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DERIVATIVE-FREE CHARACTERIZATIONS OF COMPACT GENERALIZED COMPOSITION OPERATORS BETWEEN ZYGMUND TYPE SPACES

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Abstract

We give derivative-free characterizations for bounded and compact generalized composition operators between (little) Zygmund type spaces. To obtain these results, we extend Pavlović's corresponding result for bounded composition operators between analytic Lipschitz spaces.

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1. Introduction and main results

Let $H(D)$ be the space of all analytic functions on the open unit disc D. Then, for $0 < \alpha < \infty$, we denote by $\mathcal{B}^\alpha$ the Bloch type space of all functions $f \in H(D)$ satisfying

$$\sup_{z \in D} (1 - |z|)^\alpha |f'(z)| < \infty.$$

The space $\mathcal{B}^\alpha$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in D} (1 - |z|)^\alpha |f'(z)|.$$

The little Bloch type space, denoted by $\mathcal{B}_0^\alpha$, consists of those functions $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|z| \to 1} (1 - |z|)^\alpha |f'(z)| = 0.$$

For each $0 < \alpha < 1$ the space $\mathcal{B}^{1-\alpha}$ can be identified with the analytic Lipschitz space $H\Lambda_\alpha := H(D) \cap \Lambda_\alpha(D)$, where $\Lambda_\alpha(D)$ is the Lipschitz space of order α of all functions $f \in C(D)$ with

$$p_\alpha(f) = \sup_{\substack{z,w\in D\\ z\neq w}} \frac{|f(z)-f(w)|}{|z-w|^\alpha} < \infty.$$

The space $\Lambda_\alpha(D)$, $0<\alpha\le 1$, becomes a Banach space when equipped with the norm

$$\|f\|_{\Lambda_\alpha} = \|f\|_D + p_\alpha(f),$$

and $H\Lambda_\alpha \subset A(D) := H(D)\cap C(\overline{D})$ is a closed subspace of $\Lambda_\alpha(D)$. For $0<\alpha<1$ the little Lipschitz space of order α, denoted by $\Lambda_\alpha^0(D)$, is the space of those functions $f\in\Lambda_\alpha(D)$ such that

$$\lim_{\substack{|z-w|\to 0\\ z,w\in D}} \frac{|f(z)-f(w)|}{|z-w|^\alpha} = 0. \tag{1.1}$$

The space $\Lambda_\alpha^0(D)$ is a closed subspace of $\Lambda_\alpha(D)$ and also $H\Lambda_\alpha^0 := H(D)\cap\Lambda_\alpha^0(D)$ is a closed subspace of $H\Lambda_\alpha$.

Let $\psi\in H(D)$ and φ be an analytic self-map of D. The weighted composition operator ψC_φ is the operator given by $(\psi C_\varphi f)(z) = \psi(z) f(\varphi(z))$. In the special case of $\psi = 1$ we get the composition operator $(C_\varphi f)(z) = f(\varphi(z))$. Boundedness and compactness of composition operators on Bloch spaces were first studied by Roan [15] and later by Madigan [8] and Madigan and Matheson [9]. Moreover, Ohno *et al.* studied weighted composition operators between Bloch type spaces in [12].

Using a result of Madigan [8], Pavlović gave derivative-free characterizations of bounded composition operators between analytic Lipschitz spaces $H\Lambda_\alpha$ in [14]. His proof is also based on a consequence of the Schwarz lemma which he had used earlier in [13] to give a simple proof of Dyakonov's characterization of analytic weighted Lipschitz functions in terms of their moduli [5]. Indeed, he proved that for each $f\in H(D)$ and each $z\in D$, there exists $w_z\in D$ such that $|w_z - z|\le \frac{1}{2}(1-|z|) < 1-|z|$ and

$$|f'(z)| \le 4\frac{|f(w_z)|-|f(z)|}{1-|z|}. \tag{1.2}$$

In this paper, by using the above-mentioned consequence of the Schwarz lemma, we give derivative-free characterizations of boundedness and compactness for the recently introduced concept of generalized composition operators on (little) Zygmund type spaces.

