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CLASSICAL PROPERTIES OF COMPOSITION OPERATORS ON HARDY–ORLICZ SPACES ON PLANAR DOMAINS

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Abstract

In this paper we study composition operators on Hardy–Orlicz spaces on multiply connected domains whose boundaries consist of finitely many disjoint analytic Jordan curves. We obtain a characterization of order-bounded composition operators. We also investigate weak compactness and the Dunford–Pettis property of these operators.

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1. Introduction

Let $H(\Omega)$ denote the space of all holomorphic functions on Ω, where Ω is a domain on the Riemann sphere. For an analytic self-map φ of Ω ($\varphi \in H(\Omega)$, $\varphi(\Omega) \subset \Omega$), the composition operator is defined by the formula

$$C_\varphi f := f \circ \varphi, \quad f \in H(\Omega).$$

Composition operators play an important role in the study of many problems appearing in analysis in many cases. For example, a classical result due to Forelli (see [4]) states that all surjective isometries of the Hardy space $H^p(\mathbb{D})$, $1 < p < \infty$, $p \neq 2$, are weighted composition operators. The study of composition operators on various spaces of holomorphic functions was initiated at the beginning of the 20th century by the works of Hardy, Littlewood, and Riesz (see [1]). Over the years, questions related to analytic properties of C_φ have been a great motivation for the development of complex and functional analysis.

One of the famous problems was to characterize compact composition operators on Hardy spaces $H^p(\mathbb{D})$. In the 1980s, MacCluer (see [12] or [1]) proved that

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$C_\varphi\colon H^p(\mathbb{D}) \to H^p(\mathbb{D})$ is compact if and only if the pullback measure μ_φ defined by the formula

$$\mu_\varphi(B) := m(\varphi^{*-1}(B))$$

(where m is normalized Lebesgue measure on $\partial\mathbb{D}$ and φ^* is the radial limit of φ) is a vanishing Carleson measure, that is, $\mu_\varphi(W(a,h)) = o(h)$ as $h \to 0$ for any Carleson window $W(a,h) := \{z \in \mathbb{D} : 1 - h < |z| < 1, |\arg(\overline{a}z)| < h\}$, $a \in \partial\mathbb{D}$. We note that this problem was also solved by Shapiro [18, 19]. It should be emphasized that MacCluer's result (together with the famous Carleson lemma) contributes to the study of another class of operators—inclusion operators $j_\mu\colon H^p(\mathbb{D}) \to L^p(\mathbb{D}, \mu)$, where μ is a finite Borel measure. This idea is very useful and often used in contemporary research—recently composition operators acting between Hardy-type spaces have been studied thoroughly. The problems of characterizing compactness, weak compactness, absolute p-summability, and other properties were considered on various variants of Hardy spaces. We refer to [8–11], where the authors extended the results of Shapiro and MacCluer to the case of composition operators on Hardy–Orlicz spaces.

In this paper we investigate the composition operators on Hardy–Orlicz spaces on multiply connected domains Ω whose boundaries consist of finitely many disjoint analytic Jordan curves. These spaces are generalizations of classical Hardy spaces $H^p(\Omega)$ on multiply connected domains Ω introduced by Rudin in [15]. Notice that in spite of many similarities there are significant differences between the theory of Hardy spaces on the unit disc and on multiply connected domains (we refer to the paper of Sarason [17] and the book of Fisher [2], where $H^p(\Omega)$ spaces are studied). Our goal is to study order boundedness, weak compactness, and the Dunford–Pettis property of composition operators acting between Hardy–Orlicz spaces.

2. Preliminaries

Orlicz functions and Hardy–Orlicz spaces on discs. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be an Orlicz function, that is, a continuous and nondecreasing function such that $\lim_{t\to\infty} \Phi(t) = \infty$ and $\Phi(t) = 0$ if and only if $t = 0$.

We will consider Hardy–Orlicz spaces generated by Orlicz functions belonging to certain classes defined below. These classes have appeared in the monographs [6, 14] and in the paper [10].

The Orlicz function Φ satisfies the Δ_1-condition ($\Phi \in \Delta_1$) if there exist $x_0 > 0$ and $c > 0$ such that the inequality

$$\Phi(xy) \leqslant c\,\Phi(x)\Phi(y)$$

is satisfied for $x, y \geqslant x_0$. The Orlicz function Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$) if

$$\Phi(2x) \leqslant K\Phi(x)$$

for some constant $K > 1$ and x large enough. Notice that the condition $\Phi \in \Delta_1$ implies $\Phi \in \Delta_2$.

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The Orlicz function Φ satisfies the Δ^0-condition ($\Phi \in \Delta^0$) if there exists some $\beta > 1$ such that

$$\lim_{x \to +\infty} \frac{\Phi(\beta x)}{x} = +\infty.$$

The Orlicz function Φ satisfies the Δ^1-condition ($\Phi \in \Delta^1$) if there is some $\beta > 1$ such that the following inequality is satisfied:

$$x\Phi(x) \leqslant \Phi(\beta x)$$

for large x.

