ON THE DISTRIBUTION OF THE SURPLUS PRIOR AND AT RUIN

By

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Abstract

Consider a classical compound Poisson model. The safety loading can be positive, negative or zero. Explicit expressions for the distributions of the surplus prior and at ruin are given in terms of the ruin probability. Moreover, the asymptotic behaviour of these distributions as the initial capital tends to infinity are obtained. In particular, for positive safety loading the Cramér case, the case of subexponential distributions and some intermediate cases are discussed.

Keywords

Ruin, asymptotic distribution, change of measure, Laplace transform, subexponential distribution, Cramér condition, generalized Pareto distribution, maximum domain of attraction, Gumbel distribution

1. Introduction

We consider here the classical risk model

$$X_t = u + ct - \sum_{i=1}^{N_t} Y_i$$

where $u \geq 0$ is the initial capital, $c > 0$ is the premium rate, (N_t) is a Poisson process with rate λ and $(Y_i : i \in \mathbb{N})$ are iid positive random variables independent of (N_t). We denote the distribution function of Y by G, its moments by $\mu_n = E[Y^n]$, its moment generating function by $M_Y(r) = E[\exp\{rY\}]$. For simplicity we let $\mu = \mu_1$. Here all stochastic objects are supposed to be defined on a complete probability space $(\Omega, \mathcal{F}, P)$. By $(\mathcal{F}_t)$ we denote the smallest right-continuous filtration such that (X_t) is adapted.

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This model was introduced by Lundberg (1903) and extensively studied by Cramér (1930). It is therefore often called Cramér-Lundberg model. This model is a good approximation to reality in cases where the portfolio of single contracts considered is large. It has, however, to be regarded as a technical model. For instance, time has to be considered as operational time because the size of the portfolio will change with time. Moreover, in reality premiums may not be constant over time and dividends paid will depend on the surplus. But in order to take decisions, analysis of the classical model will be helpful.

Let $\tau = \inf\{t \geq 0 : X_t < 0\}$ denote the time of ruin. As usual $\inf \emptyset = \infty$. The ruin probability is $\psi(u) = P[\tau < \infty | X_0 = u]$. In this paper we are interested in the quantity

$$f(u; x, y) = P[\tau < \infty, -X_\tau > x, X_{\tau-} > y] \ ,$$

the probability that ruin occurs, the surplus prior to ruin is larger than y and the surplus at ruin is smaller than $-x$. In particular, this gives information on the claim causing ruin,

$$P[\tau < \infty, X_{\tau-} - X_\tau \leq z] = \int_0^z \int_0^{z-x} f(u; dx, dy).$$

The function $f(u; x, y)$ is interesting to study because one would like to get information on how ruin occurs. If the capital prior to ruin $X_{\tau-}$ is known one can be sure to be "safe" as long as the surplus is far from this capital. One is also interested in the capital at ruin. The ruin time τ is a technical term. The initial capital u is the money a company is willing to risk for a certain branch of insurance. If ruin occurs and $-X_\tau$ is not too large, the company will not become bankrupt. Indeed, the surplus from other branches will cover the losses.

The classical quantity studied in literature is $\psi(u) = f(u; 0; 0)$. Results on $\psi(u)$ can be found in the text books Gerber (1979), Grandell (1991) or Rolski et al. (1999). The function $f(u; x, 0)$ was introduced by Gerber et al. (1987). For $u = 0$ and positive safety loading $c > \lambda\mu$ the formula $cf(0; x, 0) = \lambda \int_x^\infty (1 - G(z))dz$ is well-known.

The functions $f(u; x, y)$ have been investigated by Dufresne and Gerber (1988) and Dickson (1992) in the case of positive safety loading and under the assumption that the claim size distribution is absolutely continuous. These results can also be found in Rolski et al. (1999). Dufresne and Gerber (1988) found the formula

$$f(u; x, y) = -\int_y^\infty \frac{1 - G(x + z)}{1 - G(z)} f(u; 0, dz)$$

based on the observation that given $X_{\tau-} = z$ a claim of at least size $x + z$ has to occur in order that $X_\tau < -x$. Dickson (1992) considered the function $f(u; 0, y)$. He solved the cases $u < y$ and $u \geq y$ separately. For $u < y$ he split the event $\{\tau < \infty, X_{\tau-} \le y\}$ into $\{\tau < \infty, \sup_{0 \le t \le \tau} X_t < y\}$ and $\{\tau < \infty, \sup_{0 \le t \le \tau} X_t \ge y, X_{\tau-} \le y\}$. For $u \ge y$ he observed that if ruin occurs and $X_{\tau-} \le y$ then the process (X_t) has to enter the set $[0, y)$ first. The probability that the latter happens is $\psi(u-y) - f(u-y; y, 0)$. These considerations led to $f(u; 0, y)$. Combining these with the results of Dufresne & Gerber (1988) the function $f(u; x, y)$ was found in the absolutely continuous case. Numerical procedures for the calculation of $f(u; x, y)$ are discussed in Dickson et al. (1995). Recently, Willmot & Lin (1998) obtained upper and lower bounds for $f(u; x, 0)$ and $f(u; 0, y)$. These inequalities can be applied in particular if the distribution function is NWUE or NBUE (see Willmot & Lin (1998) or Rolski et al. (1999) for a definition).

We will give here another proof of the results of Section 7 in Dickson (1992), not needing the absolute continuity of the claims sizes, and also investigate the cases of negative safety loading ($c < \lambda\mu$) and of no safety loading ($c = \lambda\mu$). It will be possible to find explicit formulae for $u = 0$. We will also investigate the behaviour of $f(u; x, y)$ for large initial capital u. In the case of positive safety loading we will discuss the three main classes of distribution functions.