For $g\in H(D)$ and φ an analytic self-map of D, Li and Stević in [6] introduced the generalized composition operator C_φ^g given by

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi)) g(\xi)\, d\xi.$$

If $g=\varphi'$ then $C_\varphi^{\varphi'}$ is the composition operator C_φ up to a constant. Li and Stević investigated boundedness and compactness of generalized composition operators

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between (little) Bloch type spaces and (little) Zygmund spaces, where the Zygmund space $\mathcal{Z}$ is the class of all functions $f \in H(D) \cap C(\overline{D})$ with

$$\sup_{\substack{e^{i\theta}\in\partial D\\ h>0}} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$

Boundedness of composition operators on $\mathcal{Z}$ was first studied by Choe *et al.* in [3].

By [4, Theorem 5.3], an analytic function f belongs to $\mathcal{Z}$ if and only if $f' \in \mathcal{B}^1$, or equivalently $\sup_{z\in D}(1-|z|)|f''(z)| < \infty$. For $0 < \alpha < \infty$ we denote by $\mathcal{Z}^\alpha$ the *Zygmund type space* of those functions $f \in H(D)$ satisfying

$$\sup_{z\in D}(1-|z|)^\alpha |f''(z)| < \infty.$$

The space $\mathcal{Z}^\alpha$ is a Banach space with the norm

$$\|f\|_{\mathcal{Z}^\alpha} = |f(0)| + |f'(0)| + \sup_{z\in D}(1-|z|)^\alpha |f''(z)|.$$

The *little Zygmund type space*, denoted by $\mathcal{Z}_0^\alpha$, is the closed subspace of $\mathcal{Z}^\alpha$ consisting of those functions $f \in \mathcal{Z}^\alpha$ with

$$\lim_{|z|\to 1}(1-|z|)^\alpha |f''(z)| = 0.$$

Our main results are the following derivative-free characterizations of boundedness and compactness of C_φ^g between (little) Zygmund type spaces. We will prove these theorems after first extending Pavlović's results to compact weighted composition operators between (little) analytic Lipschitz spaces. Indeed, we reduce the study of generalized composition operators on Zygmund type spaces to weighted composition operators on Bloch type spaces. This also leads to very simple proofs of Theorems 5 and 6 in [6].

THEOREM 1.1. *Let $0 < \alpha, \beta < 1$ and $g, \varphi \in H(D)$ where φ is a self-map of D.*

(i) *$C_\varphi^g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is bounded if and only if $g \in \mathcal{B}^\beta$ and $|g|(1-|\varphi|)^{1-\alpha} \in \Lambda_{1-\beta}(D)$.*

(ii) *If $g, \varphi \in \mathcal{B}_0^\beta$, then $C_\varphi^g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is compact if and only if $|g|(1-|\varphi|)^{1-\alpha} \in \Lambda_{1-\beta}^0(D)$.*

THEOREM 1.2. *Let $0 < \alpha, \beta < 1$ and $g, \varphi \in H(D)$ where φ is a self-map of D. Then $C_\varphi^g : \mathcal{Z}_0^\alpha \to \mathcal{Z}_0^\beta$ is bounded if and only if the following hold:*

(1) *$g \in \mathcal{B}_0^\beta$;*

(2) *$|g|(1-|\varphi|)^{1-\alpha} \in \Lambda_{1-\beta}(D)$;*

(3) *$g\varphi \in \mathcal{B}_0^\beta$.*

THEOREM 1.3. *Let $0 < \alpha, \beta < 1$ and $g, \varphi \in H(D)$ where φ is a self-map of D. Then the following statements are equivalent:*

(i) $C_\varphi^g : \mathcal{Z}_0^\alpha \to \mathcal{Z}_0^\beta$ *is compact;*

(ii) $\lim_{|z|\to 1} |g(z)|((1-|z|)^\beta/(1-|\varphi(z)|)^\alpha)|\varphi'(z)| = 0$ *and* $g \in \mathcal{B}_0^\beta$;

(iii) $|g|(1-|\varphi|)^{1-\alpha} \in \Lambda_{1-\beta}^0(D)$ *and* $g \in \mathcal{B}_0^\beta$.

