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MAJOR n-CONNECTED GRAPHS

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Abstract

An induced subgraph H of connectivity (edge-connectivity) n in a graph G is a *major n-connected* (*major n-edge-connected*) subgraph of G if H contains no subgraph with connectivity (edge-connectivity) exceeding n and H has maximum order with respect to this property. An induced subgraph is a *major* (*major edge-*) *subgraph* if it is a major n-connected (major n-edge-connected) subgraph for some n. Let m be the maximum order among all major subgraphs of C. Then the *major connectivity set* $K(G)$ of G is defined as the set of all n for which there exists a major n-connected subgraph of G having order m. The *major edge-connectivity set* is defined analogously. The connectivity and the elements of the major connectivity set of a graph are compared, as are the elements of the major connectivity set and the major edge-connectivity set of a graph. It is shown that every set S of nonnegative integers is the major connectivity set of some graph G. Further, it is shown that for each positive integer m exceeding every element of S, there exists a graph G such that every major k-connected subgraph of G, where $k \in K(G)$, has order m. Moreover, upper and lower bounds on the order of such graphs G are established.

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1. Introduction

The *connectivity* $\kappa(G)$ of a graph G is the minimum cardinality of a set S of vertices of G such that $G - S$ is disconnected or the trivial graph. (See [1] for other basic graph theory terminology.) Although the connectivity of a graph is considered a global parameter, it need not reveal much information about the structure of the graph. Indeed, every subgraph of a graph with small connectivity may also have small connectivity, such as $\overline{K}_p$, or a graph with small connectivity may contain subgraphs having large connectivity. For example, $K_n \cup K_1$ $(n \geq 2)$ has connectivity 0 but contains a subgraph with connectivity $n-1$.

In 1978 Matula [3] defined the *maximum subgraph connectivity*

$$\hat{\kappa}(G) = \max\{\kappa(H) \mid H \subset G\}.$$

This parameter then gives the largest connectivity of a subgraph of G. It may happen, however, that a graph containing subgraphs with large connectivity also contains subgraphs of large order having small connectivity. Consider, for example, for every positive integer n, the graph G_n obtained by identifying an end-vertex of a path H_n of order n^2 with a vertex in the complete graph of order $n+1$. Then $p(H_n)/p(G_n) = n^2/(n^2+n+1)$, so that $\lim_{n\to\infty} p(H_n)/p(G_n) = 1$. However, $\hat{\kappa}(G_n) = n$ and $\kappa(H_n) = 1$. Thus neither the connectivity nor the maximum subgraph connectivity gives a good indication, in general, of the structure of a graph. With these observations in mind we introduce our main concept.

2. Major n-connected graphs

An induced subgraph H of connectivity n in a graph G is a *major n-connected subgraph* of G if H contains no subgraph with connectivity exceeding n and H has maximum order with respect to this property. An induced subgraph is a *major subgraph* if it is a major n-connected subgraph for some n. Let m be the maximum order among all major subgraphs of G. Then the *major connectivity set* of G is defined by

$$K(G) = \{n \mid \text{there exists a major subgraph } H < G \text{ with } \kappa(H) = n \text{ and } p(H) = m\}.$$

By definition, $K(G) \neq \varnothing$.

To illustrate the above definitions, consider the graph G shown in Figure 1. For $i = 0, 1, 2, 3$, the subgraphs H_i defined below are major i-connected subgraphs. For $i \geq 4$, G contains no major i-connected subgraphs. Since $p(H_0) = 3$ and $p(H_i) = 4$ for $i = 1, 2, 3$, we conclude that $K(G) = \{1, 2, 3\}$.

$$\begin{aligned}
H_0 &= \langle\{v_2, v_5, v_6\}\rangle, \\
H_1 &= \langle\{v_1, v_4, v_5, v_6\}\rangle, \\
H_2 &= \langle\{v_2, v_3, v_4, v_5\}\rangle, \\
H_3 &= \langle\{v_1, v_2, v_3, v_4\}\rangle,
\end{aligned}$$

A graph G is called *critically n-connected* $(n \geq 1)$, if $\kappa(G) = n$ and $\kappa(H) < n$ for every proper induced subgraph H of G. If G is an n-connected graph, $n \geq 1$, then every n-connected subgraph of G having minimum order is critically n-connected. This fact yields the following lemma.