The cases $c > \lambda\mu$ and $c \le \lambda\mu$ are quite different. Positive safety loading implies that $\psi(u) < 1$ for all $u \ge 0$ and $\psi(u) \to 0$ as $u \to \infty$. Thus also $f(u; x, y) \to 0$ as $u \to \infty$. In the case of non-positive safety loading one has $\psi(u) = 1$ for all $u \ge 0$. This will make our considerations more complicated. The case $c < \lambda\mu$ can be reduced to the case $c > \lambda\mu$ by a change of measure argument. For the behaviour of $f(u; x, y)$ as $u \to \infty$ we will find $f(u; x, y)$ converges to a non-trivial limit if $\mu < \infty$ for negative safety loading and if $\mu_2 < \infty$ in the case of no safety loading. This is in contrast to the case of positive safety loading, where basically a non-trivial limit of $f(u; x, y)$ as $u \to \infty$ is obtained if the distribution tail of the claim sizes decreases exponentially fast.

Throughout the paper we will assume that the ruin probability $\psi(u)$ is known. For a discussion of $\psi(u)$ see for instance Gerber (1979), Grandell (1991) or Rolski et al. (1999). For methods how to calculate $\psi(u)$ numerically see Panjer (1981), Dufresne & Gerber (1989) or Rolski et al. (1999).

2. An integro-differential equation for $f(u; x, y)$

In this section we first derive an integro-differential equation and an integral equation for $f(u; x, y)$, as well as the Laplace transform of f. The derivation of equations (1) and (2) below is similar to the approach in Grandell (1991), see also Rolski et al. (1999). The derivation of the Laplace transform (3) follows Feller (1971).

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Consider now the process (X_t) in the interval $[0, h \wedge T_1]$ where T_1 is the epoch of the first claim. Then, either there is a claim or there are no claims in $[0, h]$. If there is a claim then either the first claim leads to ruin or not. Because the process is Markov we get the following equation

$$
\begin{aligned}
f(u;x,y) = {} & e^{-\lambda h} f(u+ch;x,y) + \\
& + \int_0^h \Bigg(\int_0^{u+ct} f(u+ct-z;x,y)\, dG(z) + \\
& + \mathbb{1}_{u+ct>y}\, \bar{G}(u+ct+x) \Bigg) \lambda e^{-\lambda t}\, dt
\end{aligned}
$$

where $\bar{G}(z) = 1 - G(z)$ denotes the distribution tail of Y. Here, $u+ch$ is the capital at time h if $T_1 > h$, $u+ct-z$ is the capital at T_1 if $T_1 = t$ and $Y_1 = z$. If $Y_1 > u + ct$ then ruin occurs, i.e. $\tau = T_1$. Hence $X_{\tau-} = u + ct$ and $X_\tau = u + ct - Y_1$. The event of interest occurs therefore if $u + ct > y$ and $Y_1 > u + ct + x$. Letting $h \to 0$ yields that $f(u;x,y)$ is right-continuous in u. Rearranging the terms gives

$$
\begin{aligned}
0 = {} & c\frac{f(u+ch;x,y) - f(u;x,y)}{ch} - \frac{1-e^{-\lambda h}}{h} f(u+ch;x,y) + \\
& + \frac{1}{h}\int_0^h \Bigg(\int_0^{u+ct} f(u+ct-z;x,y)\, dG(z) + \mathbb{1}_{u+ct>y}\, \bar{G}(u+ct+x) \Bigg) \lambda e^{-\lambda t} dt.
\end{aligned}
$$

Thus $f(u;x,y)$ is differentiable with respect to u from the right and

$$
cf'(u;x,y) + \lambda \Bigg(\int_0^u f(u-z;x,y) dG(z) + \mathbb{1}_{u \geq y}\, \bar{G}(u+x) - f(u;x,y) \Bigg) = 0. \quad (1)
$$

In order to simplify notation $'$ denotes the derivative with respect to the first argument. Replacing u by $u-ch$ yields that $f(u;\, x,\, y)$ is left-continuous in u and differentiable from the left. Denoting by d^-/du the derivative from the left, one obtains the equation

$$
\begin{aligned}
& c\frac{d^-}{du} f(u;x,y) \\
& + \lambda \Bigg(\int_0^{u-} f(u-z;x,y)\, dG(z) + \mathbb{1}_{u>y}\, \bar{G}((u+x)-) - f(u;x,y) \Bigg) = 0.
\end{aligned}
$$

Here $\int_0^{u-} = \int_{(0,u)}$ (we use the convention $\int_0^u = \int_{(0,u]}$). This shows that $f(u;x,y)$ is not differentiable at y, at points where $G(u)$ is not continuous or where $G(u + x)$ is not continuous. Because the number of points where the derivative does not exist is countable, we have that $f(u;x,y)$ is absolutely continuous in u with density given by (1). Note that $f'(u;x,y)$ has to be regarded as density and not as derivative.

Let us now integrate (1) from 0 to u_0. Using

$$\begin{aligned}\int_0^{u_0}\int_0^u f(u-z;x,y)\,dG(z)\,du &= \int_0^{u_0}\int_z^{u_0} f(u-z;x,y)\,du\,dG(z)\\ &= \int_0^{u_0}\int_0^{u_0-z} f(u;x,y)\,du\,dG(z)\\ &= \int_0^{u_0} G(u_0-u)\,f(u;x,y)\,du\end{aligned}$$

we arrive at

$$\begin{aligned}c(f(u;x,y)-f(0;x,y)) =&\lambda\int_0^u f(u-z;x,y)\,\bar{G}(z)\,dz-\\ &-\mathbb{I}_{u>y}\lambda\int_{y+x}^{u+x}\bar{G}(z)\,dz\,. \end{aligned}\tag{2}$$

(2) looks similar to a renewal equation and will be used later. We first show that (2) determines the solution uniquely.