2. Derivative-free characterizations

By the definition of little Lipschitz spaces (1.1), one can see that $H\Lambda_\alpha^0$ is the closure of polynomials in D and therefore by [17, Theorem 7.10] we have the following lemma.

LEMMA 2.1. *Let $f \in H(D)$ and $0 < \alpha < 1$. Then $f \in H\Lambda_\alpha^0$ if and only if*

$$\lim_{|z|\to 1} (1-|z|)^{1-\alpha}|f'(z)| = 0.$$

The next lemma extends this result to C^1-functions.

LEMMA 2.2. *Let u be a real valued C^1-function on D and $0 < \alpha < 1$. If*

$$\lim_{|z|\to 1} (1-|z|)^{1-\alpha}|\nabla u(z)| = 0,$$

then $u \in \Lambda_\alpha^0(D)$.

PROOF. Let $\varepsilon > 0$ and choose $\delta_1 > 0$ such that

$$|\nabla u(z)| < \frac{\varepsilon}{(1-|z|)^{1-\alpha}}, \tag{2.1}$$

for each $z \in D$ with $1-\delta_1 \le |z| < 1$. Since u is a C^1-function, by using Lagrange's theorem one can see that $u \in \Lambda_\alpha^0((1-\frac{1}{3}\delta_1)D)$ and therefore there exists $\delta_2 > 0$ such that

$$\frac{|u(z)-u(w)|}{|z-w|^\alpha} < \varepsilon, \tag{2.2}$$

for every $z, w \in (1-\frac{1}{3}\delta_1)D$ with $0 < |z-w| < \delta_2$. Choose $0 < \delta_3 < 1$ so that for each $z, w \in D\backslash(1-\frac{2}{3}\delta_1)D$ with $|z-w| < \delta_3$ the line segment between z and w lies in $D\backslash(1-\delta_1)D$. Let $\delta = \min\{\frac{1}{3}\delta_1, \delta_2, \delta_3\}$ and $z, w \in D$ with $|z-w| < \delta$. If z or w belongs to $(1-\frac{2}{3}\delta_1)D$ then $z, w \in (1-\frac{1}{3}\delta_1)D$ and $|u(z)-u(w)|/|z-w|^\alpha < \varepsilon$ by (2.2). If $z, w \in D\backslash(1-\frac{2}{3}\delta_1)D$ then by using (2.1) and the same lines of argument as in [16, Lemma 6.4.8] one can find a constant C, independent of z and w, such that $|u(z)-u(w)| < C\varepsilon|z-w|^\alpha$. This completes the proof. □

It is worth mentioning that by using (1.2) and Lemma 2.1 we have the following corollary, extending the result of [14, Theorem B] for little analytic Lipschitz spaces (see also [5, 13]).

COROLLARY 2.3. *Let $u \in H(D)$ and $0 < \alpha < 1$. Then $u \in \Lambda_\alpha^0(D)$ if and only if $|u| \in \Lambda_\alpha^0(D)$.*