The Orlicz function Φ satisfies the Δ^2-condition ($\Phi \in \Delta^2$) if there is some $\alpha > 1$ such that

$$(\Phi(x))^2 \leqslant \Phi(\alpha x)$$

for large x. The condition $\Phi \in \Delta^2$ implies that $\Phi(x) \geqslant \exp(x^\alpha)$ for some $\alpha > 0$ and large x (see [14, Proposition 6]).

We say that the Orlicz function Φ satisfies the ∇_2-condition ($\Phi \in \nabla_2$) if there exists some $\beta > 1$ such that the inequality

$$\Phi(\beta x) \geqslant 2\beta\Phi(x)$$

is satisfied for large x. It is easy to show that $\Phi \in \nabla_2$ implies $\Phi(x)/x \to +\infty$ if $x \to +\infty$.

The Orlicz function Φ satisfies the ∇_1-condition ($\Phi \in \nabla_1$) if

$$\Phi(x)\Phi(y) \leqslant \Phi(bxy)$$

for some $b > 0$ and x, y large enough. We have the following implications (see [14, page 43]):

$$(\Phi \in \Delta^2) \Rightarrow (\Phi \in \Delta^1) \Rightarrow (\Phi \in \Delta^0) \Rightarrow (\Phi \in \nabla_2),$$
$$(\Phi \in \Delta^2) \Rightarrow (\Phi \in \nabla_1) \Rightarrow (\Phi \in \nabla_2).$$

Moreover, $\Phi \in \Delta^1$ does not imply $\Phi \in \nabla_1$, and $\Phi \in \nabla_1$ does not imply $\Phi \in \Delta^0$.

Given a measure space (Ω, Σ, μ), the Orlicz space $L^\Phi(\Omega) := L^\Phi(\Omega, \Sigma, \mu)$ is the space of all (equivalence classes of) Σ-measurable functions $f : \Omega \to \mathbb{C}$ for which there is a constant $\lambda > 0$ such that

$$\int_\Omega \Phi(\lambda |f|)\, d\mu < +\infty.$$

It is easy to check that if there exists $C > 0$ such that $\Phi(x/C) \leqslant \Phi(x)/2$ for all $x > 0$, then $L^\Phi(\Omega)$ is a quasi-Banach lattice equipped with the quasi-norm

$$\|f\|_\Phi := \inf\Big\{\lambda > 0;\ \int_\Omega \Phi\Big(\frac{|f|}{\lambda}\Big)\, d\mu \leqslant 1\Big\}.$$

It is well known that $\|\cdot\|_\Phi$ is a norm in the case when Φ is a convex function. For $\Phi(x) = x^p$, $p \in (0, +\infty]$, we have $L^\Phi(\Omega) = L^p(\Omega)$ and the norms coincide. We refer the reader to [14] for more complete information about Orlicz spaces.

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Let $\mathbb{D}$ be the unit disc of the complex plane. Throughout the paper, we identify $\partial\mathbb{D}$ with $\mathbb{T} = [0, 2\pi)$. In the same way as Hardy spaces $H^p(\mathbb{D})$ are defined from the Lebesgue spaces $L^p(\mathbb{T})$, so we define the Hardy–Orlicz spaces $H^\Phi := H^\Phi(\mathbb{D})$ from the Orlicz spaces $L^\Phi(\mathbb{T})$. For $f \in H(\mathbb{D})$ and $r \in (0, 1)$, denote by $f_r : \mathbb{T} \to \mathbb{C}$ the function given by $f_r(e^{\mathrm{it}}) = f(re^{\mathrm{it}})$. Following [13], H^Φ consists of analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{H^\Phi} := \sup_{0 \leqslant r<1} \|f_r\|_{L^\Phi(\mathbb{T})} < +\infty. \tag{2.1}$$

The formula (2.1) defines a quasi-norm in H^Φ and it is a norm when Φ is a convex function. We note that for every $f \in H^\Phi$, the radial limit

$$f^*(t) := \lim_{r\to 1^-} f(re^{\mathrm{it}}), \quad t \in \mathbb{T}$$

exists almost everywhere and $\|f\|_{H^\Phi} = \|f^*\|_{L^\Phi(\mathbb{T})}$. Recall that (see [10]) the inverse is also true: for a given $f^* \in L^\Phi(\mathbb{T})$ such that its Fourier coefficients $\widehat{f^*}(n)$ vanish for $n < 0$, the analytic extension

$$f(z) = P[f^*](z) := \sum_{n=0}^{\infty} \widehat{f^*}(n) z^n, \quad z \in \mathbb{D},$$

belongs to H^Φ and $\|f\|_{H^\Phi} = \|f^*\|_{L^\Phi(\mathbb{T})}$.

We denote by HM^Φ the subspace of finite elements of H^Φ, that is, the space of all $f \in H(\mathbb{D})$ such that for every $\lambda > 0$,

$$\sup_{0\leqslant r<1} \int_{\mathbb{T}} \Phi(\lambda |f(re^{\mathrm{it}})|)\, dt < +\infty.$$

Hardy–Orlicz spaces on planar domains. In this section we recall the definition of the Hardy–Orlicz spaces on planar domains and some basic properties of these spaces (see [16]). From now on, by an Orlicz function we mean a convex Orlicz function.