Lemma 1: *There is at most one solution to (2) for any given value of* $f(0;x,y)=f_0$.

Proof. Let $f(u)$ and $\bar{f}(u)$ be solutions to (2) with $f(0)=\bar{f}(0)=f_0$ and $g(u)=\bar{f}(u)-f(u)$. Then

$$cg(u)=\lambda\int_0^u g(u-z)\,\bar{G}(z)\,dz=\lambda\int_0^u g(z)\,\bar{G}(u-z)\,dz.$$

Thus $g(u)$ is continuous and $g(0)=0$. Fix $u_0>0$ such that $c>\lambda\int_0^{u_0}\bar{G}(z)\,dz$. Let u be the point such that $|g(u)|=\sup\{|g(z)|:0\le z\le u_0\}$. Then

$$c|g(u)|\le\lambda\int_0^u|g(u-z)|\bar{G}(z)\,dz\le|g(u)|\lambda\int_0^u\bar{G}(z)\,dz\le\lambda\int_0^{u_0}\bar{G}(z)\,dz|g(u)|.$$

Thus $g(u)=0$ and $g(v)=0$ for $0\le v\le u_0$. Hence

$$cg(u)=\lambda\int_0^{u-u_0}g(u-z)\,\bar{G}(z)\,dz\,.$$

Using induction one can show similarly that $g(u)=0$ for $0\le u\le ku_0$. □

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We denote by $\hat{f}(s; x, y) = \int_0^\infty e^{-su} f(u; x, y)\, du$ the Laplace transform of f with respect to the first argument. $\hat{f}(s; x, y) < \infty$ for all $s > 0$. Multiplying (1) by e^{-su} and integrating over $(0, \infty)$ gives

$$c\Big(s\hat{f}(s; x, y) - f(0; x, y)\Big) - \lambda \hat{f}(s; x, y)(1 - M_Y(-s)) + \lambda \int_y^\infty \bar{G}(z + x) e^{-sz} dz$$

or equivalently

$$\hat{f}(s; x, y) = \frac{cf(0; x, y) - \lambda \int_y^\infty \bar{G}(z + x) e^{-sz} dz}{cs - \lambda(1 - M_Y(-s))} . \tag{3}$$

Here we used the well-known formulae

$$\hat{g'}(s) = s\hat{g}(s) - g(0)$$

and

$$\widehat{g_1 * g_2}(s) = \hat{g}_1(s)\,\hat{g}_2(s).$$

Let us have a closer look at the numerator. Differentiating $cs - \lambda(1 - M_Y(-s))$ twice gives $\lambda M_Y''(-s) > 0$, thus the numerator is convex. The first derivative at zero is $c - \lambda\mu$. Thus zero is the only non-negative root of $cs - \lambda(1 - M_Y(-s)) = 0$ if and only if $c \geq \lambda\mu$. If $c < \lambda\mu$, then there exists a unique positive root. We will use this observation to find $f(0; x, y)$.

3. Positive safety loading

Assume now $c > \lambda\mu$. Because $f(u; x, y) \leq \psi(u)$ it follows that $f(u; x, y)$ tends to zero as $u \to \infty$. Letting $u \to \infty$ in (2) yields, using the bounded convergence theorem, that

$$f(0; x, y) = \frac{\lambda}{c} \int_{y+x}^\infty \bar{G}(z)\, dz , \tag{4}$$

a result already obtained in Dufresne & Gerber (1988). Thus (2) can be written as

$$cf(u; x, y) = \lambda \int_0^u f(u - z; x, y) \bar{G}(z)\, dz + \lambda \int_{u \vee y}^\infty \bar{G}(z + x)\, dz . \tag{5}$$

We now solve the equation in terms of $G(u)$ and $\psi(u)$.

Theorem 1 *If* $c > \lambda\mu$ *then*

$$f(u;x,y) = \frac{\lambda}{c-\lambda\mu} \times \left((1-\psi(u)) \int_{x+y}^{\infty} \bar{G}(z)\,dz - \mathbb{1}_{u>y} \int_{y}^{u} (1-\psi(u-z))\,\bar{G}(z+x)\,dz \right). \tag{6}$$

Proof. Consider (3). For $x = y = 0$ one obtains

$$\int_0^{\infty} \psi(u)e^{-su}\,du = \frac{\lambda\mu s - \lambda(1-M_Y(-s))}{s(cs - \lambda(1-M_Y(-s))}$$

where we used that $\int_0^{\infty} \bar{G}(z)e^{-sz}\,dz = s^{-1}(1-M_Y(-s))$. This leads to

$$\int_0^{\infty} (1-\psi(u))e^{-su}\,du = \frac{c-\lambda\mu}{cs - \lambda(1-M_Y(-s))}$$

Equation (3) can therefore be written as

$$\frac{\lambda}{c-\lambda\mu} \int_0^{\infty} (1-\psi(u))\,e^{-su}\,du \left(\int_{x+y}^{\infty} \bar{G}(z)\,dz - \int_y^{\infty} \bar{G}(z+x)\,e^{-sz} dz \right).$$

$\int_{x+y}^{\infty} \bar{G}(z)dz$ is a constant and $\int_y^{\infty} \bar{G}(z+x)e^{-sz}dz$ is the Laplace transform of $\mathbb{1}_{u>y}\bar{G}(u+x)$. Thus the inversion of (3) yields the solution (6) noting that the inversion of $\hat{g}_1(s)\hat{g}_2(s)$ is $\int_0^u g_1(u-z)g_2(z)dz$. □

Remark. The solution could also have been guessed from Dickson (1992). A direct verification gives then that (6) solves (5) and therefore must be the unique solution to (5), i.e. the function $f(u;x,y)$. ■