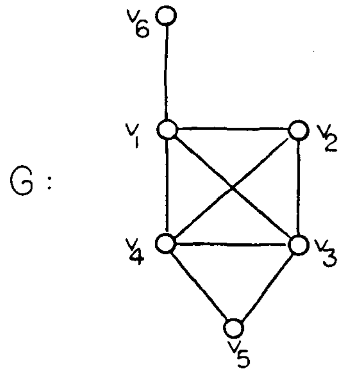

FIGURE 1

LEMMA 1. *If G is a graph with connectivity n, then G contains a major n-connected subgraph.*

As an immediate consequence we have the following.

COROLLARY. *A graph G contains a major k-connected subgraph if and only if $0 \leq k \leq \hat{\kappa}(G)$.*

We note that there are critically n-connected graphs that contain subgraphs whose connectivity exceeds n. To see this, let $m, n \geq 2$ be positive integers. Define $H_0 \cong K_{mn}$ and $H_1 \cong H_2 \cong \cdots \cong H_m \cong K_n$, with $V(H_0) = \{u_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ and $V(H_k) = \{v_{kj} \mid 1 \leq j \leq n\}$ for $1 \leq k \leq m$. Let G be obtained from $H_0 \cup H_1 \cup \cdots \cup H_m$ by adding the edges in $\{u_{ij}v_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Then G is critically n-connected but contains a subgraph with connectivity $mn - 1 > n$.

Clearly, if $k \in K(G)$, then $k \leq \hat{\kappa}(G)$. In general, the connectivity of a graph need not be an upper or lower bound for the elements belonging to the major connectivity set of the graph, as we see from the following theorem.

THEOREM 1. *For every pair n, k of positive integers, there is a graph G with $\kappa(G) = n$ and $K(G) = \{k\}$.*

SKETCH OF PROOF. If $n = k$, then $G \cong K_{n+1}$ has the desired properties. Suppose now that $0 < n < k$. Let $H_1 \cong H_2 \cong K_{k+1}$, and let $V(H_1) = \{v_1, v_2, \ldots, v_{k+1}\}$ and $V(H_2) = \{u_1, u_2, \ldots, u_{k+1}\}$. Define H to be the graph obtained from $H_1 \cup H_2$ by adding the edges in $\{u_iv_i \mid 1 \leq i \leq k\}$. Let G now be obtained from H by joining a new vertex v to each vertex in the set $\{u_1, u_2, \ldots, u_n\}$.

Assume finally that $0 < k < n$. Let $m = (2n - 2k + 1)(2n - k + 1) + 1$ and let $P_m\colon v_1, v_2, \dots, v_m$ be a path of order m. Suppose that H is the kth power of P_m (that is H is obtained from P_m by joining vertices whose distance is at most k in P_m). Next let $H_1 \cong H_2 \cong K_{n-k}$ and define G to be the graph obtained from H, H_1 and H_2 by joining every vertex of H_1 (respectively H_2) to every vertex of H that belongs to P_m except the last (first) $n-k+1$ vertices of this path. Then it can be shown that $\kappa(G) = n$ and $K(G) = \{k\}$.