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PROOF. Clearly if $u \in \Lambda_\alpha^0(D)$, then $|u| \in \Lambda_\alpha^0(D)$. Now let $|u| \in \Lambda_\alpha^0(D)$. By (1.2) for each $z \in D$ there exists $w_z \in D$ such that $|w_z - z| < 1 - |z|$ and

$$|u'(z)| \leq 4 \frac{|u(w_z)| - |u(z)|}{1 - |z|}.$$

Therefore,

$$\frac{1}{4}|u'(z)|(1 - |z|)^{1-\alpha} \leq \frac{|u(w_z)| - |u(z)|}{(1 - |z|)^\alpha} \leq \frac{||u(w_z)| - |u(z)||}{|w_z - z|^\alpha},$$

which implies that $\lim_{|z|\to 1} |u'(z)|(1 - |z|)^{1-\alpha} = 0$, since $|w_z - z| < 1 - |z|$ and

$$\frac{||u(\zeta)| - |u(\eta)||}{|\zeta - \eta|^\alpha} \to 0 \quad \text{as } |\zeta - \eta| \to 0.$$

Therefore $u \in \Lambda_\alpha^0(D)$ by Lemma 2.1. □

Let $\gamma > 0$ and consider the following linear maps between Zygmund type spaces and Bloch type spaces of order γ:

$$S : \mathcal{Z}^\gamma \to \mathcal{B}^\gamma, \quad Sh = h'; \tag{2.3}$$

$$T : \mathcal{B}^\gamma \to \mathcal{Z}^\gamma, \quad (Th)(z) = \int_0^z h(\xi)\, d\xi. \tag{2.4}$$

The operators S and T are bounded, indeed $\|Sh\|_{\mathcal{B}^\gamma} \leq \|h\|_{\mathcal{Z}^\gamma}$ and $\|Th\|_{\mathcal{Z}^\gamma} = \|h\|_{\mathcal{B}^\gamma}$.

Let φ be an analytic self-map of D and $g \in H(D)$. For the positive numbers α, β consider the generalized composition operator $C_\varphi^g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ and the following diagram:

$$\begin{array}{ccc} \mathcal{B}^\alpha & \xrightarrow{gC_\varphi} & \mathcal{B}^\beta \\ {\scriptstyle S}\uparrow & & \downarrow{\scriptstyle T} \\ \mathcal{Z}^\alpha & \xrightarrow{C_\varphi^g} & \mathcal{Z}^\beta \\ {\scriptstyle T}\uparrow & & \downarrow{\scriptstyle S} \\ \mathcal{B}^\alpha & \xrightarrow{gC_\varphi} & \mathcal{B}^\beta \end{array} \tag{2.5}$$

Applying (2.3) and (2.4) gives

$$T \circ gC_\varphi \circ S = C_\varphi^g,$$
$$S \circ C_\varphi^g \circ T = gC_\varphi,$$

which yield the following result.

PROPOSITION 2.4. *Let $\alpha, \beta > 0$ and $g, \varphi \in H(D)$ where φ is a self-map of D. Then $C_\varphi^g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is bounded (compact) if and only if $gC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded (compact).*

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Note that [6, Theorems 5 and 6] follow from Proposition 2.4 in the special case of $\alpha = \beta = 1$ along with [12, Theorems 2.1(ii) and 3.1(ii)].

Since $S(\mathcal{Z}_0^\gamma) \subseteq \mathcal{B}_0^\gamma$ and $T(\mathcal{B}_0^\gamma) \subseteq \mathcal{Z}_0^\gamma$, one can also consider the commutativity of diagram (2.5) for little Zygmund type spaces and little Bloch type spaces. Hence, we get the following proposition.

PROPOSITION 2.5. *Let $\alpha, \beta > 0$ and $g, \varphi \in H(D)$ where φ is a self-map of D. Then $C_\varphi^g : \mathcal{Z}_0^\alpha \to \mathcal{Z}_0^\beta$ is bounded (compact) if and only if $gC_\varphi : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$ is bounded (compact).*

Therefore in order to prove Theorems 1.1, 1.2, and 1.3 we need to solve the derivative-free problem for weighted composition operators between (little) Bloch type spaces or (little) analytic Lipschitz spaces.

