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NON-EXISTENCE CRITERIA FOR SMALL CONFIGURATIONS

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1. Introduction and statement of results. For graph theoretic terms, see Tutte [**1**]. A rank 2 tactical configuration of girth $2g$ and order (s, t) may be regarded as a $(1 + s, 1 + t)$-regular bipartite graph of girth $2g$. We assume $s \leqq t$. Using a technique of Friedman [**2**] we show (i) and (ii).

(i) If $g = 2k + 1$, then T can have $W = 1 + s + st + \ldots + s^k t^k$ vertices of degree $1 + t$ only if both of the following are integers:

$$B(s, t, k) = \frac{s^{k+1}t^k(1 + t)W}{4k + 2}$$

$$C(s, t, k) = \frac{s^{k+1}t^k(1 + t)(st - 2s + 1)W}{4k + 4}.$$

(ii) If $g = 2k + 2$, then T can have $W + s^{k+1}t^k$ vertices of degree $1 + t$ only if both of the following are integers:

$$B^*(s, t, k) = \frac{s^{k+1}t^{k+1}(1 + s)(1 + t)(1 + st + s^2t^2 + \ldots + s^k t^k)}{4k + 4}$$

$$C^*(s, t, k) = \frac{B^*(s, t, k)(s - 1)(t - 1)(2k + 2)}{2k + 3}.$$

2. Proof of the Theorems. In order to discuss T, we need some labels. Let the vertex set of T be $X + Y$, where the elements of X are $(1 + s)$-valent and called "red", and those of Y are "white" and have degree $(1 + t)$. Select an edge $e = (x, y)$ with x in X and y in Y and hold it fixed. There are s members of Y different from y and adjoining x; call these $W(1)$. Similarly, $R(1)$ is the set of all red vertices different from x and adjoining y. $W(2)$ is the set of all white vertices different from y and adjoining members of $R(1)$, and $R(2)$ is the set of all red vertices different from x and adjoining members of $W(1)$. Letting $R(0) = \{x\}$ and $W(0) = \{y\}$, and $T(i) = \bigcup_{j=0}^{i} (R(j) \cup W(j))$, then $T(i)$ is a set with easily determined numbers of red and white vertices for all $i = 0, 1, 2, \ldots, g - 1$, by having $R(i)$ (and $W(i)$, respectively) be the set of all red (white) vertices adjacent to members of $W(i - 1)$ $(R(i - 1))$ but not contained in $T(i - 1)$. The tree containing $T(g - 1)$ as vertices and all edges of T between classes with different numerical labels and also the edge

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e, is called the skeleton of T with respect to e, or since all skeleta of T are isomorphic, the skeleton of T, skT. We shall be considering those configurations T for which Y is a subset of skT. Feit and Higman [**3**] have shown that if g is odd, then either $s < t$ or else $g = 3$ and T is a projective plane of order $s = t$. They also show that if g is even, then X is a subset of skT and $g = 4, 6, 8$ or 12.

Case A. $g = 2k + 1$. The cardinality of $Y = W = 1 + s + \ldots + s^k t^k$ and that of $X = R = 1 + t + \ldots + s^k t^k + r$, where $(1 + s)r = s^k t^k (t - s)$, whence $(-t)^k (t + 1) \equiv 0 \pmod{1 + s}$. Denote the r unlabelled vertices by R^*. We note that each member of $W(2k)$ can be adjacent to at most s members of $R(2k)$, since otherwise at least two of these would be at a distance of at most $2k - 1$ from the same member of $W(1)$, thus exhibiting a polygon of girth at most $4k$, hence $W(2k)$ has at least $s^k t^k (t - s)$ edges to R^*. Since R^* has exactly $(1 + s)r = s^k t^k (t - s)$ edges from it, all of them must go to $W(2k)$, thus each member of $W(2k)$ adjoins exactly s members of $R(2k)$. We now enumerate the $(4k + 2)$-gons through e. Any such polygon must contain a member of $R(2k)$ and a member of $W(2k)$, since skT is a tree, and in fact these must be adjacent, for the shortest paths in skT between $R(2k)$ and $W(2k)$ have length $4k + 1$. Since every edge from $R(2k)$ is either in skT or goes to $W(2k)$, each of the edges from $R(2k)$ to $W(2k)$ must determine a unique $(4k + 2)$-gon through e. There are $s^k t^k$ members of $R(2k)$, so e has $s^{k+1} t^k$ such polygons. There are $(1 + t)W$ edges in T, so altogether T has $B(s, t, k)$ polygons of length $4k + 2$. If $k = 1$, then $B(s, t, 1)$ is trivially an integer except when $t \equiv 1 \pmod 3$ and $s \equiv 2 \pmod 3$, but then s and t do not satisfy $t(t + 1) \equiv 0 \pmod{1 + s}$.

We further consider all $(4k + 4)$-gons through e. Any such polygon has three edges between $R(2k)$ and $W(2k)$, or else one edge between $R(2k)$ and $W(2k)$ and two edges from $W(2k)$ to a single vertex in R^*. For the first type of polygon, one may pick any pair of edges from the same vertex of $R(2k)$, and any new edge back to $R(2k)$ from one of these. Since all four vertices have distinct roles relative to e, there are $s(s - 1)^2$ such selections for each vertex of $R(2k)$, and $s^{k+1} t^k (s - 1)^2$ polygons of the first type. For the second type, one may again select any pair of edges from a vertex in R^* and any edge going to $R(2k)$ from one of these. All four roles are different, so there are $r(s + 1)s^2 = s^{k+2} t^k (t - s)$ polygons of the second type through e. Thus T has $C(s, t, k)$ polygons of girth $(4k + 4)$.

Case B. $g = 2k + 2$. We note that Y is a subset of skT if and only if X is a subset of skT, whence T must be a generalized polygon. The arguments producing $B^*(s, t, k)$ and $C^*(s, t, k)$ are the same sort of arguments as produced $B(s, t, k)$ and $C(s, t, k)$, except that there are no "extra" vertices to consider. $B^*(s, t, k)$ is trivially satisfied for $k = 1, 2, 5$, but $B^*(s, t, 3)$ is integral only for certain values of s and t. $C^*(s, t, 1)$ yields that all generalized quadrangles have $s \equiv t \pmod 5$ or else at least one of s and $t \equiv 0, 1, 4 \pmod 5$.

$C^*(s, t, 2)$ says that all generalized hexagons have $s \equiv t \pmod 7$ or at least one of s and $t \equiv 0, 1, 6 \pmod 7$. Both $C^*(s, t, 3)$ and $C^*(s, t, 5)$ are trivially satisfied.

References

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