Let Ω be a domain on the Riemann sphere. Recall that a regular exhaustion of Ω is a sequence $\{\Omega_n\}_{n=1}^\infty$ of subdomains of Ω which satisfies the following conditions:

(1) $\overline{\Omega}_n \subset \Omega_{n+1}$ for $n \in \mathbb{N}$;
(2) $\bigcup_{n=1}^\infty \Omega_n = \Omega$;
(3) every component of $\partial\Omega_n$ is nontrivial for each $n \in \mathbb{N}$.

It can be proved that each domain has a regular exhaustion (see [2, Proposition 5.3]).

Recall that the Dirichlet problem is to find (if there exists) a function $\widetilde{u} : \overline{\Omega} \to \mathbb{R}$ which is continuous and satisfies two conditions:

(i) $\widetilde{u}$ is harmonic on Ω;
(ii) $\tilde{u} = u$ on Γ.

If the Dirichlet problem is solvable for the domain Ω, we write $\Omega \in (SDP)$. The Dirichlet problem can be solved for many domains. For our considerations, the following result is sufficient (see [2, Corollary 1.4.5]).

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Proposition 2.1. *If each component of $\partial\Omega$ is nontrivial, then the Dirichlet problem is solvable in Ω.*

We can define Hardy–Orlicz spaces on general domains. Let $\Phi\colon [0,+\infty) \to [0,+\infty)$ be a convex Orlicz function. Suppose that $\Omega \in (SDP)$, $\{\Omega_n\}$ is a regular exhaustion of Ω, and $z_0 \in \Omega_1$. Denote by ω_{n,z_0} the harmonic measure on $\partial\Omega_n$ for a point z_0. For $g\colon \Omega \to \mathbb{C}$, denote $g_n := g|_{\partial\Omega_n}$. We define the Hardy–Orlicz space on Ω by the following condition:

$$H^{\Phi}(\Omega) := \{f \in H(\Omega) : \|f\|_{H^{\Phi}(\Omega)} < +\infty\},$$

where

$$\|f\|_{H^{\Phi}(\Omega)} := \lim_{n\to\infty} \|f_n\|_{\Phi} = \lim_{n\to\infty} \inf\Big\{\varepsilon > 0 : \int_{\partial\Omega_n} \Phi\Big(\frac{|f_n|}{\varepsilon}\Big)\, d\omega_{n,z_0} \leqslant 1\Big\}.$$

We can also describe $H^{\Phi}(\Omega)$ in terms of a harmonic majorant. To do that, we need the following theorem (which can be easily proved by the Harnack theorem and the maximum modulus principle).

Theorem 2.2. *Let $p \in \Omega$, $\Omega \in (SDP)$, and let u be a subharmonic and continuous function on Ω. Then u has a harmonic majorant if and only if for each regular exhaustion $\{\Omega_n\}$ of Ω there exists a constant C such that*

$$\int_{\partial\Omega_n} u\, d\omega_{n,p} \leqslant C,$$

where $\omega_{n,p}$ is the harmonic measure on Ω_n for the point p.

Now we see that $H^{\Phi}(\Omega)$ is a set of all holomorphic functions f for which there exists $\lambda > 0$ such that the subharmonic function $\Phi(\lambda|f|)$ has a harmonic majorant. Moreover,

$$\|f\|_{H^{\Phi}(\Omega)} = \inf\{\varepsilon > 0 : v_{f,\varepsilon(z_0)} \leqslant 1\},$$

where $v_{f,\varepsilon}$ is the least harmonic majorant of $\Phi(|f|/\varepsilon)$. It is clear that $H^{\Phi}(\Omega)$ is a Banach space. We denote by $HM^{\Phi}(\Omega)$ the subspace of finite elements of $H^{\Phi}(\Omega)$, that is, the closure of $H^{\infty}(\Omega)$ in $H^{\Phi}(\Omega)$.

For further work we need an additional assumption on the domain Ω. Let Ω be a bounded domain whose boundary consists of $m + 1$ disjoint analytic Jordan curves, that is,

$$\Gamma := \partial\Omega = \bigcup_{k=0}^{m} \Gamma_k, \tag{2.2}$$

where Γ_k is an analytic Jordan curve and $\Gamma_k \cap \Gamma_j = \emptyset$ for $k \neq j$. Assume that Γ_0 is the boundary of the unbounded component of the complement of Ω. Denote by E_0 the bounded component of $S^2\backslash\Gamma_0$ and, for $k \in \{1, 2, \ldots, m\}$, denote by E_k the unbounded component of $\mathbb{C}_\infty\backslash\Gamma_k$, where $\mathbb{C}_\infty$ is the Riemann sphere. From now on, Ω will always be a set of this type. We also define by $H^{\Phi}_0(E_k)$ the subspace of $H^{\Phi}(E_k)$ which consists those functions which vanish at ∞. In the next step we recall some basic properties of the Hardy–Orlicz spaces on planar domains. For the proofs, we refer the reader to the paper [16].