Pavlović used [8, Theorem 4] to give a derivative-free characterization for the boundedness of a composition operator $C_\varphi : H\Lambda_\alpha \to H\Lambda_\beta$ in [14, Theorem A]. By the same method of proof as in [14, Theorem A] and using [12, Theorem 2.1(i)] one also has the following version of [14, Theorem A] for the weighted composition operators.

THEOREM 2.6. *Let $0 < \alpha, \beta \leq 1$ and $\psi, \varphi \in H(D)$ where φ is a self-map of D. Then $\psi C_\varphi : H\Lambda_\alpha \to H\Lambda_\beta$ is bounded if and only if the following hold:*

(1) $\psi \in H\Lambda_\beta$;
(2) $|\psi|(1 - |\varphi|)^\alpha \in \Lambda_\beta(D)$.

In the next theorem we obtain a result corresponding to Theorem 2.6 for little analytic Lipschitz spaces.

THEOREM 2.7. *Let $0 < \alpha, \beta < 1$ and $\psi, \varphi \in H(D)$ where φ is a self-map of D. Then $\psi C_\varphi : H\Lambda_\alpha^0 \to H\Lambda_\beta^0$ is bounded if and only if the following hold:*

(1) $\psi \in H\Lambda_\beta^0$;
(2) $|\psi|(1 - |\varphi|)^\alpha \in \Lambda_\beta(D)$;
(3) $\psi\varphi \in H\Lambda_\beta^0$.

PROOF. By [12, Theorem 4.1], $\psi C_\varphi : H\Lambda_\alpha^0 \to H\Lambda_\beta^0$ is bounded if and only if the following hold:

(1)* $\psi \in H\Lambda_\beta^0$;
(2)* $\psi C_\varphi : H\Lambda_\alpha \to H\Lambda_\beta$ is bounded;
(3)* $\lim_{|z|\to 1}(1 - |z|)^{1-\beta}|\psi(z)||\varphi'(z)| = 0$.

Let ψC_φ be bounded; then by Theorem 2.6 it is enough to show that (3) holds. Clearly,

$$|(\psi\varphi)'(z)|(1 - |z|)^{1-\beta} \leq |\psi'(z)||\varphi(z)|(1 - |z|)^{1-\beta} + |\psi(z)||\varphi'(z)|(1 - |z|)^{1-\beta}. \tag{2.6}$$

Since φ is bounded and $\psi \in H\Lambda_\beta^0$, $\lim_{|z|\to 1} |\psi'(z)||\varphi(z)|(1 - |z|)^{1-\beta} = 0$ by Lemma 2.1. Also $\lim_{|z|\to 1} |\psi(z)||\varphi'(z)|(1 - |z|)^{1-\beta} = 0$ by (3)*. Therefore (2.6) implies that $|(\psi\varphi)'(z)|(1-|z|)^{1-\beta} \to 0$ as $|z| \to 1$, and hence $\psi\varphi \in H\Lambda_\beta^0$ by Lemma 2.1.

Conversely, let (1), (2) and (3) hold. To prove that ψC_φ is bounded, we only need to show that (3)* holds. Now,

$$\begin{aligned} |(1-|z|)^{1-\beta}\psi(z)\varphi'(z)| &\leq |(1-|z|)^{1-\beta}\psi(z)\varphi'(z) + (1-|z|)^{1-\beta}\psi'(z)\varphi(z)| \\ &\quad + |(1-|z|)^{1-\beta}\psi'(z)\varphi(z)| \\ &= (1-|z|)^{1-\beta}|(\psi\varphi)'(z)| + (1-|z|)^{1-\beta}|\psi'(z)||\varphi(z)|. \end{aligned} \tag{2.7}$$

Using the boundedness of φ and (1), we have $\lim_{|z|\to 1}(1-|z|)^{1-\beta}|\psi'(z)||\varphi(z)| = 0$ by Lemma 2.1. Also (3) implies that $\lim_{|z|\to 1}(1-|z|)^{1-\beta}|(\psi\varphi)'(z)| = 0$. Therefore (2.7) implies (3)*, which completes the proof. □

Now Theorem 1.2 is an immediate consequence of Proposition 2.5 and Theorem 2.7. Also Theorem 1.3 follows from Proposition 2.5 and the next result.