In the preceding theorem, we focused on graphs whose major connectivity sets consist of a single element. However, from the graph of Figure 1, we see that the major connectivity set of a graph may contain more than one element. Observe that if $K(G) = \{n_1, n_2, \dots, n_k\}$ is the major connectivity set of some graph, where $n_i < n_{i+1}$ $(1 \le i \le k-1)$, then the order of every major n_i-connected subgraph is at least $n_k + 1$. The next result shows that every set S of positive integers is the major connectivity set of some graph G. In addition, by specifying a positive integer m exceeding every element of S, one can choose G in such a way that every major k-connected subgraph of G, where $k \in K(G)$, has order m.

Theorem 2. *Let $S = \{n_1, n_2, \dots, n_k\}$ be a set of positive integers with $n_i < n_{i+1}$ $(1 \le i \le k-1)$. If $m \ge n_k + 1$ is a positive integer, then there is a graph G with $K(G) = S$ and such that every major n_i-connected subgraph $(1 \le i \le k)$ of G has order m. Further, the minimum order of such a graph G is $m + n_k - n_1$.*

Proof. Since the result is obvious for $k = 1$, we assume $k \ge 2$. Let $G_1 \cong K_{n_k+1}$ and $G_2 \cong \overline{K}_{m-n_1-1}$, where $V(G_1) = \{v_1, v_2, \dots, v_{n_k+1}\}$ and $V(G_2) = \{u_1, u_2, \dots, u_{m-n_1-1}\}$. Set $H = G_1 \cup G_2$. Let G be that graph obtained from H by joining every vertex in $\{u_1, u_2, \dots, u_{m-n_i-1}\}$ to every vertex in $\{v_1, v_2, \dots, v_{n_i}\}$, for $1 \le i \le k-1$, and, in addition, if $m > n_k + 1$, by joining every vertex of $\{u_1, u_2, \dots, u_{m-n_k-1}\}$ to every vertex in $\{v_1, v_2, \dots, v_{n_k}\}$.

First we show that every major n_i-connected subgraph, $1 \le i \le m$, has order m. For $i = 1, 2, \dots, k$, let

$$H_i = \langle \{v_1, v_2, \dots, v_{n_i}\} \cup \{v_{n_k+1}\} \cup \{u_1, u_2, \dots, u_{m-n_i-1}\} \rangle.$$

Then $p(H_i) = m$. Because $H_i - \{v_1, v_2, \dots, v_{n_i}\}$ is disconnected for $1 \le i \le k-1$ and since $H_k - \{v_1, v_2, \dots, v_{n_k}\}$ is either trivial if $m = n_k + 1$ or disconnected if $m > n_k + 1$, we conclude that $\kappa(H_i) \le n_i$, $1 \le i \le k$. However, if U is a set of at most $n_i - 1$ vertices of H_i, then $V(H_i) - U$ contains at least one vertex of $\{v_1, v_2, \dots, v_{n_i}\}$. Let $v \in \{v_1, v_2, \dots, v_{n_i}\} - U$. Then v is joined to every vertex in $V(H_i)$, other than itself, implying that $H_i - U$ is connected. Further, $p(H_i - U) \ge 2$, so that $\kappa(H_i) \ge n_i$; hence $\kappa(H_i) = n_i$. Let H_0 be an induced subgraph of H_i. Then $H_0 \cong \overline{K}_l$ $(1 \leq l \leq m - n_i)$, $H_0 \cong K_h$ $(1 \leq h \leq n_i + 1)$ or $H_0 \cong K_r + \overline{K}_s$, where $1 \leq r \leq n_i$ and $1 \leq s \leq m - n_i$. Hence every major n_i-connected subgraph has order at least m.

Next we show that every major n_i-connected subgraph has order at most m. A major n_i-connected subgraph contains at most n_i+1 vertices that belong to $V(G_1)$ because a major n_i-connected subgraph does not contain subgraphs whose connectivity exceeds n_i. Further, a major n_i-connected subgraph, $2 \leq i \leq k$, does not contain vertices from $\{u_{m-n_i}, u_{m-n_i+1}, \dots, u_{m-n_1-1}\}$ because these vertices have degree at most $n_{i-1} < n_i$ in G_i and thus in every subgraph of G. Hence, every major n_i-connected subgraph has order at most m, so that every major n_i-connected subgraph has order m.

