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ON A CONTINUOUS-STATE POPULATION-SIZE-DEPENDENT BRANCHING PROCESS AND ITS EXTINCTION

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Abstract

A continuous-state population-size-dependent branching process $\{X_t\}$ is a modification of the Jiřina process. We prove that such a process arises as the limit of a sequence of suitably scaled population-size-dependent branching processes with discrete states. The extinction problem for the population X_t is discussed, and the limit distribution of X_t/t obtained when X_t tends to infinity.

Keywords: Population-size-dependent branching process; finite-dimensional distribution; convergence; extinction; limit distribution

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1. Introduction

A time-homogeneous, nonnegative integer-valued Markov chain $\{X(t),\ t = 0, 1, \dots\}$ is called a population-size-dependent branching process (PSDBP) with discrete states if its transition probabilities are given by

$$\mathrm{E}[s^{X(t+1)} \mid X(t) = k] = f(k, s)^k, \qquad k = 0, 1, \dots,$$

where $0 \le s \le 1$ and

$$f(k, s) = \sum_{j=0}^{\infty} {}_k p_j s^j, \quad \text{with } {}_k p_j \ge 0 \text{ such that } \sum_{j=0}^{\infty} {}_k p_j = 1, \qquad k = 0, 1, \dots,$$

is a family of probability generating functions governing the reproduction (see, e.g. [3] and [8]). In this paper, we call $f(k, s)$ the reproduction generating function of $\{X(t)\}$. Compared with the classical Galton–Watson process (see, e.g. [1]), the PSDBP takes into account the fact that the reproductive behavior may depend on the current size of the population. This process has been studied by a number of authors and has been applied in a variety of contexts. In a series of papers, Klebaner [8], [9], [10] studied the extinction problem of the process and the convergence of $X(t)/C(t)$ as $t \to \infty$, for suitable normalizing constants $C(t)$. Jagers [4] gave sufficient conditions for exponent growth for general branching processes, and as a corollary obtained the corresponding results for the PSDBP. Lu and Jagers [15] and Wang [18], [19] studied PSDBPs in random environments. See [5], [7], and [11] for more results on the discrete-state PSDBP.

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In this paper, we study a continuous-state version of the PSDBP. Let

$$\{X_n(t),\ t = 0, 1, \dots\}$$

be a sequence of PSDBPs. We prove that, under suitable assumptions, the finite-dimensional distributions of $\{X_n(t)/n\}$, $n = 1, 2, \dots$, converge to those of a $[0, \infty)$-valued Markov process $\{Y(t)\}$, as $n \to \infty$. We also discuss the extinction problem for $\{Y(t)\}$ and the limiting behavior of $Y(t)/t$ as $t \to \infty$. The process $\{Y(t)\}$ is the so-called continuous-state PSDBP, defined as follows.

A time-homogeneous Markov process $\{Y(t),\ t = 0, 1, \dots\}$ with state space $[0, \infty)$ is called a continuous-state PSDBP if the Laplace transform of its one-step transition probability $P(x, \mathrm{d}y)$ is given by

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} P(x, \mathrm{d}y) = \exp\{-xF(x, \lambda)\}, \tag{1.1}$$

where

$$F(x, \lambda) = \beta(x)\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda u})\nu(x, \mathrm{d}u), \tag{1.2}$$

with $\beta(x)$ a nonnegative Borel function on $[0, \infty)$ and $(1 \wedge s)\nu(x, \mathrm{d}s)$ a finite kernel from $[0, \infty)$ to $(0, \infty)$. In particular, if $\beta(x) \equiv \beta$ and $\nu(x, \mathrm{d}s) \equiv \nu(\mathrm{d}s)$ are both independent of $x \in [0, \infty)$, the process $\{Y(t)\}$ degenerates to the classical Jiřina process (see, e.g. [6] and [17]). Thus, we can regard $\{Y(t)\}$ as a modification of the Jiřina process whose reproductive behavior depends on the size of the population. Obviously, for each $x \in [0, \infty)$,

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q(x, \mathrm{d}y) = \exp\{-F(x, \lambda)\}, \tag{1.3}$$

defines an infinitely divisible probability measure $Q(x, \mathrm{d}y)$. In view of (1.1) and (1.3), we have

$$P(x, \cdot) = Q(x, \cdot)^x,$$

where the right-hand side denotes the x-fold convolution introduced in [16, p. 35]. We shall call the function $F(x, \lambda)$ defined by (1.2) the reproduction cumulative function (RCF) of $\{Y(t)\}$.

The continuous-state PSDBP can be used to model a number of biological situations and chemical reactions. For example, for some bacteria whose reproduction depends on their concentration in the medium, it is suitable to describe the change of the concentration using a continuous-state PSDBP. See [9], [12], and the references therein for more examples.

In the remainder of the paper, we write $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{N}_+ = \{1, 2, \dots\}$. For each $n \in \mathbb{N}_+$, let $\{X_n(t)\}_{t\in\mathbb{N}}$ be a discrete-state PSDBP with reproduction generating function $f_n(k, s)$. Then $Y_n(t) = n^{-1}X_n(t)$ defines a Markov chain $\{Y_n(t)\}_{t\in\mathbb{N}}$ on $\mathbb{Q}_n := \{k/n,\ k \in \mathbb{N}\}$. For any $i_n \in \mathbb{Q}_n$, let

$$F_n(i_n, \lambda) = n[1 - f_n(ni_n, 1 - \lambda/n)], \qquad \lambda \le n. \tag{1.4}$$

In Section 2, we show that the continuous-state PSDBP arises as the limit of a sequence of suitably scaled PSDBPs with discrete states. In Section 3, we discuss the extinction problem and prove a limit theorem for the continuous-state PSDBP.

