Canad. Math. Bull. Vol. 15 (4), 1972

ON THE FRATTINI SUBGROUPS OF GENERALIZED FREE PRODUCTS WITH CYCLIC AMALGAMATIONS(1)

BY
C. Y. TANG

1. In [1] Higman and Neumann asked the questions whether the Frattini subgroup of a generalized free product can be larger than the amalgamated subgroup and whether such groups necessarily have maximal subgroups. In [4] Whittemore gave answers to the special cases of generalized free products of finitely many free groups with cyclic amalgamation and of generalized free products of finitely many finitely generated abelian groups. In this paper we shall study the Frattini subgroups of generalized free products of any groups with cyclic amalgamation. Indeed if the amalgamated subgroup is a finite cyclic subgroup then we can completely determine the Frattini subgroup in terms of the Frattini subgroups of the generalized free factors.

Throughout this paper we shall use Φ-free as in [4] to mean a group having trivial Frattini subgroup.

2. We first note that since we place no restriction on the generalized free factors, it suffices to prove our theorems for generalized free products involving only two factors.

The following lemma is not difficult to prove.

LEMMA 2.1. *Let H be a cyclic p-subgroup of a group G. If H contains no nontrivial normal subgroup of G then there exists $x \in G$ such that $H^x \cap H=1$.*

LEMMA 2.2. *Let $P=(A * B)_H$ where H is a finite cyclic subgroup. If H contains no nontrivial normal subgroup of P then there exists an element x of P such that $H^x \cap H=1$.*

Proof. Let $H=H_1\times \cdots \times H_r$ where the H_i's are the sylow subgroups of H. Since H contains no nontrivial normal subgroup of P, for each i, at least in one of A and B, H_i contains no nontrivial normal subgroup of that group. Thus by Lemma 2.1, for each i there exists an element y of A or B such that $H_i^y \cap H_i=1$, whence $H_i^y \cap H=1$. Let $H_1, \ldots, H_k$ be the set of H_i's for which there exists $a_i \in A$ such that $H_i^{a_i} \cap H=1$, $i=1,\ldots,k$. Thus for each of H_i, $i=k+1,\ldots,r$, there exists $b_i \in B$ such that $H_i^{b_i} \cap H=1$. Let $a \in A$ and $b \in B$ such that $a, b \notin H$. Let $x=a_1ba_2b\cdots ba_kb_{k+1}a\cdots b_{r-1}ab_r$. Let $1\neq h_i \in H_i$, $i=1,\ldots,r$. It is not difficult to see that the length of h_i^x is not zero for each i. It follows that $H^x \cap H=1$.

Received by the editors October 28, 1970 and, in revised form, March 10, 1971.
(1) This work was partly supported by National Research Council Grant A-4064.

https://doi.org/10.4153/CMB-1972-099-5 Published online by Cambridge University Press

Applying Lemma 2.2 and Proposition 2.3 of [4] the following theorem is immediate.

THEOREM 2.3. *If P is a generalized free product amalgamating a finite cyclic subgroup H which contains no nontrivial normal subgroup of P then P is Φ-free.*

THEOREM 2.4. *Let $P=(A*B)_H$ where H is any cyclic subgroup. If H has a nontrivial subgroup normal in P then $\Phi(P)\subseteq N$ where N is the maximal subgroup of H normal in P.*

Proof. Consider $\bar{P}=P/N=(\bar{A}*\bar{B})_{\bar{H}}$ where $\bar{A}$, $\bar{B}$, $\bar{H}$ are respectively A/N, B/N, and H/N. Since N is the maximal subgroup of H normal in P it follows that $\bar{H}$ contains no nontrivial normal subgroup of $\bar{P}$. Moreover $\bar{H}$ is finite cyclic. Therefore by Theorem 2.3 $\Phi(\bar{P})=1$. But $\Phi(P)N/N\subseteq\Phi(\bar{P})$, whence $\Phi(P)\subseteq N$.

We shall now determine the Frattini subgroup of a generalized free product amalgamating a finite cyclic subgroup in terms of the Frattini subgroups of the generalized free factors.