Theorem 2.3. *For every $f \in H^{\Phi}(\Omega)$, we have the following decomposition:*

$$f(z) = f_0(z) + f_1(z) + \cdots + f_m(z), \quad z \in \Omega,$$

where $f_0 \in H^{\Phi}(E_0)$ and $f_k \in H_0^{\Phi}(E_k)$ for each $1 \leqslant k \leqslant m$. Moreover, the map $f \mapsto f_0$ is a bounded linear projection of $H^{\Phi}(\Omega)$ onto $H^{\Phi}(E_0)$ and $f \mapsto f_k$ is a bounded linear projection of $H^{\Phi}(\Omega)$ onto $H_0^{\Phi}(E_k)$.

Let ω_p be a harmonic measure on $\partial\Omega$. Notice that ω_p depends on the point $p \in \Omega$ but it can be shown that, for p and $q \in \Omega$, ω_p and ω_q are boundedly mutually absolutely continuous. Further, if K is a compact subset of Ω, then there is a constant M such that $\omega_q(E) \leqslant M\omega_p(E)$ for all $q \in K$ and all measurable sets $E \subset \partial\Omega$ (see [3] for more details). Thus, we sometimes leave out the lower index indicating the point which defines the harmonic measure.

Theorem 2.4. *Every $f \in H^{\Phi}(\Omega)$ has the boundary value f^* ω-almost everywhere on Γ and $f^* \in L^{\Phi}(\Gamma, \omega)$. Moreover,*

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f^*(w)}{w - z}\, dw, \quad z \in \Omega,$$

$$0 = \int_{\Gamma} \frac{f^*(w)}{w - z}\, dw, \quad z \notin \bar{\Omega},$$

$$f(z) = \int_{\Gamma} f^*(\zeta)\, d\omega_z(\zeta), \quad z \in \Omega.$$

Finally, the mapping $f \mapsto f^*$ is an isomorphism of $H^{\Phi}(\Omega)$ onto a closed subspace of $L^{\Phi}(\Gamma, \omega)$ and it is an isometry of $HM^{\Phi}(\Omega)$ onto a closed subspace of $L^{\Phi}(\Gamma, \omega)$.

Let us remark that for a domain that satisfies condition (2.2), we have $d\omega_z(\zeta) = -(1/2\pi)(\partial/\partial n_\zeta)g(\zeta, z)\, ds$, where $g(\cdot, z)$ is the Green's function for Ω with a pole at z, $(\partial/\partial n)$ is the derivative in the direction of the outward normal at $\partial\Omega$, and ds is the arc length (see [3] or [5]). Moreover, we have $c_1 < (d\omega_z(\zeta))/ds < c_2$, $\zeta \in \partial\Omega$, for some positive constants c_1, c_2. The function $P_z(\zeta) := -(1/2\pi)(\partial/\partial n_\zeta)g(\zeta, z)$, $\zeta \in \partial\Omega$, is called the *Poisson kernel*.

Notice that in the case when $\Phi \in \Delta_2$, we have $H^{\Phi}(\Omega) = HM^{\Phi}(\Omega)$. Thus, Theorem 2.3 improves the result of Rudin [15, Theorem 3.2] for the Hardy spaces $H^p(\Omega)$, $1 \leqslant p < \infty$, because, as we know, each power function $\Phi(t) = t^p$ satisfies the Δ_2-condition.

Using the characterization of $H^{\Phi}(\Omega)$ in terms of harmonic majorants, it is easy to show that conformal maps generate isometries between Hardy–Orlicz spaces on conformally equivalent domains. Let us remark that each domain Ω of the type (2.2) is conformally equivalent to the so-called *circular domain*, that is, the domain of the form

$$G := \mathbb{D} \setminus \bigcup_{i=1}^{m} (a_i + r_i\overline{\mathbb{D}}),$$

where the a_i belong to $\mathbb{D}$, $r_i \in (0, 1)$, and the circles $\partial\mathbb{D}, a_1 + r_1\partial\mathbb{D}, \ldots, a_m + r_m\partial\mathbb{D}$ do not intersect. Thus, we may assume that Ω is a circular domain. In this case, we have $E_i = \mathbb{C}_\infty\backslash(a_i + r_i\overline{\mathbb{D}})$ for $i = 1, \ldots, m$ and $E_0 = \mathbb{D}$. For further work we will need conformal maps from the unit disc onto E_i, $i \in \{0, \ldots, m\}$, which are of the form

$$\eta_i(z) = \begin{cases} \dfrac{r_i}{z} + a_i & \text{for } z \in \mathbb{D}\backslash\{0\}, \\ \infty & \text{for } z = 0, \end{cases}$$

$$\eta_i^{-1}(z) = \begin{cases} \dfrac{r_i}{z - a_i} & \text{for } z \in E_i\backslash\{\infty\}, \\ 0 & \text{for } z = \infty \end{cases}$$

for $1 \leqslant i \leqslant m$ and we put $\eta_0 = \mathrm{id}_{\mathbb{D}}$.