Since $\hat{\kappa}(G) = n_k$, the graph G does not contain major t-connected subgraphs for $t > n_k$. It remains to be shown that if $0 \leq t < n_k$ and $t \notin S$, then every major t-connected subgraph of G has order less than m. If $t < n_1$, then every major t-connected subgraph of G contains at most $t + 1 (\leq n_1)$ vertices of $V(G_1)$, so that a major t-connected subgraph has order at most $|V(G_2)| + t + 1 = m - n_1 + t < m$. Suppose now that $n_i < t < n_{i+1}$ for some $i \in \{1, 2, \dots, k-1\}$. Then every major t-connected subgraph contains at most $t + 1$ vertices of G_1. Since the vertices in $\{u_{m-n_{i+1}}, u_{m-n_{i+1}+1}, \dots, u_{m-n_1-1}\}$ have degree at most n_i $(< t)$ in G (and therefore in every induced subgraph of G), these vertices do not belong to any major t-connected subgraph. Hence a major t-connected subgraph contains at most $m - n_{i+1} - 1$ vertices of G_2 and thus at most $m - n_{i+1} + t (< m)$ vertices of G. Therefore, $K(G) = \{n_1, n_2, \dots, n_k\}$ and every major n_i-connected subgraph of G has order m $(1 \leq i \leq k)$.

Since $p(G) = m + n_k - n_1$, it follows that the minimum order of a graph having S as its major connectivity set and for which every major n_i-connected subgraph $(1 \leq i \leq k)$ has order m is at most $m + n_k - n_1$. Suppose now that H is any graph with $K(H) = S$ such that every major n_i-connected subgraph $(1 \leq i \leq k)$ has order m. Then a major n_1-connected subgraph of H contains at most $m - (n_k - n_1)$ vertices of any major n_k-connected subgraph. However, then, H contains at least $m + n_k - n_1$ vertices. Hence $p(H) = m + n_k - n_1$.

If, in the preceding theorem, we allow $0 \in S$, so that $n_1 = 0$, then a minor modification of the proof yields the same conclusion. Consequently, if G is a graph with $K(G) = \{n_1, n_2, \dots, n_k\}$, where $0 \leq n_1 < n_2 < \cdots < n_k$, every major n_i-connected subgraph $(1 \leq i \leq k)$ of which has order m, then $p(G) \geq m + n_k - n_1$. However, the order of such a graph G cannot be arbitrarily large. Clearly if $m = 1$, then $K(G) = \{0\}$, implying that $p(G) = 1$. for $m \geq 2$, we present the following result.

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THEOREM 3. *Let G be a graph of order p, every major n-connected subgraph of which has order $m \geq 2$ for $n \in K(G)$. Then*

$$p < \frac{m^{m+1} - m}{m - 1}.$$

PROOF. Observe first that $(m^{m+1} - m)/(m-1) = m^m + m^{m-1} + \cdots + m$. Let G be a graph satisfying the hypothesis of the theorem. If H is a subgraph of G having connectivity k, then $k \leq m - 1$; otherwise, G contains a subgraph H with connectivity $k \geq m$. However, then, by Lemma 1, H and therefore G contains a major k-connected subgraph of order at least $k + 1 \geq m + 1$, which is not possible. In particular, this implies that $\kappa(G) \leq m - 1$. Assume, to the contrary, that

$$p \geq m^m + m^{m-1} + \cdots + m.$$

Suppose $S_0 \subseteq V(G)$, where $|S_0| = \kappa(G)$ and $G - S_0$ is disconnected. Then, by a preceding observation, $|S_0| \leq m - 1$. If $G - S_0$ contains at least $m + 1$ components, then $\beta(G) \geq m + 1$, which is not possible since every major 0-connected subgraph of G has order at most m. Further, $p > m$ so that G is not complete. Hence $G - S_0$ has at least two but at most m components. Therefore, $G - S_0$ contains at least one component G_1 with order at least

$$\frac{p - (m-1)}{m} > \frac{p - m}{m} \geq m^{m-1} + m^{m-2} + \cdots + m. \tag{1}$$

Let $v_1 \in V(G - S_0) - V(G_1)$. Then v_1 is not adjacent to any vertex of G_1.