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2. The sequential limit theorems of the discrete-state PSDBP

In this section, we discuss the relation between the discrete-state PSDBP and the continuous-state PSDBP (see Theorems 2.1–2.2). At the end of the section we give two examples to illustrate the theorems.

Proposition 2.1. *Let $F_n(x, \lambda)$ be defined by (1.4) and let $F(x, \lambda)$ be a continuous function of $(x, \lambda) \in [0, \infty) \times [0, \infty)$. Suppose that*

(A1) *for any i_n, $\mathbb{Q}_n \ni i_n \to x$, we have $F_n(i_n, \lambda) \to F(x, \lambda)$ as $n \to \infty$.*

Then $F(x, \lambda)$ is the RCF of a continuous-state PSDBP.

Proof. It is sufficient to prove that $F(x, \lambda)$ has the representation (1.2). Since $F(x, 0) = \lim_{n\to\infty} F_n(i_n, 0) = 0$, for any $x \in [0, \infty)$ we may apply Corollary 2 of [13] to obtain the representation

$$F(x, \lambda) = \int_{[0,\infty)} (1 - \mathrm{e}^{-\lambda u})(1 - \mathrm{e}^{-u})^{-1} G(x, \mathrm{d}u), \tag{2.1}$$

where $G(x, \mathrm{d}u)$ is a finite measure on $[0, \infty)$ and the value of the integrand at $u = 0$ is defined as λ. Clearly, the representation (2.1) can be written as (1.2) with $\beta(x) = G(x, \{0\}) \geq 0$ and $\nu(x, \mathrm{d}u) = (1 - \mathrm{e}^{-u})^{-1} G(x, \mathrm{d}u)$. It follows that

$$\int_{(0,\infty)} (1 \wedge u)\nu(x, \mathrm{d}u) < \infty.$$

By the continuity assumption on $F(x, \lambda)$, the measure $G(x, \cdot)$ depends continuously on $x \in [0, \infty)$, by the weak convergence topology. In particular, $G(x, \mathrm{d}u)$ is a kernel from $[0, \infty)$ to $[0, \infty)$. Then $\beta(\cdot)$ is a measurable function on $[0, \infty)$ and $\nu(x, \mathrm{d}u)$ is a kernel from $[0, \infty)$ to $(0, \infty)$.

Lemma 2.1. *Under the conditions of Proposition 2.1, assume in addition that*

(A2) *for any i_n, $\mathbb{Q}_n \ni i_n \to x$, we have $f_n'(ni_n, 1-)/n \to 0$ as $n \to \infty$, where $f_n'(k, s) = \mathrm{d} f_n(k, s)/\mathrm{d}s$.*

Then

$$H_n(i_n, \lambda) := -n \log f_n(ni_n, \mathrm{e}^{-\lambda/n}) \to F(x, \lambda) \tag{2.2}$$

as $n \to \infty$.

Proof. The assumption implies that $f_n(ni_n, 1 - \lambda/n) \to 1$. By the mean value theorem, we have

$$\begin{aligned} 0 &\leq |H_n(i_n, \lambda) - n\log f_n(ni_n, 1 - \lambda/n)| \\ &\leq \frac{n|f_n(ni_n, \mathrm{e}^{-\lambda/n}) - f_n(ni_n, 1 - \lambda/n)|}{f_n(ni_n, 1 - \lambda/n)} \\ &\leq \frac{nf_n'(ni_n, 1-)|\mathrm{e}^{-\lambda/n} - 1 + \lambda/n|}{f_n(ni_n, 1 - \lambda/n)}, \end{aligned}$$

where the right-hand side goes to 0 because $f_n'(ni_n, 1-)/n \to 0$. Consequently,

$$|H_n(i_n, \lambda) - n\log f_n(ni_n, 1 - \lambda/n)| \to 0. \tag{2.3}$$

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By using a Taylor expansion, we find that there is a constant $\theta_n \in (f_n(ni_n, 1-\lambda/n), 1)$ such that

$$\log f_n(ni_n, 1-\lambda/n) = f_n(ni_n, 1-\lambda/n) - 1 - \tfrac{1}{2}\theta_n^{-2}[f_n(ni_n, 1-\lambda/n) - 1]^2.$$

It follows that

$$\lim_{n\to\infty} n\log f_n(ni_n, 1-\lambda/n) = \lim_{n\to\infty} n[f_n(ni_n, 1-\lambda/n) - 1] = -F(x, \lambda). \tag{2.4}$$

By combining (2.3) with (2.4) we obtain the desired result.

Theorem 2.1. *Suppose that conditions (A1) and (A2) hold, and that the distribution of $Y_n(0)$ converges weakly to some distribution μ on $[0, \infty)$ as $n \to \infty$. The finite-dimensional distributions of $\{Y_n(t)\}$ then converge to those of a continuous-state PSDBP $\{Y(t)\}$ with RCF $F(x, \lambda)$ and initial distribution μ.*

Proof. By the Kolmogorov existence theorem, we can construct a Markov process $\{Y(t)\}_{t\in\mathbb{N}}$ on $[0, \infty)$ with initial distribution μ and one-step transition probability $P(x, \mathrm{d}y)$ characterized by (1.1). Let '$\xrightarrow{\mathrm{D}}$' denote convergence in distribution. By assumption, we have $Y_n(0) \xrightarrow{\mathrm{D}} Y(0)$. In the sequel, we assume that $(Y_n(0), \dots, Y_n(m)) \xrightarrow{\mathrm{D}} (Y(0), \dots, Y(m))$ and prove that

$$(Y_n(0), \dots, Y_n(m+1)) \xrightarrow{\mathrm{D}} (Y(0), \dots, Y(m+1)),$$

from which the theorem follows by induction.