Example 1. If $\bar{G}(z) = e^{-z/\mu}$, i.e. the claim sizes are exponentially distributed, then $\psi(u) = \lambda\mu/ce^{-Ru}$, where $R = \mu^{-1} - \lambda/c$. This gives

$$f(u;x,y) = \frac{\lambda\mu}{c-\lambda\mu} e^{-(x+y)/\mu} \left(e^{-R(u-y)^+} - \frac{\lambda\mu}{c} e^{-Ru} \right).$$

If we condition on $\{\tau < \infty\}$ we get

$$P[-X_\tau > x, X_{\tau-} > y | \tau < \infty] = e^{-x/\mu} \frac{c}{c-\lambda\mu} e^{-y/\mu} \left(e^{R(y\wedge u)} - \frac{\lambda\mu}{c} \right).$$

As one expects from the lack of memory property of the exponential distribution, $-X_\tau$ is conditionally independent of X_τ given $\{\tau < \infty\}$ and exponentially distributed with mean μ. ■

We now want to investigate as $u \to \infty$ the asymptotic conditional joint distribution of $(-X_\tau, X_{\tau-})$ given $\{\tau < \infty\}$. We keep x and y fixed. Assume $u > y$. Consider first the last term of (6). We have

$$\begin{aligned}\int_y^u \psi(u-z)\,\bar{G}(z+x)\,dz &= \int_{y+x}^{u+x} \psi(u+x-z)\,\bar{G}(z)\,dz \\ &= \int_0^{u+x} \psi(u+x-z)\,\bar{G}(z)\,dz \\ &\quad - \int_0^{y+x} \psi(u+x-z)\,\bar{G}(z)\,dz\end{aligned}$$

and using (5) with $x = y = 0$ we find

$$\int_0^{u+x} \psi(u+x-z)\,\bar{G}(z)\,dz = \frac{c}{\lambda}\psi(u+x) - \int_{u+x}^{\infty} \bar{G}(z)\,dz\,.$$

Putting the above together we have

$$\frac{c-\lambda\mu}{\lambda} f(u;x,y) = \frac{c}{\lambda}\psi(u+x) - \int_0^{y+x} \psi(u+x-z)\bar{G}(z)dz - \psi(u)\int_{x+y}^{\infty} \bar{G}(z)dz$$

if $u > y$. This leads to the following

Theorem 2 *Let $c > \lambda\mu$. Assume for each $z \in \mathbb{R}$ the limit*

$$\gamma(z) = \lim_{u\to\infty} \frac{\psi(u+z)}{\psi(u)}$$

exists. Then

$$\begin{aligned}&\lim_{u\to\infty} P[-X_\tau > x, X_{\tau-} > y | \tau < \infty] \\ &= \frac{1}{c-\lambda\mu}\left(c\gamma(x) - \lambda\int_0^{y+x} \gamma(x-z)\bar{G}(z)dz - \int_{x+y}^{\infty} \bar{G}(z)dz\right)\,. \end{aligned} \tag{7}$$

Proof. It remains to show that we can interchange limit and integration in the middle term. Because $\psi(u)$ is decreasing we find

$$\frac{\psi(u+x-z)}{\psi(u)} \le \frac{\psi(u-y)}{\psi(u)}$$

which is bounded because it is continuous and converges to $\gamma(-y)$ as $u \to \infty$. Thus we have an integrable upper bound and the theorem follows from the bounded convergence theorem. □

Example 2. Assume that the Cramér condition is fulfilled, i.e. there is $R > 0$ such that $\lambda(M_Y(R) - 1) = cR$. Then $\psi(u) \sim Ce^{-Ru}$ for some $C > 0$ and $R > 0$. This case is for instance discussed in Gerber (1979), Grandell (1991) and Rolski et al. (1999). The assumption of Theorem 2 is fulfilled with $\gamma(z) = e^{-Rz}$. Thus the asymptotic distribution is

$$\frac{e^{-Rx}}{c - \lambda\mu}\left(c - \lambda\int_0^{y+x} e^{Rz}\,\bar{G}(z)\,dz - \lambda e^{Rx}\int_{x+y}^{\infty}\bar{G}(z)\,dz\right). \qquad (8)$$

Because $\lambda\int_0^\infty e^{Rx}\,\bar{G}(z)\,dz = c$, (8) can be written as

$$\frac{\lambda}{c - \lambda\mu}\int_y^\infty (e^{Rz} - 1)\,\bar{G}(z + x)\,dz\,.$$

■

Let $R \geq 0$ and define the class $\mathcal{S}(R)$ of distribution functions G fulfilling

i) $\lim_{u\to\infty} \overline{G^{*2}}(u)/\bar{G}(u) = \kappa < \infty$,

ii) $\lim_{u\to\infty} \bar{G}(u - z)/\bar{G}(u) = e^{Rz}$,

iii) $M_Y(R) < \infty$

$\mathcal{S}(0)$ is the class of subexponential distributions, including the Pareto, the lognormal and the heavy-tailed Weibull distributions. For a discussion of subexponential distributions see for instance Embrechts et al. (1997), Rolski et al. (1999) and references in these two books. The classes $\mathcal{S}(R)$ are discussed in Embrechts & Goldie (1982). If $G \in \mathcal{S}(R)$ then $M_Y(r) = \infty$ for all $r > R$. That means that the moment generating function jumps to infinity at R. All distributions of interest with a moment generating function jumping to infinity at R are included in this class. Embrechts & Goldie (1982) show that for $G \in \mathcal{S}(R)$ there exists $\tilde{G} \in \mathcal{S}(0)$ such that

$$G(u) = \frac{\int_0^u e^{-Rz}d\tilde{G}(z)}{\int_0^\infty e^{-Rz}d\tilde{G}(z)}\,.$$