LEMMA 2.5. *Let $P=(A*B)_H$ where H is cyclic. If N is a subgroup of H normal in P then $\Phi(A)\cap N$ and $\Phi(B)\cap N$ are contained in $\Phi(P)$.*

Proof. Let $x\in\Phi(A)\cap N$. Let S be any set of elements of P such that $\langle S, x\rangle=P$. Now $\langle x\rangle$ characteristic in N and $N\lhd P$ imply that $\langle x\rangle\lhd P$. Thus for each element $a\in A$ there exists $S_a\in\langle S\rangle$ such that $a=S_ax^\alpha$. Now $S_a\in A$. It follows that $A=\langle x, S_a; a\in A\rangle$. But $x\in\Phi(A)$. Therefore $A=\langle S_a; a\in A\rangle\subseteq\langle S\rangle$, whence $x\in\langle S\rangle$. Thus $P=\langle S\rangle$. Hence $x\in\Phi(P)$.

The following lemma is easy to prove:

LEMMA 2.6. *Let G be any group and H a normal subgroup of prime order. Then $\Phi(G)\cap H=1$ if and only if G splits over H.*

LEMMA 2.7. *Let $H=\langle h\rangle$ be a finite cyclic subgroup of order n in a group G. If K is a subgroup of order p, p a prime, such that $K\lhd G$ and G splits over K with complement C, then $C\cap H=\langle h^p\rangle$, and $p^2\nmid n$.*

Proof. Since C complements K in G, $C\cap H$ complements K in H. Hence $H=(C\cap H)\times K$, since H is abelian. Moreover since H is cyclic the orders of $C\cap H$ and K must then be coprime.

THEOREM 2.8. *Let $P=(A*B)_H$ where $H=\langle h\rangle$ is a finite cyclic subgroup. Let N be the maximal subgroup of H normal in P. If $\Phi(A)\cap N=\Phi(B)\cap N=1$, then $\Phi(P)=1$.*

Proof. If H has no nontrivial normal subgroup of P then by Theorem 2.3 our theorem is true. Thus we need only prove the case when $N\neq 1$. Now, by [3], $\Phi(A)\cap N=1$ implies that $\Phi(N)=1$. Since N is cyclic, it follows that the order of N is a product of distinct primes. Let $|N|=n=p_1\cdots p_r$, where the p_i's are distinct primes. Let $q_i=p_1\cdots p_{i-1}p_{i+1}\cdots p_r$. By Theorem 2.4 $\Phi(P)\subseteq N$. Let $N=\langle h^\alpha\rangle$ and $\Phi(P)=\langle h^\beta\rangle$. If $\Phi(P)\neq 1$, then, for some q_i, $h^{\alpha q_i}\in\Phi(P)$. Now the order of $\langle h^{\alpha a_i}\rangle = H_i$ is p_i. Moreover $H_i \lhd A$ and $\Phi(A) \cap H_i = 1$. Thus by Lemma 2.6 A splits over H_i. In the same way B splits over H_i. Let C and D be complements of H_i in A and B respectively. Then by Lemma 2.7 $C \cap H = D \cap H$. But this means $\langle C, D\rangle$ is a complement of H_i in P. Hence $h^{\alpha a_i} \notin \Phi(P)$ contradicting the hypothesis that $h^{\alpha a_i} \in \Phi(P)$. This completes the proof.

THEOREM 2.9. *Let $P=(A * B)_H$ where H is a finite cyclic subgroup. If N is the maximal subgroup of H normal in P then $\Phi(P) = \langle \Phi(A) \cap N, \Phi(B) \cap N\rangle$.*

Proof. Let $U=\Phi(A) \cap N$ and $V=\Phi(B) \cap N$. Let θ and ψ be homomorphisms mapping P onto $\bar{P}=P/U=(\bar{A} * \bar{B})_{\bar{H}}$ and $\tilde{P}=P/V=(\tilde{A} * \tilde{B})_{\tilde{H}}$ respectively where $\bar{A}=A\theta$, $\bar{B}=B\theta$, $\bar{H}=H\theta$, $\tilde{A}=A\psi$, $\tilde{B}=B\psi$, and $\tilde{H}=H\psi$. Let $N=N_1\times \cdots \times N_r$ where the N_i's are the cyclic sylow subgroups of N. Then $U=M_1 \times \cdots \times M_k$, where $M_i \subseteq N_i$. Let $\bar{N}=N\theta$ and $\tilde{N}=N\psi$. Now $\Phi(\bar{A}) \cap \bar{N}=1$ implies that $\bar{N}$ is Φ-free. Thus order of $\bar{N}$ is a product of distinct primes, whence the order of $\bar{N}_i = N_i\theta$ is a prime for each i. In the same way we have that the order of $\tilde{N}_i=N_i\psi$ is a prime for each i. We can let $V=M_1' \times \cdots \times M_s' \times M_{k+1} \times \cdots \times M_t$ where $M_i' \subseteq N_i$ and $s \leq k$. Indeed $M_i'=M_i$ possibly for some i. Moreover the order of N_i is a prime for each $i>s$.