It is sufficient to show that, for every $\lambda_i \geq 0$, $i = 0, 1, \dots, m+1$,

$$\mathrm{E}\biggl[\exp\biggl\{-\sum_{i=0}^{m+1} \lambda_i Y_n(i)\biggr\}\biggr] \to \mathrm{E}\biggl[\exp\biggl\{-\sum_{i=0}^{m+1} \lambda_i Y(i)\biggr\}\biggr] \tag{2.5}$$

for every $\lambda_i \geq 0$. Since $Y_n(t) = X_n(t)/n$, for any $i_n \in \mathbb{Q}_n$ we have

$$\begin{aligned}
&\mathrm{E}[\exp\{-\lambda_{m+1} Y_n(m+1)\} \mid Y_n(m) = i_n] \\
&\quad = \mathrm{E}[\exp\{-\lambda_{m+1} X^{(n)}(m+1)/n\} \mid X_n(m) = ni_n] \\
&\quad = [f_n(ni_n, \mathrm{e}^{-\lambda_{m+1}/n})]^{ni_n} \\
&\quad = \exp\{-i_n H_n(i_n, \lambda_{m+1})\},
\end{aligned} \tag{2.6}$$

where $H_n(\cdot, \cdot)$ is defined by (2.2). Let $\mathcal{F}_m^{(n)} = \sigma(\{Y_n(0), Y_n(1), \dots, Y_n(m)\})$. From (2.6) and the Markov property, we obtain

$$\mathrm{E}\biggl[\exp\biggl\{-\sum_{i=0}^{m+1} \lambda_i Y_n(i)\biggr\}\biggr] = \mathrm{E}\biggl[\exp\biggl\{-\sum_{i=0}^{m} \lambda_i Y_n(i)\biggr\} \exp\{-Y_n(m) H_n(Y_n(m), \lambda_{m+1})\}\biggr]. \tag{2.7}$$

Since $(Y_n(0), \dots, Y_n(m)) \xrightarrow{\mathrm{D}} (Y(0), \dots, Y(m))$, we can use the Skorokhod representation theorem (see [2, p. 102]) to construct random variables $(Z_n(0), \dots, Z_n(m))$ and $(Z(0), \dots, Z(m))$ such that

$$\begin{aligned}
(Z_n(0), \dots, Z_n(m)) &\stackrel{\mathrm{D}}{=} (Y_n(0), \dots, Y_n(m)), \\
(Z(0), \dots, Z(m)) &\stackrel{\mathrm{D}}{=} (Y(0), \dots, Y(m)),
\end{aligned} \tag{2.8}$$

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and $Z_n(i) \to Z(i)$ almost surely (a.s.) for every $i = 0, 1, \dots, m$, where '$\stackrel{\mathrm{D}}{=}$' means the related random variables have identical distributions. Since $\mathrm{P}((Y_n(0), \dots, Y_n(m)) \in \mathbb{Q}_n^{m+1}) = 1$, we have $\mathrm{P}((Z_n(0), \dots, Z_n(m)) \in \mathbb{Q}_n^{m+1}) = 1$. From (2.7) and (2.8), we obtain

$$\mathrm{E}\biggl[\exp\biggl\{-\sum_{i=0}^{m+1} \lambda_i Y_n(i)\biggr\}\biggr] = \mathrm{E}\biggl[\exp\biggl\{-\sum_{i=0}^{m} \lambda_i Z_n(i)\biggr\} \exp\{-Z_n(m) H_n(Z_n(m), \lambda_{m+1})\}\biggr]. \tag{2.9}$$

By Lemma 2.1 and the dominated convergence theorem, we see that the right-hand side of (2.9) converges to

$$\begin{aligned}
&\mathrm{E}\biggl[\exp\biggl\{-\sum_{i=0}^{m} \lambda_i Z(i)\biggr\} \exp\{-Z(m) F(Z(m), \lambda_{m+1})\}\biggr] \\
&\quad = \mathrm{E}\biggl[\exp\biggl\{-\sum_{i=0}^{m} \lambda_i Y(i)\biggr\} \exp\{-Y(m) F(Y(m), \lambda_{m+1})\}\biggr] \\
&\quad = \mathrm{E}\biggl[\exp\biggl\{-\sum_{i=0}^{m+1} \lambda_i Y(i)\biggr\}\biggr],
\end{aligned}$$

where the last equality follows from the Markov property of $\{Y(t)\}$. We thus obtain (2.5).

Proposition 2.2. *Let $F(x, \lambda)$ be a continuous RCF defined by (1.2). Suppose that one of the following conditions holds.*

(B1) *For a sufficiently small $\delta > 0$, we have*

$$\int_{(0,\infty)} (\delta \wedge u) \nu(x, \mathrm{d}u) \le \mathrm{e}^{-\beta(x)}$$

for all $x \in [0, \infty)$.