Since $m \geq 2$, it follows from (1) that $p(G_1) > m$. Further, because $G_1 \subset G$, the connectivity of G_1 is at most $m - 1$. Let $S_1 \subseteq V(G_1)$, where $|S_1| = \kappa(G_1)$ and $G_1 - S_1$ is disconnected. As in the case of $G - S_0$, the graph $G_1 - S_1$ has at most m components. Therefore, $G - S_1$ contains a component G_2 having order at least

$$\frac{p(G_1) - (m-1)}{m} > \frac{p(G_1) - m}{m} > m^{m-2} + m^{m-3} + \cdots + m.$$

Let $v_2 \in V(G_1 - S_1) - V(G_2)$. then $\{v_1, v_2\}$ is an independent set of vertices disjoint from $V(G_2)$ and such that no vertex in G_2 is adjacent to a vertex in $\{v_1, v_2\}$.

Continuing in this fashion for k steps, where $k < m$, we obtain a sequence $G_1, G_2, \ldots, G_k$ of subgraphs of G, where

$$p(G_i) > m^{m-i} + m^{m-i-1} + \cdots + m,$$

$1 \leq i \leq k$, and an independent set $\{v_1, v_2, \ldots, v_k\}$ of vertices of G disjoint from $V(G_k)$ such that no vertex of G_k is adjacent to a vertex in $\{v_1, v_2, \ldots, v_k\}$. In particular, if $k = m - 1$, then $\{v_1, v_2, \ldots, v_{m-1}\}$ is an independent set of vertices disjoint from $V(G_{m-1})$ and such that no vertex in G_{m-1} is adjacent to any vertex in $\{v_1, v_2, \ldots, v_{m-1}\}$. Since $p(G_{m-1}) > m$ and $\kappa(G_{m-1}) \le m-1$, it follows that G_{m-1} is not complete and therefore contains a pair v_m, v_{m+1} of nonadjacent vertices. Then $\{v_1, v_2, \ldots, v_{m+1}\}$ is an independent set of vertices of G, implying that $\beta(G) \ge m + 1$. However, this contradicts the hypothesis that every major 0-connected subgraph of G has order less than m.

3. Major n-edge-connected graphs

We now consider the edge analogue of major n-connected graphs. An induced subgraph H of a graph G with $\kappa_1(H) = n$ is a *major n-edge-connected subgraph* of G if H has maximum order with respect to this property. An induced subgraph is a *major edge-subgraph* if it is a major n-edge-connected subgraph for some n. Let m be the maximum order among all major edge-subgraphs of G. Then the *major edge-connectivity set* of$G$ is defined by

$$K_1(G) = \{n \mid \text{there exists a major edge-subgraph } H < G \text{ with } \kappa_1(H) = n \text{ and } p(H) = m\}.$$

To illustrate these definitions, we consider the graph G of Figure 1. For $i = 0, 1, 2, 3$, the subgraphs H_i defined in Figure 1 are major i-edge-connected subgraphs. For $i \ge 4$, G contains no major i-edge-connected subgraphs. We observe that $K_1(G) = K(G) = \{1, 2, 3\}$.

Mader [2] defined a graph G to be *critically n-edge-connected* ($n \ge 1$) if G is n-edge-connected and if for every vertex v of G, the graph $G - v$ is not n-edge-connected. The following two results are due to Mader.