THEOREM 2.8. *Let* $0 < \alpha, \beta < 1$ *and* $\psi, \varphi \in H(D)$ *where* φ *is a self-map of* D. *Then the following statements are equivalent:*

(i) $\psi C_\varphi : H\Lambda_\alpha^0 \to H\Lambda_\beta^0$ *is compact;*

(ii) $\lim_{|z|\to 1} |\psi(z)|((1-|z|)^{1-\beta}/(1-|\varphi(z)|)^{1-\alpha})|\varphi'(z)| = 0$ *and* $\psi \in H\Lambda_\beta^0$*;*

(iii) $|\psi|(1-|\varphi|)^\alpha \in \Lambda_\beta^0(D)$ *and* $\psi \in H\Lambda_\beta^0$.

PROOF. (i) ⇔ (ii). This holds by [12, Theorem 5.1], where little Bloch type spaces are considered instead of little analytic Lipschitz spaces.

(iii) ⇒ (ii). By (1.2), for each $z \in D$ there exists $w_z \in D$ such that $|w_z - z| < 1 - |z|$ and

$$|\varphi'(z)| \leq 4\frac{|\varphi(w_z)| - |\varphi(z)|}{1-|z|}. \tag{2.8}$$

On the other hand, using (2.8) and the inequality $\alpha x^{\alpha-1}(x-y) \leq x^\alpha - y^\alpha$ for $0 < y < x$,

$$\begin{aligned} \frac{\alpha}{4}|\varphi'(z)|\frac{(1-|z|)^{1-\beta}}{(1-|\varphi(z)|)^{1-\alpha}} &\leq \frac{1}{4}|\varphi'(z)|(1-|z|)\frac{\alpha(1-|\varphi(z)|)^{\alpha-1}}{|w_z-z|^\beta} \\ &\leq \frac{\alpha(1-|\varphi(z)|)^{\alpha-1}(|\varphi(w_z)|-|\varphi(z)|)}{|w_z-z|^\beta} \\ &\leq \frac{|(1-|\varphi(z)|)^\alpha - (1-|\varphi(w_z)|)^\alpha|}{|w_z-z|^\beta}. \end{aligned} \tag{2.9}$$

Let $h(z) := |\psi(z)|(1-|\varphi(z)|)^\alpha$; then (2.9) gives

$$\begin{aligned} &\frac{\alpha}{4}|\psi(z)|\frac{(1-|z|)^{1-\beta}}{(1-|\varphi(z)|)^{1-\alpha}}|\varphi'(z)| \\ &\quad \leq |\psi(z)|\frac{|(1-|\varphi(z)|)^\alpha - (1-|\varphi(w_z)|)^\alpha|}{|w_z-z|^\beta} \end{aligned}$$

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$$\begin{aligned}&\le \left|\frac{|\psi(z)|\{(1-|\varphi(z)|)^{\alpha}-(1-|\varphi(w_z)|)^{\alpha}\}-(1-|\varphi(w_z)|)^{\alpha}(|\psi(w_z)|-|\psi(z)|)}{|w_z-z|^{\beta}}\right|\\&\quad+(1-|\varphi(w_z)|)^{\alpha}\frac{||\psi(w_z)|-|\psi(z)||}{|w_z-z|^{\beta}}\\&\le \frac{|h(z)-h(w_z)|}{|z-w_z|^{\beta}}+(1-|\varphi(w_z)|)^{\alpha}\frac{|\psi(w_z)-\psi(z)|}{|w_z-z|^{\beta}}.\end{aligned} \tag{2.10}$$