(B2) *The function*

$$x \mapsto \beta(x) + \int_{(0,\infty)} (1 \wedge u) \nu(x, \mathrm{d}u)$$

is bounded on $[0, \infty)$.

Then there exist probability generating functions $\{f_n(i, \cdot)\}$ such that condition (A1) is satisfied.

Proof. If condition (B1) is satisfied, we let

$$f_n(i, s) = \exp\biggl\{\beta\biggl(\frac{i}{n}\biggr)(s-1)\biggr\} + \frac{1}{n} \int_{(0,\infty)} (\mathrm{e}^{un(s-1)} - 1) \nu\biggl(\frac{i}{n}, \mathrm{d}u\biggr). \tag{2.10}$$

Then $f_n(i, 1) = 1$ and

$$\begin{aligned}
f_n(i, 0) &= \exp\biggl\{-\beta\biggl(\frac{i}{n}\biggr)\biggr\} - \frac{1}{n} \int_{(0,\infty)} (1 - \mathrm{e}^{-un}) \nu\biggl(\frac{i}{n}, \mathrm{d}u\biggr) \\
&\ge \exp\biggl\{-\beta\biggl(\frac{i}{n}\biggr)\biggr\} - \frac{1}{n} \int_{(0,\infty)} (1 \wedge un) \nu\biggl(\frac{i}{n}, \mathrm{d}u\biggr) \\
&= \exp\biggl\{-\beta\biggl(\frac{i}{n}\biggr)\biggr\} - \int_{(0,\infty)} (n^{-1} \wedge u) \nu\biggl(\frac{i}{n}, \mathrm{d}u\biggr).
\end{aligned}$$

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It follows that $f_n(i,0) \geq 0$ for sufficiently large $n \geq 1$. On the other hand, $f_n(i,s)$ is clearly analytic in $s \in (-1,1)$, with

$$\frac{\mathrm{d}^k f_n(i,s)}{\mathrm{d}s^k} = \beta\left(\frac{i}{n}\right)^k \exp\left\{\beta\left(\frac{i}{n}\right)(s-1)\right\} + n^{k-1}\int_{(0,\infty)} u^k \mathrm{e}^{un(s-1)}\nu\left(\frac{i}{n}, \mathrm{d}u\right) \geq 0.$$

Thus, $f_n(i,s)$ is a probability generating function. Let $F_n(i_n,\lambda)$ be defined by (1.4). It is easy to check that

$$\begin{aligned} F_n(i_n,\lambda) &= n(1-\mathrm{e}^{-\beta(i_n)\lambda/n}) + \int_{(0,\infty)} (1-\mathrm{e}^{-\lambda u})\nu(i_n,\mathrm{d}u) \\ &= n[1-\beta(i_n)\lambda/n - \mathrm{e}^{-\beta(i_n)\lambda/n}] + F(i_n,\lambda), \end{aligned}$$

which converges to $F(x,\lambda)$ if $i_n \to x$.

If condition (B2) holds, we choose an integer

$$K > \sup_{x\in[0,\infty)}\left[\beta(x) + \int_{(0,\infty)} (1\wedge u)\nu(x,\mathrm{d}u)\right],$$

and let

$$h_n(i,s) = 1 + \frac{\beta(i/n)}{K}(s-1) + \frac{1}{nK}\int_{(0,\infty)} (\mathrm{e}^{nu(s-1)}-1)\nu\left(\frac{i}{n},\mathrm{d}u\right). \tag{2.11}$$

It is easy to check that $h_n(i,s)$ is a probability generating function and that, if $i_n \to x$,

$$n\left[1 - h_n\left(ni_n, 1-\frac{\lambda}{n}\right)\right] = \frac{1}{K}F(i_n,\lambda) \to \frac{1}{K}F(x,\lambda). \tag{2.12}$$

Moreover, $f_n(i,s) := [h_n(i,s)]^K$ is also a probability generating function. Let $F_n(i_n,\lambda)$ again be defined by (1.4). By a Taylor expansion,

$$F_n(i_n,\lambda) = nK[1-h_n(ni_n,1-\lambda/n)] - \tfrac{1}{2}nK(K-1)\eta^{K-2}[h_n(ni_n,1-\lambda/n)-1]^2,$$

where $\eta \in (h_n(ni_n,1-\lambda/n),1)$. From (2.12) we have

$$\lim_{n\to\infty} F_n(i_n,\lambda) = \lim_{n\to\infty} nK[1-h_n(ni_n,1-\lambda/n)] = F(x,\lambda),$$

which completes the proof.

Remark 2.1. An analogue of condition (B2) was used in [14] in the study of the convergence of branching particle systems to measure-valued processes. Although this condition is more natural and easier to understand than condition (B1), it is not always satisfied, even in some simple cases, as we will see in Example 2.1. We also remark that if

$$\sup_{x\in[0,\infty)}\left[\beta(x) + \int_{(0,\infty)} \nu(x,\mathrm{d}u)\right] < \infty,$$

then both (B1) and (B2) are satisfied.

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Proposition 2.3. *Let $F(x,\lambda)$ be a continuous RCF defined by (1.2), and assume that the conditions of Proposition 2.2 hold. Suppose in addition that*

(B3) *the function $x \mapsto \int_{(0,\infty)} u\nu(x, \mathrm{d}u)$ is bounded on $[0,\infty)$.*

Then there exist probability generating functions $\{f_n(i,\cdot)\}$ such that conditions (A1) and (A2) are satisfied.