Now we introduce a useful family of functions defined on a circular domain. Recall that (see [10]) for $a \in \partial\mathbb{D}$ and $r \in (0, 1)$,

$$u_{a,r}(z) = \left(\frac{1 - r}{1 - \bar{a}rz}\right)^2$$

is a holomorphic function on $\mathbb{D}$ with $\|u_{a,r}\|_{H^1} \leqslant 1 - r$, $\|u_{a,r}\|_{H^\infty} = 1$, and $\|u_{a,r}\|_{H^\Phi(\mathbb{D})} \approx 1/(\Phi^{-1}(1/(1 - r)))$. We note also that if $z \in \mathbb{D}$ satisfies the inequality $|z - a| \leqslant 1 - r$, then

$$|u_{a,r}(z)| \geqslant \tfrac{1}{4}.$$

For $1 \leqslant i \leqslant m$ and a and r as above, we define

$$u^i_{a,r}(z) = (u_{a,r} \circ \eta_i^{-1})(z) = \left(\frac{1 - r}{1 - \frac{\bar{a}rr_i}{z - a_i}}\right)^2, \quad z \in \Omega,$$

and $u^0_{a,r} = u_{a,r}|_\Omega$. Note that for each $0 \leqslant i \leqslant m$, the function $u^i_{a,r}$ extends to a holomorphic function on E_i. It is also clear that we have an analogous norm estimation: $\|u^i_{a,r}\|_{H^1(E_i)} \leqslant 1 - r$, $\|u^i_{a,r}\|_{H^\infty(E_i)} = 1$, $\|u^i_{a,r}\|_{H^\Phi(E_i)} \approx 1/(\Phi^{-1}(1/(1 - r)))$, and

$$\|u^i_{a,r}\|_{H^1(\Omega)} \leqslant C(1 - r), \tag{2.3}$$

$$\|u^i_{a,r}\|_{H^\infty(\Omega)} \approx 1,$$

$$\|u^i_{a,r}\|_{H^\Phi(\Omega)} \approx \frac{1}{\Phi^{-1}(\frac{1}{1-r})} \tag{2.4}$$

for a positive constant C.

For further considerations, we will need estimations of the norm of the *evaluation functional* defined on the Hardy–Orlicz space on a circular domain. Recall that for any $z \in \Omega$, the *evaluation functional* $\delta_z \colon H^\Phi(\Omega) \mapsto \mathbb{C}$ is defined as follows:

$$\delta_z f := f(z), \quad f \in H^\Phi(\Omega).$$

It was proved in [10] that the norm of the evaluation functional $\delta_z \colon H^\Phi \mapsto \mathbb{C}$ at $z \in \mathbb{D}$ satisfies the inequalities

$$\frac{1}{4}\Phi^{-1}\left(\frac{1}{1 - |z|}\right) \leqslant \|\delta_z\| \leqslant 4\Phi^{-1}\left(\frac{1}{1 - |z|}\right). \tag{2.5}$$

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We denote this fact in short writing $\|\delta_z\| \approx \Phi^{-1}(1/(1-|z|))$. We will show that similar estimations are true in the case of $\delta_z\colon H^{\Phi}(\Omega) \mapsto \mathbb{C}$, $z \in \Omega$, where Ω is a circular domain.

Proposition 2.5. *Let Ω be a circular domain and let Φ be a convex Orlicz function. Then for each $z \in \Omega$ we have the following estimation:*

$$\|\delta_z\|_{H^{\Phi}(\Omega)^*} \approx \Phi^{-1}\Big(\frac{1}{\operatorname{dist}(z, \partial\Omega)}\Big). \tag{2.6}$$

Proof. First we prove that if $w \in E_i$, $1 \leqslant i \leqslant m$, then for every $g \in B_{H^{\Phi}(E_i)}$,

$$|g(w)| \leqslant C\Phi^{-1}\Big(\frac{1}{\operatorname{dist}(w, \partial E_i)}\Big) \tag{2.7}$$

for a constant $C > 0$. Indeed, put $f = g \circ \eta_i$; then we have $\|f\|_{H^{\Phi}(\mathbb{D})} = \|g\|_{H^{\Phi}(E_i)}$, so, if $z = \eta_i^{-1}(w) = r_i/(w - a_i)$, then

$$|g(w)| = |(f \circ \eta_i^{-1})(w)| = \Big|f\Big(\frac{r_i}{w - a_i}\Big)\Big|.$$

Now, using (2.5),

$$\begin{aligned}|g(w)| \leqslant 4\Phi^{-1}\Big(\frac{1}{1 - |r_i/(w - a_i)|}\Big) &= 4\Phi^{-1}\Big(\frac{|w - a_i|}{|w - a_i| - r_i}\Big)\\ &\leqslant 8\Phi^{-1}\Big(\frac{1}{\operatorname{dist}(w, \partial E_i)}\Big).\end{aligned}$$

Fix $z \in \Omega$. By Theorem 2.3 and inequality (2.7), for each $f \in B_{H^{\Phi}(\Omega)}$,

$$\begin{aligned}|f(z)| \leqslant |f_0(z)| + \cdots + |f_m(z)| &\leqslant 8\sum_{i=0}^{m} \Phi^{-1}\Big(\frac{1}{\operatorname{dist}(z, \partial E_i)}\Big)\\ &\leqslant 8\sum_{i=0}^{m} \Phi^{-1}\Big(\frac{1}{\operatorname{dist}(z, \partial\Omega)}\Big)\end{aligned}$$

and this proves the upper estimate in (2.6). To prove the lower estimate, let us notice that if $z \in \Omega$ satisfies the inequality $|z - p| \leqslant 1 - s$, where $s \in (0, 1)$ and $p \in \partial\mathbb{D}$, then