Example 3. Assume $G \in \mathcal{S}(R)$ for some $R > 0$ and that $\lambda\int_0^\infty e^{Rz}\bar{G}(z)dz < c$. Embrechts and Veraverbeke (1982) showed that there exists a constant $C > 0$ such that $\psi(u) \sim C\bar{G}(u)$. Thus $\gamma(z) = e^{-Rz}$ and (8) holds. At first sight the result may be surprising. However, in both Examples 2 and 3 the ruin probability is exponentially decreasing

$$\lim_{u\to\infty} -\frac{1}{u}\log\psi(u) = R.$$

It is the exponent R that determines $f(u; x, y)$ as $u \to \infty$. We will now give an intuitive explanation why one would expect that $\lim_{u\to\infty} f(u; x, y)/\psi(u)$ should be determined by (8). Let $\theta(R) = \lambda(M_Y(R) - 1) - cR$. Then $\left(e^{-R(X_t - u) - \theta(R)t}\right)$

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is a martingale, see Embrechts et al. (1993) or Rolski et al. (1999). This martingale can be used to change the measure $dQ/dP = e^{-R(X_t-u)-\theta(R)t}$ on $\mathcal{F}_t$. The measure Q can be extended to a measure on the whole probability space $(\Omega, \mathcal{F})$. For an introduction to change of measure techniques in risk theory see for instance Schmidli (1995) or Rolski et al. (1999). Under Q the process (X_t) is a classical risk process with claim arrival intensity $\lambda_Q = \lambda M_Y(R)$ and claim size distribution $G_Q(u) = \int_0^u e^{Rz} dG(z)/M_Y(R)$. Expressing the quantities of interest under the measure Q gives

$$\frac{f(u;x,y)}{\psi(u)} = \frac{E_Q[e^{RX_\tau}e^{\theta(R)\tau}; -X_\tau > x, X_{\tau-} > y, \tau < \infty]e^{-Ru}}{E_Q[e^{RX_\tau}e^{\theta(R)\tau}; \tau < \infty]e^{-Ru}} .$$

Intuitively for large u the variables τ and $(X_{\tau-}, X_\tau)$ become nearly independent, so

$$\frac{f(u;x,y)}{\psi(u)} \approx \frac{E_Q[e^{RX_\tau}; -X_\tau > x, X_{\tau-} > y, \tau < \infty]}{E_Q[e^{RX_\tau} | \tau < \infty]} .$$

The latter expression is the same for $\theta(R) = 0$ and $\theta(R) < 0$. Thus one would expect (8) to hold in all cases where $\bar{G}(u)$ is exponentially decreasing and $\int_0^\infty ze^{Rz} dG(z) < \infty$. ■

Example 4. Assume that the distribution function $\mu^{-1} \int_0^u \bar{G}(z)dz$ is in $\mathcal{S}(0)$. Then Embrechts and Veraverbeke (1982) showed that $(c - \lambda\mu)\psi(u) \sim \sim \lambda \int_u^\infty \bar{G}(z)dz$. Thus $\gamma(z) = 1$. Then it follows from (7) that

$$\lim_{u\to\infty} P[-X_\tau > x, X_{\tau-} > y | \tau < \infty] = 1 .$$

Let $x = 0$ and $y = u$. Then (6) reads

$$(c - \lambda\mu) f(u; 0, u) = \lambda(1 - \psi(u)) \int_u^\infty \bar{G}(z)dz .$$

Dividing by $\psi(u)$ and letting $u \to \infty$ yields

$$\lim_{n\to\infty} P[X_{\tau-} > u | \tau < \infty] = 1 .$$

This is not surprising. Asmussen and Klüppelberg (1996) showed that basically for large u the process (X_t) conditioned on $\{\tau < \infty\}$ behaves like an unconditioned process until the time of ruin, and there an enormous claim will happen. Recall that τ depends on u. Thus $X_{\tau-}$ will very likely be above any fixed level for u large enough, and most likely above u. This also implies that X_τ will very likely be below any fixed level for u large enough.

Assume now that $y(u) \geq u$ and $x(u)$ are some functions. Then we find

$$f(u; x(u),\ y(u)) = \frac{\lambda}{c - \lambda\mu}(1 - \psi(u)) \int_{x(u)+y(u)}^\infty \bar{G}(z)\, dz \sim \psi(x(u) + y(u)) .$$

If $x(u)$ and $y(u)$ are differentiable then

$$\frac{f(u; x(u), y(u))}{\psi(u)} \sim (x'(u) + y'(u)) \frac{\bar{G}(x(u) + y(u))}{\bar{G}(u)} .$$

If for example $\bar{G}(z) \sim L(z)z^{-\alpha}$ for some $\alpha \geq 1$ and some slowly varying function $L(z)$, i.e. $L(tz)/L(z) \to 1$ as $z \to \infty$ for all $t > 0$, then $\psi(u) \sim CL(z)u^{1-\alpha}$ for some constant $C > 0$. Thus

$$\lim_{u\to\infty} \frac{f(u; au, bu)}{\psi(u)} = (a+b)^{1-\alpha}$$

provided $a \geq 0$ and $b \geq 1$.