Proof. By Proposition 2.2, there exists a family of probability generating functions $\{f_n(i,\cdot)\}$ such that (A1) holds. If (B1) holds then $f_n(i,s)$ is defined by (2.10) and we have

$$f_n'(ni_n, 1-) = \beta(i_n) + \int_{(0,\infty)} u\nu(i_n, \mathrm{d}u).$$

We have the same equality if (B2) holds and $f_n(i,s) = h_n(i,s)^K$, with h_n given by (2.11). By (B3) and the continuity of $F(x,\lambda)$, it follows that $f_n'(ni_n, 1-)/n \to 0$ for any i_n such that $\mathbb{Q}_n \ni i_n \to x \in [0,\infty)$.

Theorem 2.1 and Proposition 2.3 imply the following result.

Theorem 2.2. *Let $\{Y(t)\}$ be a continuous-state PSDBP with continuous RCF $F(x,\lambda)$ satisfying the conditions of Proposition 2.3. Then there exists a sequence of discrete-state PSDBPs $\{X_n(t)\}$ such that the finite-dimensional distributions of $\{X_n(t)/n\}$ converge to those of $\{Y(t)\}$.*

Recall that a Jiřina process is the limit of a sequence of Galton–Watson processes (see, e.g. [6]). Thus, the above theorems generalize the classical result in some sense. In the remainder of this section, we give two examples to illustrate the theorems.

Example 2.1. Let $\{X_n(t)\}_{t\in\mathbb{N}}$ be a sequence of discrete-state PSDBPs with reproduction distributions

$${}_kp_j^{(n)} = \frac{\mathrm{e}^{-k/n}(k/n)^j}{j!}, \qquad j \ge 0.$$

Then

$$f_n(k,s) = \sum_{j=0}^{\infty} {}_kp_j^{(n)} s^j = \exp\left\{-\frac{k(1-s)}{n}\right\}$$

and, as $i_n \to x$, we have

$$F_n(i_n,\lambda) := n(1 - f_n(ni_n, 1-\lambda/n)) = n[1 - \mathrm{e}^{-i_n\lambda/n}] \to x\lambda,$$

and $f_n'(ni_n, 1)/n = i_n/n \to 0$. Suppose that $X_n(0) = n$. Then $Y_n(0) = X_n(0)/n \equiv 1$, and Theorem 2.1 implies that the finite-dimensional distributions of $\{Y_n(t)\}$ converge to those of a continuous-state PSDBP process $\{Y(t)\}$ with RCF $F(x,\lambda) = \lambda x$ and $Y(0) = 1$. Clearly, the limit process $\{Y(t)\}$ is deterministic. It is not hard to see that condition (B1) holds for $\{Y(t)\}$, but condition (B2) does not hold.

Example 2.2. Let $\alpha \in (0,1]$, $\beta > 0$, and $\gamma > 0$, with $\beta \le 1+\gamma$. Let $\{X_n(t)\}_{t\in\mathbb{N}}$ be a sequence of discrete-state PSDBPs with reproduction distributions given by

$${}_kp_j^{(n)} = \begin{cases} 1 - \dfrac{\beta\alpha^{k/n}}{1+k\gamma}, & j = 0, \\ \dfrac{\beta\alpha^{k/n}(k\gamma)^{j-1}}{(1+k\gamma)^{j+1}}, & j > 0. \end{cases}$$

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Then

$$f_n(k,s) = \sum_{j=0}^{\infty} {}_k p_j^{(n)} s^j = 1 - \frac{\beta\alpha^{k/n}(1-s)}{1+k\gamma-k\gamma s}.$$

If $i_n \to x$, we have

$$F_n(i_n,\lambda) = \frac{\beta\alpha^{i_n}\lambda}{1+i_n\gamma\lambda} \to F(x,\lambda) := \frac{\beta\alpha^x\lambda}{1+\gamma\lambda x},$$

and

$$\frac{f_n'(ni_n,1-)}{n} = \frac{\beta\alpha^{i_n}}{n} \to 0.$$

By Theorem 2.1, if $Y_n(0)$ converges in distribution then the finite-dimensional distributions of $\{Y_n(t)\}$ converge to those of a continuous-state PSDBP with RCF $F(x,\lambda)$. It is not hard to show that $F(x,\lambda)$ has the decomposition (1.2) with $\beta(x) = 0$ and

$$\nu(x,\mathrm{d}u) = \frac{\beta\alpha^x}{\gamma^2x^2}\mathrm{e}^{-u/\gamma x}\,\mathrm{d}u$$

for $x > 0$.

3. The extinction of the continuous-state PSDBP

In this section, we discuss the extinction problem for the continuous-state PSDBP $\{Y_t\}$. Since $\{Y_t\}$ has continuous states and its Laplace transform cannot be expressed explicitly, we cannot infer that Y_t tends to either 0 or ∞, as we do for the discrete-state PSDBP (see, e.g. [9]) and the classical continuous-state branching processes (see, e.g. [1]). However, we can still find some interesting properties (see Theorems 3.1–3.3) of the extinction, under certain conditions. We can also study the limit distribution of Y_t/t as Y_t tends to infinity (see Theorem 3.4).