$$|u^0_{p,s}(z)| \geqslant \tfrac{1}{4}.$$

On the other hand, for $i \in \{1, \ldots, m\}$, we have $u^i_{p,s} = u^0_{p,s} \circ \eta_i^{-1}$, so the inequalities

$$|u^i_{p,s}(w)| \geqslant \tfrac{1}{4}$$

remain true for $w \in A^i_{p,s} := \{z \in \Omega\colon |a_i + r_i\overline{p} - z| \leqslant (1 - s)r\}$, where $r = \min_{1\leqslant i\leqslant m} r_i$. Now, if we write $z \in \Omega$ in the form $z = a_i + ((1 - s)r + r_i)\bar{p}$,

$$\frac{1}{4} \leqslant |u^i_{p,s}(z)| \leqslant \|\delta_z\|_{H^{\Phi}(\Omega)^*}\|u^i_{p,s}\|_{H^{\Phi}(\Omega)} \leqslant \frac{\|\delta_z\|_{H^{\Phi}(\Omega)^*}}{\Phi^{-1}(\frac{1}{1-s})}.$$

Using this fact and the following inequality:

$$\begin{aligned}\Phi^{-1}\Big(\frac{1}{1-s}\Big) &= \Phi^{-1}\Big(\frac{r}{(1-s)r}\Big)\\ &\geqslant \Phi^{-1}\Big(\frac{r}{|a_i + r_i\overline{p} - z|}\Big)\\ &\geqslant r\Phi^{-1}\Big(\frac{1}{\operatorname{dist}(z, \partial E_i)}\Big),\end{aligned}$$

it is easy to obtain the lower estimation in (2.6). □

3. Composition operators on $H^\Phi(\Omega)$

In this section we study the properties of composition operators on Hardy–Orlicz spaces on circular domains. Recall that if φ is a holomorphic self-map of Ω (in this case we write $\varphi \in \Upsilon := \Upsilon_\Omega$), then for $f \in H(\Omega)$ we define C_φ as follows:

$$(C_\varphi f)(z) := (f \circ \varphi)(z), \quad z \in \Omega.$$

In the paper [16], we investigated compactness of $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$. For a (convex) Orlicz function satisfying the ∇_2-condition, we characterized compact composition operators on $H^\Phi(\Omega)$ in terms of Carleson measures. Recall that the Carleson window on $\Gamma_i \subset \Gamma = \partial\Omega$ with center $\xi \in \Gamma_i$ and radius $0 < h < \min_{i\neq j} \operatorname{dist}(\Gamma_i, \Gamma_j)$ is the set

$$W_0(\xi, h) = \{z \in \overline{\Omega} : 1 - h < |z|, |\arg(\overline{\xi}z)| < h\}, \quad i = 0,$$
$$W_i(\xi, h) = \Big\{z \in \overline{\Omega} : |z - a_i| < \frac{r_i}{1-h}, |\arg(\overline{\xi}z)| < h\Big\}, \quad 1 \leqslant i \leqslant m.$$

A Carleson function ρ_φ is defined on the interval $(0, \min_{i\neq j} \operatorname{dist}(\Gamma_i, \Gamma_j))$ by the formula

$$\rho_\varphi(h) := \max_{0\leqslant i\leqslant m} \sup_{\xi\in\Gamma_i} \mu_\varphi(W_i(\xi, h)),$$

where μ_φ is the pullback measure, that is, $\mu_\varphi(B) = \omega((\varphi^*)(B))$ for any Borel set $B \subset \Omega$. Since the pullback measure is regular (see [16, Theorem 4.11]), we can replace in the definition of ρ_φ the Carleson windows by circles centered at $\xi \in \Gamma$ and with radii $h > 0$, intersected with $\overline{\Omega}$, that is, the sets $S(\xi, h) = \{z \in \overline{\Omega} \colon |z - \xi| < h\}$.

Theorem 3.1. *Let $\varphi \in \Upsilon$ and let $\Phi \in \nabla_2$ be an Orlicz function. The composition operator $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is compact if and only if*

$$\lim_{h\to 0^+} \frac{\Phi^{-1}(\frac{1}{h})}{\Phi^{-1}(\frac{1}{\rho_\varphi(h)})} = 0.$$

As a consequence, we obtain the following result.

Proposition 3.2. *Let $\varphi \in \Upsilon$ and let $\Phi \in \nabla_2$ be an Orlicz function. The composition operator $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is compact if and only if, for each $0 \leqslant i \leqslant m$,*

$$\lim_{s\to 1^-} \sup_{p\in\partial\mathbb{D}} \Phi^{-1}\Big(\frac{1}{1-s}\Big)\|C_\varphi u^i_{p,s}\|_{H^\Phi(\Omega)} = 0. \tag{3.1}$$

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Order-bounded composition operators. We recall that an operator $T: X \to Z$ from a Banach space X into a Banach subspace Z of a Banach lattice Y is order bounded if there is some positive $y \in Y$ such that $|Tx| \leqslant y$ for every x in the unit ball B_X of X.