Assume that $\bar{G}(z)$ is not regularly varying. Goldie and Resnick (1988) showed that under quite mild assumptions the distribution G is in the maximum domain of attraction of a Gumbel distribution $\exp\{-e^{-x}\}$, i.e. there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that $\lim_{u\to\infty} (G(a_n z + b_n))^n = \exp\{-e^{-x}\}$. Then, see Balkema & de Haan (1974), there exists a function $a(u)$ such that $\bar{G}(u + za(u))/\bar{G}(u) \to e^{-z}$. The function $a(u)$ can be chosen as $E[Y - u | Y > u]$. The distribution function G is then of the form

$$\bar{G}(u) = c(u) \exp\left(- \int_0^u 1/g(z)\, dz \right)$$

where $g(u) > 0$ is absolutely continuous, $g'(u) \to 0$ and $c(u) \to c > 0$ as $u \to \infty$. It follows that $a(u) \sim g(u)$. This gives

$$\lim_{u\to\infty} \frac{f(u; x(u), u + zg(u) - x(u))}{\psi(u)} = \lim_{u\to\infty} (1 + zg'(u)) \frac{\bar{G}(u + zg(u))}{\bar{G}(u)} = e^{-z}$$

provided $x(u) \leq zg(u)$. If the claims are Weibull distributed $\bar{G}(u) = \exp\{-\alpha x^\beta\}$ with $\alpha > 0$ and $0 < \beta < 1$, then $c(u) = e^{-\alpha}$ and $g(u) = u^{1-\beta}/(\alpha\beta)$. This gives

$$\lim_{u\to\infty} \frac{f\left(u; x(u), u + zu^{1-\beta}/(\alpha\beta) - x(u)\right)}{\psi(u)} = e^{-z}$$

provided $x(u) \leq u^{1-\beta}/(\alpha\beta)$. ■

4. Negative safety loading

In practice the mean value of the claims and the claim arrival intensity have to be estimated from data. Estimation of the claim arrival intensity is usually no problem. But observation of the claims only gives information about the distribution on a finite interval. Thus it may happen that the estimate of μ is far from the true value. In such a situation it is possible that $c < \lambda\mu$, even that $\mu = \infty$. The latter case, however, usually is excluded in the contract by

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defining a maximal loss. If now a wrong premium has been specified it would be interesting to know how ruin will occur. Namely, if the capital prior to ruin will be small, as in Examples 2 and 3, one has time to observe the business until action has to be taken. If the capital prior to ruin will be large, as in Example 4, it would not be possible to observe negative safety loading before ruin occurs. We will see below that the latter can only happen if $\mu = \infty$.

Assume now $c < \lambda\mu$. This includes in particular the case $\mu = \infty$. The equation $cs - \lambda(1 - M_Y(-s))$ has then a strictly positive solution R. We define the new measure Q under which (X_t) is a classical risk process with premium rate c, intensity $\lambda_Q = \lambda M_Y(-R)$ and claim size distribution $G_Q(x) = \int_0^x e^{-Ry} dG(y)/M_Y(-R)$. The expected claim size is $M'_Y(-R)/M_Y(-R)$ giving $\lambda_Q\mu_Q = \lambda M'_Y(-R)$. Because $s \mapsto cs - \lambda(1 - M_Y(-s))$ is a convex function with derivative $c - \lambda M'_Y(-R)$ at $s = R$ we find $c - \lambda M'_Y(-R) > 0$. That means under Q the safety loading is positive. We denote by $\psi_Q(u) = Q[\tau < \infty]$ the ruin probability under measure Q.

Because $\hat{f}(R; x, y) < \infty$ and the numerator in (3) is zero also the denominator has to be zero, yielding

$$f(0; x, y) = \frac{\lambda}{c} \int_y^\infty \bar{G}(z + x) e^{-Rz} dz\ . \tag{9}$$

Remark. We could also have obtained (9) from the following consideration. Note that $\lambda c^{-1} \int_0^\infty e^{-Rz} \bar{G}(z) dz = 1$. Let $g(u) = \lambda c^{-1} e^{-Ru} \bar{G}(u)$. Multiplying (2) by e^{-Ru} yields the renewal equation

$$f(u; x, y) e^{-Ru} = \int_0^u f(u - z; x, y) e^{-R(u-z)} g(z)\, dz + z(u)$$

with

$$z(u) = f(0; x, y) e^{-Ru} - \mathbb{1}_{u>y} \frac{\lambda}{c} e^{-Ru} \int_{y+x}^{u+x} \bar{G}(z)\, dz\ .$$

By the key renewal theorem, see for instance Feller (1971),

$$\int_0^\infty \left(f(0; x, y) - \mathbb{1}_{u>y} \frac{\lambda}{c} \int_{y+x}^{u+x} \bar{G}(z)\, dz \right) e^{-Ru} du = \lim_{u\to\infty} f(u; x, y)\, e^{-Ru} = 0\ ,$$

which also yields (9). ■

We now are ready to invert the Laplace transform (3).

Theorem 3 *Assume* $c < \lambda\mu$. *Then*

$$f(u;x,y) = \frac{\lambda e^{Ru}}{c - \lambda M'_Y(-R)} \left((1 - \psi_Q(u)) \int_y^\infty \bar{G}(z+x)e^{-Rz}dz - \right.$$
$$\left. - \mathbb{I}_{u>y} \int_y^u (1 - \psi_Q(u-z))\bar{G}(x+z)e^{-Rz}dz \right) . \tag{10}$$

Proof. Fix x and y. Let $g(u) = f(u;x,y)e^{-Ru}$ and $\hat{g}(s) = \int_0^\infty g(u)e^{-su}du = \hat{f}(R+s;x,y)$. Then we find from (3)

$$\hat{g}(s) = \frac{\lambda \int_y^\infty \bar{G}(z+x)e^{-Rz}(1 - e^{-sz})dz}{cs - \lambda(M_Y(-R) - M_Y(-s-R))}$$

using that $cR = \lambda(1 - M_Y(-R))$. Note that

$$\int_0^\infty (1 - \psi_Q(u))e^{-su}du = \frac{c - \lambda M'_y(-R)}{cs - \lambda(M_Y(-R) - M_Y(-s-R))} .$$

Thus $\hat{g}(s)$ can be expressed as

$$\hat{g}(s) = \frac{\lambda}{c - \lambda M'_Y(-R)} \int_0^\infty (1 - \psi_Q(u))e^{-su}du \int_y^\infty \bar{G}(z+x)e^{-Rz}(1 - e^{-sz})dz .$$