Let $\{Y_t\}_{t\in\mathbb{N}}$ be a continuous-state PSDBP with RCF $F(x,\lambda)$ and initial state $Y_0 = x_0 \geq 0$. Let

$$F^{(k)}(x,\lambda) := \mathrm{d}^k F(x,\lambda)/\mathrm{d}\lambda^k,$$

and write $m_k(x) = (-1)^{k-1}F^{(k)}(x,0+)$ if the required derivative exists. Obviously, $m_k(x) \geq 0$ and

$$F(x,\lambda) \leq m_1(x)\lambda. \tag{3.1}$$

Lemma 3.1. *Suppose that $m_k(x) < M_k(x^{(k-3)\vee 0} \vee 1)$ for all $x > 0$ and $k \in \mathbb{N}_+$, where M_k is a positive constant depending only on k. Then, for any $k \geq 3$, there exists a positive constant A_k such that*

$$|\,\mathrm{E}[\{Y_{t+1} - m_1(Y_t)Y_t\}^k \mid Y_t]| \leq A_k(Y_t^{k-2} \vee 1). \tag{3.2}$$

Proof. We shall prove the inequality by induction on k. Note that

$$|\,\mathrm{E}[\{Y_{t+1} - m_1(Y_t)Y_t\}^k \mid Y_t]| = \left|\frac{\mathrm{d}^k \exp\{Y_t(\lambda m_1(Y_t) - F(Y_t,\lambda))\}}{\mathrm{d}\lambda^k}\right|_{\lambda=0+}. \tag{3.3}$$

It is easy to show that

$$|\,\mathrm{E}[\{Y_{t+1} - m_1(Y_t)Y_t\}^3 \mid Y_t]| = m_3(Y_t)Y_t < M_3Y_t.$$

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Now suppose that (3.2) holds for all i, $3 \le i < k$. From (3.3), by tedious calculations, we obtain

$$\begin{aligned}
&|\mathrm{E}[\{Y_{t+1} - m_1(Y_t)Y_t\}^k \mid Y_t]| \\
&\quad \le Y_t \sum_{j=2}^{k-3} \binom{k-1}{j-1} m_j(Y_t) |\mathrm{E}[\{Y_{t+1} - m_1(Y_t)Y_t\}^{k-j} \mid Y_t]| \\
&\qquad + Y_t m_k(Y_t) + \frac{(k-1)(k-2)}{2} Y_t^2 m_2(Y_t) m_{k-2}(Y_t) \\
&\quad \le \left(M_k + \sum_{j=2}^{k-3} \binom{k-1}{j-1} M_j A_{k-j} + \frac{(k-1)(k-2)}{2} M_2 M_{k-2} \right) (Y_t^{k-2} \vee 1), \qquad (3.4)
\end{aligned}$$

where the term containing the summation disappears if $k = 4$. Let

$$A_k = M_k + \sum_{j=2}^{k-3} \binom{k-1}{j-1} M_j A_{k-j} + \frac{(k-1)(k-2)}{2} M_2 M_{k-2}.$$

Then (3.2) follows from (3.4) for $i = k$.

Lemma 3.2. *Under the assumption of Lemma 3.1, for any $k \ge 2$ we have*

$$\mathrm{E}[Y_{t+1}^k \mid Y_t] = (m_1(Y_t)Y_t)^k + \binom{k}{2} m_1^{k-2}(Y_t) m_2(Y_t) Y_t^{k-1} + R(Y_t, k), \qquad (3.5)$$

where

$$\begin{aligned}
|R(Y_t, k)| &:= \left| \sum_{i=3}^{k} \binom{k}{i} \mathrm{E}[\{Y_{t+1} - m_1(Y_t)Y_t\}^i \mid Y_t](m_1(Y_t)Y_t)^{k-i} \right| \qquad (3.6) \\
&< B_k(Y_t^{k-2} \vee 1), \quad \textit{for some } B_k > 0.
\end{aligned}$$

Proof. When $k = 2$, we obtain (3.5) from

$$\mathrm{E}[Y_{t+1}^2 \mid Y_t] = m_1^2(Y_t) Y_t^2 + m_2(Y_t) Y_t.$$

Note that, for $k > 2$,

$$\begin{aligned}
\mathrm{E}[Y_{t+1}^k \mid Y_t] &= \mathrm{E}[\{Y_{t+1} - m_1(Y_t)Y_t + m_1(Y_t)Y_t\}^k \mid Y_t] \\
&= (m_1(Y_t)Y_t)^k + \binom{k}{2} m_1^{k-2}(Y_t) m_2(Y_t) Y_t^{k-1} + R(Y_t, k). \qquad (3.7)
\end{aligned}$$

Equations (3.2) and (3.6) imply that

$$|R(Y_t, k)| \le \sum_{i=3}^{k} \binom{k}{i} A_i M_1^{k-i} (Y_t^{i-2} \vee 1) Y_t^{k-i} \le (Y_t^{k-2} \vee 1) \sum_{i=3}^{k} \binom{k}{i} A_i M_1^{k-i}. \qquad (3.8)$$

By combining (3.7) with (3.8), we recover (3.5) with

$$B_k = \sum_{i=3}^{k} \binom{k}{i} A_i M_1^{k-i},$$

for all $k > 1$.

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Theorem 3.1. *If* $\sup_{x\in(0,\infty)} m_1(x) < 1$ *or* $m_1(x) \nearrow 1$ *as* $x \to \infty$, *then* $Y_t \to 0$ *a.s.*

Proof. Note that

$$\mathrm{E}[Y_{t+1} \mid Y_t] = Y_t m_1(Y_t) \le Y_t.$$

By the supermartingale convergence theorem, we know there exists a nonnegative integrable random variable Y such that

$$Y_t \to Y \quad \text{a.s.} \tag{3.9}$$

Consider the following two cases.