THEOREM A. *For a positive integer n, every critically n-edge-connected graph G contains at least two vertices of degree n.*

THEOREM B. *For a positive integer n, every n-edge-connected graph G contains at least* $\min\{p(G), 2\lfloor n/2 \rfloor + 2\}$ *vertices v such that* $\kappa_1(G - v) \ge n - 1$.

From Theorem A, it now follows that a graph G is critically n-edge-connected if and only if $\kappa_1(G) = n$ and $\kappa_1(G - v) < n$ for every vertex v of G. The following three lemmas will prove to be useful.

LEMMA 2. *If G is a graph with edge-connectivity* $n \ge 1$, *then G contains an induced critically n-edge-connected subgraph.*

PROOF. If $\kappa_1(G - v) < n$ for all $v \in V(G)$, then G is itself critically n-edge-connected. Suppose, therefore, that G contains a vertex v_0 such that

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$\kappa_1(G - v_0) \geq n$. Let $G_1 = G - v_0$. If $\kappa_1(G_1 - v) < n$ for all $v \in V(G_1)$, then G_1 is an induced critically n-edge-connected subgraph of G. Otherwise, G_1 contains a vertex v_1 such that $\kappa_1(G_1 - v_1) \geq n$. Let $G_2 = G_1 - v_1$. If G_2 is critically n-edge-connected, then the proof is complete; otherwise, we continue in this fashion to produce an induced subgraph G_i of G such that G_i is critically n-edge-connected.

LEMMA 3. *If G is a critically n-edge-connected graph* ($n \geq 1$), *then G contains a vertex v such that $\kappa_1(G - v) = n - 1$.*

PROOF. Since G is critically n-edge-connected, $\kappa_1(G_v) \leq n - 1$ for every vertex v of G. By Theorem B, G contains a vertex u such that $\kappa_1(G - u) \geq n - 1$. Consequently, $\kappa_1(G - u) = n - 1$.

LEMMA 4. *If G is a graph with edge-connectivity $n \geq 1$, then for every integer k $(0 \leq k \leq n)$, G contains an induced subgraph G_k with edge-connectivity k.*

PROOF. Clearly $G_n = G$ is an induced subgraph of G having edge-connectivity n. By Lemma 2, G_n contains an induced critically n-edge-connected subgraph G'_n. By Lemma 3, G'_n contains a vertex v_n such that $\kappa_1(G'_n - v_n) = n - 1$. Let $G_{n-1} = G'_n - v_n$. If $n - 1 \geq 1$, then, by Lemma 2, G_{n-1} contains a critically $(n-1)$-edge-connected subgraph G'_{n-1}. By Lemma 4, G contains a vertex v_{n-1} such that $\kappa_1(G'_{n-1} - v_{n-1}) = n - 2$. Let $G_{n-2} = G'_{n-1} - v_{n-1}$. Proceeding in this fashion, we produce induced subgraphs $G_n, G_{n-1}, \ldots, G_0$ with $\kappa_1(G_i) = i$ $(0 \leq i \leq n)$.

THEOREM 4. *If G is a graph with edge-connectivity n, then G contains a major n-edge-connected subgraph.*

PROOF. To prove the theorem, we need only show that every graph G with edge-connectivity n contains an induced subgraph having edge-connectivity n and which contains no subgraphs whose edge-connectivity exceeds n.

If $n = 0$, then a maximum independent set of vertices induces a major 0-connected subgraph. Assume then that $n \geq 1$. By Lemma 2, G contains an induced critically n-edge-connected subgraph. We show that if H is an induced critically n-edge-connected subgraph of G having minimum order, then $\kappa_1(H) = n$ and every (induced) subgraph of H has edge-connectivity at most n. The observation preceding Lemma 2 shows that $\kappa_1(H) = n$. It remains to be shown that H contains no subgraph whose edge-connectivity exceeds n. Assume, to the contrary, that H contains an induced subgraph H' with $\kappa_1(H') = m > n$. By Lemma 3, for every k $(0 \leq k \leq m)$, H' contains

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an induced subgraph with edge-connectivity k. In particular, H' contains an induced subgraph F with $\kappa_1(F) = n$. However, by Lemma 2, F contains an induced critically n-edge-connected subgraph F'. Clearly $p(F') < p(H)$, which contradicts our choice of H.