Let Ω be a circular domain. For $p \in \Omega$, denote by $\omega := \omega_p$ the harmonic measure on $\partial\Omega$ with respect to p. By Theorem 2.4, we conclude that for every $f \in H^{\Phi}(\Omega)$, the boundary function f^* exists ω-almost everywhere on $\partial\Omega$ and $f^* \in L^{\Phi}(\partial\Omega, \omega)$. Let φ be a holomorphic self-map of Ω and let $C_\varphi: H^{\Phi}(\Omega) \to H^{\Phi}(\Omega)$ be a composition operator. Since each such operator is bounded (see [16]), the operator $\widetilde{C}_\varphi: H^{\Phi}(\Omega) \to L^{\Phi}(\partial\Omega, \omega)$ given by $\widetilde{C}_\varphi f = (C_\varphi f)^*$ is well defined.

Theorem 3.3. *Let Φ be an Orlicz function and $\varphi \in \Upsilon$. The composition operator $C_\varphi: H^{\Phi}(\Omega) \to H^{\Phi}(\Omega)$ induces an operator $\widetilde{C}_\varphi: H^{\Phi}(\Omega) \to H^{\Phi}(\partial\Omega)$ given by $\widetilde{C}_\varphi f = (C_\varphi f)^*$, which is order bounded in $L^{\Phi}(\partial\Omega)$ (respectively in $M^{\Phi}(\partial\Omega)$) if and only if $\operatorname{dist}(\varphi^*, \partial\Omega) > 0$ ω-almost everywhere and the function $\Phi^{-1}(1/(\operatorname{dist}(\varphi^*, \partial\Omega)))$ belongs to $L^{\Phi}(\partial\Omega)$ (respectively belongs to $M^{\Phi}(\partial\Omega)$).*

Proof. We present the proof in the case of $M^{\Phi}(\partial\Omega)$ (in the case of $L^{\Phi}(\partial\Omega)$, the same argument can be applied). Suppose that $g(\xi) := \Phi^{-1}(1/(\operatorname{dist}(\varphi^*(\xi), \partial\Omega)))$ belongs to $M^{\Phi}(\partial\Omega)$. By the estimate (2.6), we conclude that there exists a constant $C > 0$ such that, for every $f \in B_{H^{\Phi}(\Omega)}$, the inequality

$$|(C_\varphi f)^*(\xi)| = |f(\varphi^*(\xi))| \leqslant \|\delta_{\varphi^*(\xi)}\|_{(H^{\Phi}(\Omega))^*} \leqslant Cg(\xi)$$

is satisfied for ω-almost all $\xi \in \partial\Omega$, that is, C_φ is order bounded in $M^{\Phi}(\partial\Omega)$. Assume, conversely, that C_φ is order bounded in $M^{\Phi}(\partial\Omega)$. Then by the definition there exists a function $g \in M^{\Phi}(\partial\Omega)$ such that $|C_\varphi f| \leqslant g$. Using the harmonic measure ω_z to express $|f(\varphi(z))|$,

$$|\delta_{\varphi(z)} f| = |f(\varphi(z))| = \left| \int_{\partial\Omega} (C_\varphi f)^* \, d\omega_z \right| = \left| \int_{\partial\Omega} \widetilde{C_\varphi} f \, d\omega_z \right| \leqslant \int_{\partial\Omega} g \, d\omega_z$$

for every $z \in \Omega$ and every $f \in B_{H^{\Phi}(\Omega)}$. Now taking the supremum over $f \in B_{H^{\Phi}(\Omega)}$,

$$\|\delta_{\varphi(z)}\| \leqslant \int_{\partial\Omega} g \, d\omega_z$$

and, by (2.6),

$$\Phi^{-1}\Big(\frac{1}{\operatorname{dist}(\varphi^*(z), \partial\Omega)}\Big) \leqslant C \int_{\partial\Omega} g \, d\omega_z.$$

Notice that the function $z \mapsto \int_{\partial\Omega} g \, d\omega_z$ has the boundary function g for ω_z-almost all $\xi \in \partial\Omega$. Since $\Phi^{-1}(t) \to \infty$ as $t \to \infty$, we obtain that $\operatorname{dist}(\varphi^*(\xi), \partial\Omega) > 0$ for ω-almost all $\xi \in \partial\Omega$ and $|\Phi^{-1}(1/(\operatorname{dist}(\varphi^*, \partial\Omega)))| \leqslant Cg$ ω-almost everywhere, that is, $|\Phi^{-1}(1/(\operatorname{dist}(\varphi^*, \partial\Omega)))| \in M^{\Phi}(\partial\Omega)$. □

Theorem 3.4. *Let Ω be a circular domain, $\varphi \in \Upsilon$, and let Φ be an Orlicz function. Suppose that $\widetilde{C_\varphi}: H^{\Phi}(\Omega) \to M^{\Phi}(\partial\Omega)$ is order bounded. Then $C_\varphi: H^{\Phi}(\Omega) \to H^{\Phi}(\Omega)$ is compact.*

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Proof. Let $\{f_n\}$ be a sequence in the unit ball of $H^\Phi(\Omega)$ such that $f_n \to 0$ uniformly on compact subsets of Ω as $n \to \infty$. Let $g \in M^\Phi(\partial\Omega)$ be such a function that $|\widetilde{C_\varphi}h| \leqslant g$ for each $h \in B_{H^\Phi(\Omega)}$. By Theorem 3.3, we have that $\operatorname{dist}(\varphi^*, \partial\Omega) > 0$ ω-almost everywhere on $\partial\Omega$, which implies that $|f_n \circ \varphi^*| \to 0$ ω-almost everywhere on $\partial\Omega$. Since we also have $|\widetilde{C_\varphi}f_n| \leqslant g$, $f_n \circ \varphi^* \to 0$ in the norm of $L^\Phi(\partial\Omega, \omega)$. Using a variant of the classical Schwartz criterion (see [10, Proposition 3.8]), we deduce that C_φ is compact on $H^\Phi(\Omega)$. □

Now we show that for fast-growing Orlicz functions Φ the classes of compact and order-bounded composition operators coincide.