Since $h,\ \psi\in\Lambda^0_\beta(D)$ and $|w_z-z|<1-|z|$, then

$$\frac{|h(z)-h(w_z)|}{|z-w_z|^{\beta}}\to 0$$

and

$$\frac{|\psi(w_z)-\psi(z)|}{|w_z-z|^{\beta}}\to 0\quad\text{as } |z|\to 1.$$

Therefore (2.11) implies that

$$|\psi(z)|\frac{(1-|z|)^{1-\beta}}{(1-|\varphi(z)|)^{1-\alpha}}|\varphi'(z)|\to 0\quad\text{as } |z|\to 1.$$

(ii) $\Rightarrow$ (iii). First note that since ψ and φ are analytic, $|\nabla|\psi||=|\psi'|$ and $|\nabla(1-|\varphi|)^{\alpha}|=\alpha|\varphi'|(1-|\varphi|)^{\alpha-1}$. Therefore with the previous notation for $h(z)$ we get

$$\begin{aligned}&|\nabla h(z)|(1-|z|)^{1-\beta}\\&\quad=\big((1-|\varphi(z)|)^{\alpha}|\nabla|\psi(z)||+|\psi(z)||\nabla(1-|\varphi(z)|)^{\alpha}|\big)\,(1-|z|)^{1-\beta}\\&\quad=(1-|\varphi(z)|)^{\alpha}|\psi'(z)|(1-|z|)^{1-\beta}\\&\qquad+\alpha|\psi(z)|\frac{(1-|z|)^{1-\beta}}{(1-|\varphi(z)|)^{1-\alpha}}|\varphi'(z)|.\end{aligned} \tag{2.11}$$

Since $\psi\in H\Lambda^0_\beta$, by Lemma 2.1 we have $\lim_{|z|\to 1}|\psi'(z)|(1-|z|)^{1-\beta}=0$. Thus, the hypothesis of part (ii), the boundedness of φ, and (2.11) imply that

$$\lim_{|z|\to 1}|\nabla(|\psi(z)|(1-|\varphi(z)|)^{\alpha})|(1-|z|)^{1-\beta}=0,$$

and thus by Lemma 2.2, $|\psi|(1-|\varphi|)^{\alpha}\in\Lambda^0_\beta(D)$. □

Let $0<\alpha\le 1$, $0<\beta<1$ and $\varphi: D\to D$ belong to $H\Lambda^0_\beta$. Then by a similar argument as in [10, Proposition 2.2], one can prove that

$$\lim_{|z|\to 1}|\psi(z)|\frac{(1-|z|)^{1-\beta}}{(1-|\varphi(z)|)^{1-\alpha}}|\varphi'(z)|=0$$

if and only if

$$\lim_{|\varphi(z)|\to 1}|\psi(z)|\frac{(1-|z|)^{1-\beta}}{(1-|\varphi(z)|)^{1-\alpha}}|\varphi'(z)|=0,$$

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where ψ is an arbitrary bounded map on D. Using this fact, along with [12, Theorem 3.1], leads to the following result for analytic Lipschitz spaces and therefore, by Proposition 2.4, for Zygmund type spaces.

COROLLARY 2.9. *Let* $0 < \alpha \leq 1$, $0 < \beta < 1$ *and* $\psi, \varphi \in H\Lambda^0_\beta$ *where* φ *is a self-map of* D. *Then* $\psi C_\varphi : H\Lambda_\alpha \to H\Lambda_\beta$ *is compact if and only if* $|\psi|(1-|\varphi|)^\alpha \in \Lambda^0_\beta(D)$.

Note that if $g = \varphi'$, then

$$(C_\varphi^{\varphi'} f)(z) = (C_\varphi f)(z) - (C_\varphi f)(0).$$

Therefore, C_φ is bounded (compact) if and only if $C_\varphi^{\varphi'}$ is bounded (compact). So one can apply Theorems 1.1, 1.2, and 1.3 to obtain necessary and sufficient conditions for boundedness and compactness of composition operators on Zygmund type spaces. In particular, we have the following derivative-free conditions for boundedness and compactness of composition operators on little Zygmund type spaces.