Suppose now $\Phi(\tilde{A}) \cap \tilde{N} \neq (\Phi(A) \cap N)\psi$. Since $(\Phi(A) \cap N)\psi \subseteq \Phi(\tilde{A}) \cap \tilde{N}$, this implies that for some $i>k$, we have $\tilde{N}_i \subseteq \Phi(\tilde{A}) \cap \tilde{N}$. Now $\Phi(\bar{A}) \cap \bar{N}=1$ and the order of $\bar{N}_i$ is a prime. Thus by Lemma 2.6, $\bar{A}$ splits over $\bar{N}_i$. Let C be a complement of $\bar{N}_i$ in $\bar{A}$. Let K and L be respectively the sets of all pre-images of C and $\bar{N}_i$ in A. Then $K \cap L = \ker\theta$. Thus $K\psi \cap L\psi = \tilde{M}_1 \times \cdots \times \tilde{M}_k$ where $\tilde{M}_i=M_i\psi$. Also $L\psi=\tilde{M}_1\times \cdots \times \tilde{M}_k \times \tilde{N}_i$. Therefore $K\psi \cap \tilde{N}_i=1$. But $\langle K, L\rangle = A$ implies that $\langle K\psi, L\psi\rangle=\tilde{A}$. Thus $\langle K\psi, \tilde{N}_i\rangle=\tilde{A}$. Since the order of $\tilde{N}_i$ is a prime, we have by Lemma 2.6 $\tilde{A}$ splits over $\tilde{N}_i$ whence $\tilde{N}_i \not\subseteq \Phi(\tilde{A})$ contradicting the choice of $\tilde{N}_i$. Thus $\Phi(\tilde{A}) \cap \tilde{N}=(\Phi(A) \cap N)\psi$.

Now let $Q=\tilde{P}/(\Phi(A) \cap N)\psi$. Then Q is Φ-free, whence $\Phi(P)=\langle U, V\rangle$.

3. In this section we shall discuss the Frattini subgroups of generalized free products amalgamating infinite cyclic subgroups. The nature of these Frattini subgroups is much more elusive. Indeed we can construct generalized free products of two Φ-free groups amalgamating an infinite cyclic subgroup H where in one instance $\Phi(P) \subsetneq H$ and in another $\Phi(P)=$H. Even in the case when H contains no nontrivial normal subgroup of P we can only give a partial result.

LEMMA 3.1. *Let $P=(A * B)_H$. Let K be a subgroup of H. If θ is a homomorphism of P onto $\bar{P}$ with $\ker\theta=K^P$ where K^P is the normal closure of K in P, then $\bar{P}\approx (\bar{A} * \bar{B})_{\bar{H}}$ where $\bar{A}=A/A \cap K^P$, $\bar{B}=B/B \cap K^P$, and $\bar{H}=H/H \cap K^P$.*

Proof. Let θ_1 and θ_2 be the homomorphisms mapping A and B onto $\bar{A}$ and $\bar{B}$ respectively with $\ker\theta_1=A \cap K^P$ and $\ker\theta_2=B \cap K^P$. Since $H \cap (A \cap K^P)= H \cap K^P$, it follows that $H_1=H\theta_1\approx H/H \cap K^P$. In the same way $H_2=H\theta_2\approx H/H \cap K^P$. Thus $H_1 \approx H_2$. Let $Q=(\bar{A} * \bar{B})_{H_1=H_2}$ where H_1 and H_2 are identified under the natural isomorphism. Now by [2] θ_1 and θ_2 can be extended to a homomorphism ψ of P onto Q. It is not difficult to verify that $\ker \psi = \ker \theta$ whence the lemma follows.

THEOREM 3.2. *Let $P=(A * B)_H$ where $H=\langle h\rangle$ is an infinite cyclic subgroup. If for every strictly descending sequence of subgroups $H=K_0 \supset K_1 \supset \cdots \supset K_i \supset \cdots$, we have $\langle K_{i-1}, K_i^P\rangle \subsetneqq K_{i-1}^P$ for all i, then P is Φ-free.*

Proof. Let p be any prime. Let $K_i=\langle h^{p^i}\rangle$ with $i=0, 1, 2, \ldots$ and $N_i=K_i^P$. Also let $P_i=P/N_i$, $A_i=A/A \cap N_i$, $B_i=B/B \cap N_i$ and $H_i=H/H \cap N_i$. Now by Lemma 3.1, $P_i \approx (A_i * B_i)_{H_i}=Q_i$. Clearly $|H_i|=p^i$. Moreover $\langle K_{i-1}, K_i^P\rangle \subsetneqq K_{i-1}^P$ implies that K_{i-1} is not normal in P. It follows that H_i contains no nontrivial normal subgroup of Q_i, whence by Theorem 2.3, Q_i is Φ-free. But this implies that P_i is Φ-free. Thus $\Phi(P) \subseteq N_i$ for all i. Hence $\Phi(P) \subseteq \bigcap_{i=1}^\infty N_i=(\bigcap_{i=1}^\infty K_i)^P=1$.