Similar to the proof of Theorem 1, (10) follows by inverting $\hat{g}(s)$. □

Next we find the asymptotic behaviour of $f(u;x,y)$. We first show that the Cramér condition is fulfilled under the measure Q and that the adjustment coefficient is R. The moment generating function of the claims is

$$E_Q[e^{rY}] = \frac{\int_0^\infty e^{rz}e^{-Rz}dG(z)}{M_Y(-R)} = \frac{M_Y(r-R)}{M_Y(-R)} .$$

The equation determining the adjustment coefficient is $\lambda(M_Y(r-R) - M_Y(-R)) - cr = 0$. By the definition of R we find $r = R$ is a solution. If $\mu < \infty$ then

$$\psi_Q(u) \approx \frac{c - \lambda M'_Y(-R)}{\lambda\mu - c} e^{-Ru} .$$

If $\mu = \infty$ then $\psi_Q(u)e^{Ru} \to 0$ as $u \to \infty$. In particular, $\psi_Q(u)e^{Ru}$ is bounded uniformly in u.

The following result gives the asymptotic behaviour as $u \to \infty$.

Theorem 4 *Assume $c < \lambda\mu$. If $\mu < \infty$ then*

$$\lim_{u\to\infty} f(u;x,y) = \frac{\lambda}{\lambda\mu - c}\int_y^\infty (1-e^{-Rz})\,\bar G(x+z)\,dz\,. \tag{11}$$

If $\mu = \infty$ then $\lim_{u\to\infty} f(u;x,y) = 1$.

Proof. Assume $u > y$. We consider the limits of the factors given in (10) separately. We start with two factors $\int_y^\infty - \int_y^u = \int_u^\infty$ and consider $e^{Ru}\int_u^\infty \bar G(z+x)e^{-Rz}dz$. This can be written as $\int_{0-}^\infty \bar G(u+z+x)e^{-Rz}\,dz$ which converges to zero. Consider next $\psi_Q(u)e^{Ru}\int_y^\infty \bar G(x+z)e^{-Rz}dz$. This factor converges to $\big(c-\lambda M_Y'(-R)\big)/(\lambda\mu - c)\int_y^\infty \bar G(x+z)e^{-Rz}dz$, interpreted as zero if $\mu = \infty$. Let now $\mu < \infty$ and consider $\int_y^u \psi_Q(u-z)e^{R(u-z)}\bar G(x+z)dz$. Because $\mu < \infty$ we can interchange limit and integration, and obtain the limit $\big(c-\lambda M_Y'(-R)\big)/(\lambda\mu - c)\int_y^\infty \bar G(x+z)dz$. Putting these limits together proves (11).

Let now $\mu = \infty$. Consider first $f(u;x,0) - f(u;x,y)$. Then it remains to consider the limit of

$$\int_0^y \psi_Q(u-z)\,e^{R(u-z)}\,\bar G(x+z)dz\,.$$

But here limit and integration can be interchanged, yielding that $\lim_{u\to\infty} f(u;x,y) = \lim_{u\to\infty} f(u;x,0)$ provided the latter limit exists. Thus we can assume $y = 0$. Consider now $1 - f(u;x,0) = f(u;0,0) - f(u;x,0)$. Then it remains to show that

$$\lim_{u\to\infty}\int_0^u \psi_Q(u-z)\,e^{R(u-z)}\big(\bar G(z) - \bar G(x+z)\big)\,dz = 0\,.$$

Because

$$\int_0^\infty \big(\bar G(z) - \bar G(x+z)\big)\,dz = \int_0^\infty\int_z^{z+x} dG(v)\,dz \le x\int_0^\infty dG(v) = x$$

we can interchange limit and integration, yielding the result. □

The above result shows that in the case $\mu = \infty$ ruin will be caused very likely by a very large claim.

Remarks.

i) Assume $c > \lambda\mu$ and that $\psi(u) \sim Ce^{Ru}$ for some $R > 0$ and $C > 0$. Asmussen (1982) showed that, conditioned on $\{\tau < \infty\}$ the classical risk model converges weakly to a classical risk model $\tilde X$ with intensity $\tilde\lambda = \lambda M_Y(R)$ and claim size distribution $\tilde G(u) = \int_0^u e^{Rz}dG(z)/M_Y(R)$. Thus one could think that (8) for the model (X_t) and (11) for the model $\tilde X$ coincide. This is not the case. For example, in the case of exponentially

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distributed claim sizes we find $-X_\tau$ conditioned on $\tau < \infty$ is exponentially distributed with mean μ. But in the model $\tilde{X}$ the claim sizes are exponentially distributed with mean c/λ, i.e. $-\tilde{X}_{\tilde{\tau}}$ is exponentially distributed with mean c/λ. The reason is that as $u \to \infty$ also $\tau \to \infty$. For weak convergence events near infinity do not necessarily converge.

ii) The result can also be obtained by a change of measure. However, the calculations become more complicated. As in Gerber (1973) it follows then that $\left(e^{R(X_t-u)}\right)$ is a martingale. Define the new measure Q via

$$Q[A] = E_P\left[e^{R(X_t-u)}; A\right] = E_P\left[e^{R(X_t-u)}\mathbb{1}_A\right]$$

for all $A \in \mathcal{F}_t$ and all $t \geq 0$. It is shown in Schmidli (1995), see also Rolski et al. (1999), that under the measure Q the process (X_t) is a classical risk process with intensity $\lambda_Q = \lambda M_Y(-R)$ and claim size distribution $G_Q(x) = \int_0^x e^{-Ry}dG(y)/M_Y(-R)$. Thus the new measure coincides with the measure Q used above. It follows then, see Schmidli (1995) or Rolski et al. (1999), that

$$\begin{aligned} f(u;x,y) &= E_Q\left[e^{-R(X_\tau-u)}; \tau < \infty, -X_\tau > x, X_{\tau-} > y\right] \\ &= \left(\int_x^\infty f_Q(u,z,y)Re^{Rz}dz + e^{Rx}f_Q(u,x,y)\right)e^{Ru} \end{aligned}$$

where we use the subscript Q to denote the quantities under the measure Q. By straightforward calculations (9) is recovered from (4) and (10) is recovered from (6). ■