Consequently, G contains an induced subgraph whose edge-connectivity is n and which contains no subgraphs with edge-connectivity exceeding n. This implies that G contains a major n-edge-connected subgraph.

The construction immediately following the corollary to Lemma 1 also illustrates that there are critically n-edge-connected graphs containing (induced) subgraphs with edge-connectivity exceeding n.

Matula [2] defined the *maximum subgraph edge-connectivity* of a graph G by $\hat{\kappa}_1(G) = \max\{\kappa_1(H) \mid H \subset G\}$

We now show for which integers k, a graph G contains a major k-edge-connected subgraph.

Theorem 5. *A graph G contains a major k-edge-connected subgraph if and only if $0 \le k \le \hat{\kappa}_1(G)$.*

Proof. Let H be an induced subgraph of G having edge-connectivity $\hat{\kappa}_1(G) = m$. By Lemma 3, H contains induced subgraphs $G_0, G_1, \ldots, G_m$ such that $\kappa_1(G_k) = k$ for $0 \le k \le m$. By Theorem 4, G_k contains a major k-edge-connected subgraph for every k $(0 \le k \le m)$, implying that for every k $(0 \le k \le \hat{\kappa}_1(G))$, the graph G contains an induced subgraph having edge-connectivity k and which contains no subgraph with edge-connectivity exceeding k $(0 \le k \le \hat{\kappa}_1(G))$. Consequently, G contains a major k-edge-connected subgraph for every k satisfying $0 \le k \le \hat{\kappa}_1(G)$.

Clearly, since G contains no (induced) subgraphs whose edge-connectivity exceeds $\hat{\kappa}_1(G)$, G does not contain major k-edge-connected subgraphs for $k > \hat{\kappa}_1(G)$.

The following result may be regarded as the edge analogue to Theorem 1.

Theorem 6. *For every pair, n, k of positive integers, there is a graph G with $\kappa_1(G) = n$ and $K_1(G) = \{k\}$.*

We omit the proof of this theorem, since the graphs described in the proof of Theorem 1 have the desired properties.

Theorem 2 shows that every set of positive integers can be realized as the major connectivity set of some graph. Similarly the next result shows that every set of positive integers can also be realized as a major edge-connectivity set of some graph. This proof too is omitted since the graphs described in Theorem 2 have the desired properties.

THEOREM 7. *Let $S = \{n_1, n_2, \ldots, n_k\}$ be a set of positive integers with $n_i < n_{i+1}$ $(1 \le i \le k-1)$. If $m \ge n_k + 1$ is a positive integer, then there is a graph G with $\kappa_1(G) = n$ and $K_1(G) = \{k\}$.*

The next result parallels Theorem 4.

THEOREM 8. *Let G be a graph of order p, every major n-edge-connected subgraph of which has order $m \ge 2$, where $n \in K_1(G)$. Then*

$$p < \frac{m^{m+1} - m}{m - 1}.$$

The proof of this result is very similar to the proof of Theorem 4. We need only observe, by Theorem 4, that if G is a graph satisfying the hypothesis of the theorem, then every (induced) subgraph of G has edge-connectivity at most $m-1$, implying that every (induced) subgraph of G has connectivity at most $m-1$.

Our next result, which we state without proof, shows that there is no relationship, in general, between an element of the major connectivity set of a graph and an element of its major edge-connectivity set.

THEOREM 9. *For every pair m, n of positive integers, there exists a graph G with $K(G) = \{m\}$ and $K_1(G) = \{n\}$.*

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