COROLLARY 3.3. *The generalized free product P of any free groups amalgamating a cyclic subgroup which is not normal in P is Φ-free.*

Corollary 3.3 is also implicitly proved in [4].

In the case when the amalgamated subgroup contains a nontrivial normal subgroup of the generalized free product, then Theorem 2.4 is about the most we can say about its Frattini subgroup. The following example will illustrate the point. This example in fact is a modification of Baumslag's example in [4], which was also constructed independently by the author before he came to know Whittemore's work [4].

Let $A_i=\langle a_i\rangle$ be infinite cyclic groups. Let p_i denote the i^{th} prime. For each i define $H_i=\langle a_i^{p_{2i}}\rangle$. Let

$$A = \left(\prod_{i=1}^{\infty} * A_i\right)_H$$

where H is the amalgamated subgroup under the identification $a_i^{p_{2i}}=a_j^{p_{2j}}$ for all i, j. We shall show that $\Phi(A)=1$.

Since $H \lhd A$ we have $\Phi(A) \subseteq H$. Let $H=\langle h\rangle$ where $h=a_i^{p_{2i}}$. We shall show that $h^\alpha \notin \Phi(A)$ for any nonzero α. For any $\alpha\neq 0$, there exists k such that $p_{2k+1}\nmid\alpha$. Consider the group $S=\langle h^\alpha, a_i^{p_{2k+1}}; i=1, 2, \ldots\rangle$. Since $(\alpha, p_{2k+1})=1$, it follows that $\langle h^\alpha, a_i^{p_{2k+1}}\rangle=\langle a_i^{\alpha p_{2i}}, a_i^{p_{2k+1}}\rangle=A_i$. Thus $S=A$. Suppose now $h^\alpha=a_{\alpha_1}^{\beta_1 p_{2k+1}} \cdots a_{\alpha_n}^{\beta_n p_{2k+1}}$. It is not difficult to see that $a_{\alpha_i}^{\beta_i p_{2k+1}} \in H$. Thus $p_{2\alpha_i} \mid \beta_i$. Let $\gamma_i \cdot p_{2\alpha_i}=\beta_i$. Then $a_{\alpha_i}^{\beta_i p_{2k+1}}=h^{\gamma_i p_{2k+1}}$. Hence $h^\alpha=h^{\gamma p_{2k+1}}$ where $\gamma=\sum_{i=1}^n \gamma_i$. Since H is infinite cyclic this means $\alpha=p_{2k+1}\gamma$ contradicting the choice of p_{2k+1} whence $\langle a_i^{p_{2k+1}}; i=1, 2, \ldots\rangle\neq A$. Hence $\Phi(A)=1$.

Now let $B_i=\langle b_i\rangle$ be infinite cyclic groups. Let $K_i=\langle b_i^{p_{2i-1}}\rangle$. Let $B=(\prod_{i=1}^\infty * B_i)_K$ where K is the amalgamated subgroup under the identification $b_i^{p_{2i-1}}=b_j^{p_{2j-1}}$ for all i, j. Then $\Phi(B)=1$. Let $P=(A * B)_{H=K}$ with $a_i^{p_{2i}}=b_j^{p_{2j-1}}$. It is easy to see that P is the same group as in Baumslag's example in §2.5 [4]. Thus $\Phi(P)=H$. By modifying the construction, we can make $\Phi(P) \subsetneqq H$.

References

1. G. Higman and B. H. Neumann, *On two questions of Itô*, J. London. Math. Soc. **29** (1954), 84–88.

2. B. H. Neumann, *An essay on free products of groups with amalgamations*, Philos. Trans. Roy. Soc. London Ser. A, **246** (1954), 503–554.

3. W. R. Scott, *Group theory*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

4. A. Whittemore, *On the Frattini subgroups*, Trans. Amer. Math. Soc. (406) **141** (1969), 323–334.

University of Waterloo,
Waterloo, Ontario

https://doi.org/10.4153/CMB-1972-099-5 Published online by Cambridge University Press