COROLLARY 2.10. *Let* $0 < \alpha, \beta < 1$ *and* φ *be an analytic self-map of* D. *Then:*

(i) $C_\varphi : \mathcal{Z}_0^\alpha \to \mathcal{Z}_0^\beta$ *is bounded if and only if* $\varphi \in \mathcal{Z}_0^\beta \cap \mathcal{B}_0^{\beta/2}$ *and* $(1-|\varphi|)^{1-\alpha/2} \in \Lambda_{1-\beta/2}(D)$;

(ii) $C_\varphi : \mathcal{Z}_0^\alpha \to \mathcal{Z}_0^\beta$ *is compact if and only if* $\varphi \in \mathcal{Z}_0^\beta$ *and* $(1-|\varphi|)^{1-\alpha/2} \in \Lambda^0_{1-\beta/2}(D)$.

PROOF. To prove (ii), it is enough to use the equivalence of (i) and (ii) in Theorem 1.3 for $g = \varphi'$ and then the equivalence of (ii) and (iii) for $g = 1$. The proof of part (i) follows from a similar argument using Theorem 1.2, the equivalence of (B) and (C) in [14, Theorem A], Theorem 2.6, and

$$(1-|z|)^\beta(\varphi'\varphi)'(z) = (1-|z|)^\beta \varphi''(z)\varphi(z) + ((1-|z|)^{\beta/2}\varphi'(z))^2. \qquad \square$$

As a final remark, let us for $\alpha > 0$ consider the standard weighted Banach spaces of analytic functions defined as follows:

$$H_\alpha^\infty = \Big\{ f \in H(D) : \|f\|_{H_\alpha^\infty} = \sup_{z \in D}(1-|z|)^\alpha |f(z)| < \infty \Big\},$$
$$H_\alpha^0 = \Big\{ f \in H_\alpha^\infty : \lim_{|z|\to 1}(1-|z|)^\alpha |f(z)| = 0 \Big\}.$$

For more details about spaces of this type we refer to [1, 2, 7, 11] and the references therein. Let $\gamma > 0$ and consider the bounded operators

$$S : \mathcal{B}^\gamma \to H_\gamma^\infty, \quad Sh = h';$$
$$T : H_\gamma^\infty \to \mathcal{B}^\gamma, \quad (Th)(z) = \int_0^z h(\xi)\, d\xi.$$

Then by the same arguments as in Propositions 2.4 and 2.5, one can reduce the study of generalized composition operators $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ and $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$ to weighted composition operators $gC_\varphi : H_\alpha^\infty \to H_\beta^\infty$ and $gC_\varphi : H_\alpha^0 \to H_\beta^0$. Indeed, the following diagram commutes:

$$\begin{array}{ccc} H_\alpha^\infty & \xrightarrow{gC_\varphi} & H_\beta^\infty \\ \uparrow S & & \downarrow T \\ \mathcal{B}^\alpha & \xrightarrow{C_\varphi^g} & \mathcal{B}^\beta \\ \uparrow T & & \downarrow S \\ H_\alpha^\infty & \xrightarrow{gC_\varphi} & H_\beta^\infty \end{array}$$

Therefore, the following proposition holds.

PROPOSITION 2.11. *Let $\alpha, \beta > 0$ and $g, \varphi \in H(D)$ where φ is a self-map of D. Then:*

(i) *$C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded (compact) if and only if $gC_\varphi : H_\alpha^\infty \to H_\beta^\infty$ is bounded (compact);*

(ii) *$C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$ is bounded (compact) if and only if $gC_\varphi : H_\alpha^0 \to H_\beta^0$ is bounded (compact).*

As a result of Proposition 2.11, one can easily obtain several of the results in [6, Section 5] by applying [11, Theorems 2.1 and 2.2].

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