5. No safety loading

This section is for completeness only. Indeed, if an insurance company fixes the premium based on the estimates for λ and μ it is very unlikely that $c = \lambda\mu$. Because as in the case of negative safety loading $\psi(u) = 1$ one would expect similar results for $f(u;x,y)$ as in the case $c < \lambda\mu$. However, as $u \to \infty$ it is not any more the mean value μ that determines whether the distribution of $(-X_\tau, X_{\tau-})$ converges to a proper distribution or not. In the case $c = \lambda\mu$ the above distribution will converge if and only if $\mu_2 = E[Y^2] < \infty$. This is due to the distribution of the descending ladder heights. In the case $c < \lambda\mu$ the expected value of the ladder height is finite if and only if $E[Y] < \infty$, see (9). In the case $c < \lambda\mu$ the ladder height distribution is $G_I(u) = \mu^{-1}\int_0^u \bar{G}(z)dz$, see (12) below. This distribution has a finite mean if and only if $\mu_2 < \infty$.

The change of measure method used in the case of a negative safety loading does not work anymore. The function $cs - \lambda(1 - M_Y(-s))$ has a minimum at $s = 0$ and therefore only one root $s = 0$. We therefore do not have an interpretation of $(cs - \lambda(1 - M_Y(-s)))^{-1}$ in terms of ruin probabilities. Indeed, the solution (14) below is not as explicit as (6) and (10) obtained above.

We start by finding $f(0;x,y)$. Note that $f(u;x,y) \le 1$ and therefore $\hat{f}(s;x,y) \le s^{-1}$. This gives

$$0 \le s\hat{f}(s;x,y) = \frac{cf(0;x,y) - \lambda \int_y^\infty \bar{G}(z+x)e^{-sz}\,dz}{c - s^{-1}\lambda(1 - M_Y(-s))} \le 1 .$$

The numerator converges to zero as $s \to 0$, thus also the denominator has to converge to zero. This gives

$$f(0;x,y) = \frac{1}{\mu}\int_{y+x}^\infty \bar{G}(z)\,dz . \tag{12}$$

As a consequence (2) can be written as

$$f(u;x,y) = \frac{1}{\mu}\int_0^u f(u-z;x,y)\,\bar{G}(z)\,dz + \frac{1}{\mu}\int_{u\vee y}^\infty \bar{G}(z+x)\,dz . \tag{13}$$

This is a renewal equation. Denote by $U(u) = \sum_{k=0}^\infty G_I^{*k}(u)$ the corresponding renewal measure. Then from renewal theory, see for instance Feller (1971), we find

Theorem 5 *Let $c = \lambda\mu$. Then*

$$f(u;x,y) = \frac{1}{\mu}\left(\int_{y+x}^\infty \bar{G}(z)\,dz\,U(u) - \mathbb{1}_{u>y}\int_y^u U(u-z)\,\bar{G}(z+x)\,dz\right) . \tag{14}$$

Proof. It follows immediately from renewal theory that

$$\begin{aligned}\mu f(u;x,y) &= \int_{0-}^u \int_{(u-z)\vee y}^\infty \bar{G}(v+x)\,dv\,dU(z)\\ &= U(u)\int_y^\infty \bar{G}(v+x)dv - \mathbb{1}_{u>y}\int_{0-}^{u-y}\int_y^{u-z}\bar{G}(v+x)\,dv\,dU(z) .\end{aligned}$$

The result follows now by changing the order of integration. □

The behaviour of $f(u;x,y)$ for large u follows now readily from the key renewal theorem.

Theorem 6 *Let $c = \lambda\mu$. If $\mu_2 < \infty$, i.e. the claim sizes have finite variance, then*

$$\lim_{u\to\infty} f(u;x,y) = \frac{\int_y^\infty z\bar{G}(z+x)\,dz}{\int_0^\infty z\bar{G}(z)\,dz} . \tag{15}$$

If $\mu_2 = \infty$ then $\lim_{u\to\infty} f(u;x,y) = 1$.

Proof. (15) follows readily from the key renewal theorem. Assume now $\mu_2 = \infty$. Then the distribution function $G_I(u)$ has infinite mean. Let $g(u) = f(u;x,0) - f(u;x,y)$. Then it follows from (13) that $g(u)$ fulfils

$$g(u) = \frac{1}{\mu}\left(\int_0^u g(u-z)\,\bar{G}(z)\,dz + \int_u^{u\vee y} \bar{G}(z+x)\,dz\right).$$

The key renewal theorem yields then that $g(u)$ tends to zero as $u \to \infty$ because

$$\int_0^\infty \int_u^{u\vee y} \bar{G}(z+x)\,dz\,du = \int_0^y \int_u^y \bar{G}(z+x)\,dz\,du < \infty .$$

We can therefore assume $y = 0$. Let now $g(u) = 1 - f(u;x,0) = f(u;0,0) - f(u;x,0)$. Then (13) gives

$$g(u) = \frac{1}{\mu}\left(\int_0^u g(u-z)\,\bar{G}(z)\,dz + \int_u^{u+x} \bar{G}(z)\,dz\right).$$

Again the key renewal theorem yields $g(u) \to 0$ because

$$\int_0^\infty \int_u^{u+x} \bar{G}(z)\,dz\,du \le \int_0^\infty x\bar{G}(z)\,dz = x\mu < \infty .$$

This proves the result. □

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