Case 1. $\sup_{x\in(0,\infty)} m_1(x) < 1$. Let $m = \sup_{x\in(0,\infty)} m_1(x) < 1$. For any $\lambda \ge 0$, by (3.1) and the dominated convergence theorem we have

$$\begin{aligned} 1 \ge \mathrm{E}[\mathrm{e}^{-\lambda Y}] &= \lim_{t\to\infty} \mathrm{E}[\mathrm{e}^{-\lambda Y_{t+1}}] = \lim_{t\to\infty} \mathrm{E}[\mathrm{E}[\mathrm{e}^{-\lambda Y_{t+1}} \mid Y_t]] = \lim_{t\to\infty} \mathrm{E}[\mathrm{e}^{-Y_t F(Y_t,\lambda)}] \\ &\ge \lim_{t\to\infty} \mathrm{E}[\mathrm{e}^{-m Y_t \lambda}] \ge \lim_{t\to\infty} \mathrm{E}[\mathrm{e}^{-m^2 Y_{t-1}\lambda}] \ge \cdots \ge \lim_{t\to\infty} \mathrm{E}[\mathrm{e}^{-x_0 m^{t+1}\lambda}] = 1, \end{aligned}$$

which implies that $Y = 0$ a.s.

Case 2. $m_1(x) \nearrow 1$. There exists a $\delta(x) > 0$ such that $m_1(x) + \delta(x) \nearrow 1$. From (3.9) we find that

$$\prod_{i=0}^{n-1} [m_1(Y_i) + \delta(Y_i)] \to 0 \quad \text{a.s.},$$

as $n \to \infty$. Let

$$X_t = \frac{Y_t}{\prod_{i=0}^{t-1}[m_1(Y_i) + \delta(Y_i)]}.$$

Obviously, X_t is a nonnegative supermartingale. There thus exists an integrable random variable X such that $X_t \to X$ a.s. Hence, $Y_t = X_t \prod_{i=0}^{t-1}[m_1(Y_i) + \delta(Y_i)] \to 0$ a.s.

Theorem 3.2. (i) *Suppose that* $\bar{m} := \inf_{x\in(0,\infty)} m_1(x) > 1$ *and* $m_2(x) < M$ *for some* $M > 0$. *Then* $Y_t \to Y \in [0,\infty]$ *a.s.*

(ii) *If* $\beta := \inf_{x\in(0,\infty)} \beta(x) > 1$ *and* $x_0 > 0$, *then* $Y_t \to \infty$ *a.s.*

(iii) *If* $\bar{\beta} := \sup_{x\in(0,\infty)} \beta(x) < 1$ *and, as* $\lambda \to \infty$,

$$\sup_{x\in(0,\infty)} \int_{(0,\infty)} \left(\frac{1}{\lambda} \wedge u\right) \nu(x, \mathrm{d}u) \to 0, \tag{3.10}$$

then there exist an r, $0 < r < 1$, *and a* λ_0, $\lambda_0 > 0$, *such that, for any* $\lambda \ge \lambda_0$, *we have*

$$F(x,\lambda) \le r\lambda. \tag{3.11}$$

(iv) *If, for* $\lambda \ge \lambda_0 > 0$, $F(x,\lambda)$ *satisfies (3.11), then* $Y_t \to Y \in [0,\infty]$ *a.s.*

Proof. (i) Since $F^{(3)}(x,\lambda) > 0$, by a Taylor expansion it is easy to show that

$$F(x,\lambda) \ge (m_1(x) - \tfrac{1}{2} m_2(x)\lambda)\lambda. \tag{3.12}$$

By combining (3.12) with the assumptions, we obtain

$$F(x,\lambda) \ge \alpha\lambda, \tag{3.13}$$

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for a sufficiently small $\lambda > 0$ and some constant $\alpha > 1$. From (3.13), we have

$$\mathrm{E}[\mathrm{e}^{-\lambda Y_{t+1}} \mid Y_t] = \mathrm{e}^{-Y_t F(Y_t,\lambda)} \le \mathrm{e}^{-\alpha\lambda Y_t} \le \mathrm{e}^{-\lambda Y_t}.$$

Thus, $\{\mathrm{e}^{-\lambda Y_t}\}$ is a nonnegative supermartingale. There exists a nonnegative random variable $X \in [0, 1]$ such that $\mathrm{e}^{-\lambda Y_t} \to X$ a.s. Hence, $Y_t \to Y = -\lambda^{-1}\log X \in [0, \infty]$ a.s.

(ii) Note that $F(x, \lambda) \ge \beta(x)\lambda \ge \beta\lambda$. By an argument similar to that used in proving (i), we have

$$Y_n \to Y \in [0, \infty] \quad \text{a.s.}$$

Now note that, for $\lambda > 0$,

$$\begin{aligned} 0 \le \mathrm{E}[\mathrm{e}^{-\lambda Y}] &= \lim_{t\to\infty} \mathrm{E}[\mathrm{e}^{-\lambda Y_{t+1}}] = \lim_{t\to\infty} \mathrm{E}[\mathrm{E}[\mathrm{e}^{-\lambda Y_{t+1}} \mid Y_t]] = \lim_{t\to\infty} \mathrm{E}[\mathrm{e}^{-Y_t F(Y_t,\lambda)}] \\ &\le \lim_{t\to\infty} \mathrm{E}[\mathrm{e}^{-\lambda Y_t \beta}] \le \lim_{t\to\infty} \mathrm{E}[\mathrm{e}^{-\lambda Y_{t-1}\beta^2}] \le \cdots \le \lim_{t\to\infty} \mathrm{E}[\mathrm{e}^{-x_0\beta^{t+1}\lambda}] = 0. \end{aligned}$$

Thus, $Y = \infty$ a.s.

