the next section we will describe both QED and QCD in two dimensions using the bosonization language. This will enable us to solve for the baryonic spectrum in the strong coupling limit. In Chapter 14 the string tension of several two-dimensional dynamical systems will be computed. An analysis of the spectrum of these theories will be derived using the BRST quantization approach in Chapter 15. In Chapter 10 we present the seminal 't Hooft solution of two-dimensional QCD in the large N limit. A current algebra generalization of the latter approach will enable us to solve the mesonic spectra of certain models. Finally in Chapter 12 we will implement a discrete light-cone quantization approach to solve QCD in two dimensions with quarks in the fundamental as well as the adjoint representation.