(iii) Since $\bar{\beta} < 1$, there exists a $\delta_1 > 0$ such that $\bar{\beta} + \delta_1 < 1$. From (3.10), there exists a $\lambda_0 > 0$ such that, for $\lambda \ge \lambda_0$,

$$\sup_{x\in(0,\infty)} \int_{(0,\infty)} \left(\frac{1}{\lambda} \wedge u\right) \nu(x, \mathrm{d}u) < \delta_1.$$

Thus, (3.11) holds for $\lambda \ge \lambda_0$.

(iv) Suppose that (3.11) holds for $\lambda \ge \lambda_0 > 0$. Then

$$\mathrm{E}[\mathrm{e}^{-\lambda_0 Y_{t+1}} \mid Y_t] = \mathrm{e}^{-Y_t F(Y_t,\lambda_0)} \ge \mathrm{e}^{-r\lambda_0 Y_t} \ge \mathrm{e}^{-\lambda_0 Y_t}.$$

Hence, there exists a random variable $Y \in [0, \infty]$ such that $Y_t \to Y$ a.s.

The main ideas of the following two theorems come from [9].

Theorem 3.3. *If, for any $x \ge 0$,*

$$\int_0^\infty \mathrm{e}^{-xF(x,\lambda)-\lambda}\, \mathrm{d}\lambda \le \frac{1}{x+1},$$

then there exists a random variable Y such that $Y_t \to Y$ a.s. and $\mathrm{P}\{Y = 0\} \le 1/(x_0 + 1)$.

Proof. The proof is similar to that of Theorem 1.2 of [9]. We omit it here.

Remark 3.1. Theorem 3.1 and Theorem 3.2 discuss extinction for $m_1(x) < 1$ and $m_1(x) > 1$, respectively. Theorem 3.3 gives a nontrivial upper bound for the extinction probability.

In the following we will consider the distribution of $\lim_{t\to\infty} Y_t/t$ when $m_1(x) \to 1$ as $x \to \infty$ (with $m_1(x) > 1$) and $Y_t \to \infty$ as $t \to \infty$. To this end we let $\varepsilon(x) = m_1(x) - 1$ and consider the following two conditions:

(E1) $\qquad x\varepsilon(x) = c + \eta(x), \quad 0 < c < \infty, \qquad \lim_{x\to\infty} \eta(x) = 0,$

(E2) $\qquad \lim_{x\to\infty} m_2(x) = \sigma^2, \qquad 0 < \sigma < \infty.$

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Theorem 3.4. *Suppose that (E1) and (E2) hold. Under the assumption of Lemma 3.1, if $Y_t \to \infty$ a.s., then the distribution of Y_t/t converges to the $\Gamma(r,\alpha)$ distribution on $[0,\infty)$, where $\alpha = 2/\sigma^2$ and $r = 2c/\sigma^2$; that is,*

$$\lim_{t\to\infty} \mathrm{P}\{Y_t/t \le x\} = \int_0^x \frac{\alpha^r}{\Gamma(r)} s^{r-1} \mathrm{e}^{-\alpha s}\,\mathrm{d}s.$$

Proof. The proof is very similar to that of Theorem 1.5 of [9]. In the following we emphasize the differences.

By the method of moments, we shall prove that

$$\beta_k = \lim_{n\to\infty} \mathrm{E}[(Y_t/t)^k] < \infty,$$

with $\beta_1 = c$ and, for $k > 1$,

$$\beta_k = (c + \tfrac{1}{2}(k-1)\sigma^2)\beta_{k-1}.$$

We will use induction on k. For $k = 1$, by starting from (E1) and iterating, we find that

$$\mathrm{E}[Y_t] = x_0 + tc + \sum_{i=0}^{t-1} \mathrm{E}[\eta(Y_i)]. \tag{3.14}$$

From (3.14) it is easy to obtain $\beta_1 = \lim_{t\to\infty} \mathrm{E}[Y_t/t] = c$. Now suppose that, for all $i \le k-1$, $\beta_i = \lim_{t\to\infty} \mathrm{E}[(Y_t/t)^i] < \infty$. Let $\alpha_t^{(k)} = \mathrm{E}[Y_t^k]$ and $\gamma(x) = k\eta(x) + \binom{k}{2}(\sigma^2 - m_2(x))$. Then, by Lemma 3.2 and the argument in the proof of Theorem 1.5 of [9], we have

$$\alpha_{t+1}^{(k)} = \alpha_t^{(k)} + \left[kc + \binom{k}{2}\sigma^2\right]\alpha_t^{(k-1)} + \mathrm{E}[Y_t^{k-1}\gamma(Y_t)] + \mathrm{E}[D_{t,k}],$$

where $D_{t,k}$ is a random variable such that $|D_{t,k}| \le C_k(Y_t^{k-2} \vee 1)$ for some positive constant C_k. We point out that, in our case,

$$0 \le \lim_{t\to\infty} \mathrm{E}[|D_{t,k}/t^{k-1}|] \le C_k \lim_{t\to\infty} (\mathrm{E}[Y_t^{k-2}/t^{k-1}] + 1/t^{k-1}) = 0.$$

The remainder of the proof is similar to that of Theorem 1.5 of [9].

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