Theorem 3.5. *Let Ω be a circular domain, $\varphi \in \Upsilon$, and let Φ be an Orlicz function such that $\Phi \in \Delta^2$. The following assertions are equivalent:*

(1) $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ *is compact;*

(2) $\widetilde{C_\varphi} \colon H^\Phi(\Omega) \to M^\Phi(\partial\Omega)$ *is order bounded.*

Proof. We proved in the previous theorem that condition (2) implies (1). Now we show the inverse implication. More precisely, we prove that condition (3.1) in Proposition 3.2 implies (when $\Phi \in \Delta^2$) order boundedness of $\widetilde{C_\varphi} \colon H^\Phi(\Omega) \to M^\Phi(\partial\Omega)$. Assume that $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is a compact operator. In particular, we have (3.1), that is, for each $0 \leqslant i \leqslant m$,

$$\lim_{s\to 1^-} \sup_{p\in\partial\mathbb{D}} \Phi^{-1}\Big(\frac{1}{1-s}\Big)\|C_\varphi u^i_{p,s}\|_{H^\Phi(\Omega)} = 0.$$

This equality means that for every $\varepsilon > 0$, there exists $s_0 \in (0,1)$ such that for $s_0 < s < 1$, $p \in \partial\mathbb{D}$, and $0 \leqslant i \leqslant m$,

$$\int_{\partial\Omega} \Phi\Big(\frac{1}{\varepsilon}\Phi^{-1}\Big(\frac{1}{1-s}|u^i_{p,s} \circ \varphi^*|\Big)\Big)\,d\omega \leqslant 1. \tag{3.2}$$

Notice that the inequality $|u^0_{p,s}(z)| \geqslant \frac{1}{4}$ holds for all $z \in \Omega$ which satisfy $|z - p| \leqslant 1 - s$. Analogously, for $1 \leqslant i \leqslant m$ and $z \in \{z \in \Omega \colon |a_i + r_i\overline{p} - z| \leqslant (1-s)r_i\}$,

$$|u^i_{p,s}(z)| \geqslant \tfrac{1}{4}.$$

For $r = \min_{0\leqslant i\leqslant m} r_i$, we define

$$D^{p,s}_i := \{\xi \in \partial\Omega \colon |a_i + r_i\overline{p} - \varphi^*(\xi)| \leqslant (1-s)r\}, \quad 1 \leqslant i \leqslant m,$$
$$D^{p,s}_0 := \{\xi \in \partial\Omega \colon |p - \varphi^*(\xi)| \leqslant (1-s)r\}.$$

For s close to 1, the family $\{D^{p,s}_i\}_{i=0}^m$ contains pairwise-disjoint subsets of $\partial\Omega$. Using (3.2),

$$1 \geqslant \omega(D^{p,s}_i)\Phi\Big(\frac{1}{4\varepsilon}\Phi^{-1}\Big(\frac{1}{1-s}\Big)\Big).$$

Let $\Omega_h := \{z \in \Omega \mid \operatorname{dist}(z, \partial\Omega) < h\}$. Notice that Ω_h (for small $h > 0$) can be covered by fewer than C/h balls with radius $2h$ and centers at $\partial\Omega$ for a certain constant C (we can

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put $C = (2\pi/h)\sum_{i=0}^{m}(r_i/r))$. Using this fact and the equivalence of harmonic and arc length measures, and taking $2h = (1-s)r$,

$$\frac{C'}{h} \geqslant \omega(\{\xi \in \partial\Omega\colon\ \mathrm{dist}(\varphi^*(\xi), \partial\Omega) < h\})\Phi\Big(\frac{1}{4\varepsilon}\Phi^{-1}\Big(\frac{r}{2h}\Big)\Big).$$

By concavity of Φ^{-1},

$$\frac{C'}{h} \geqslant \omega\Big(\Big\{\xi \in \partial\Omega\colon\ \frac{1}{\mathrm{dist}(\varphi^*(\xi), \partial\Omega)} > \frac{1}{h}\Big\}\Big)\Phi\Big(\frac{r}{8\varepsilon}\Phi^{-1}\Big(\frac{1}{h}\Big)\Big).$$

Now put $g(\xi) = \Phi^{-1}(1/(\mathrm{dist}(\varphi^*(\xi), \partial\Omega)))$, $\xi \in \partial\Omega$, and $x = \Phi^{-1}(1/h)$. For x large enough,

$$\begin{aligned} C'\Phi(x) &\geqslant \omega(\{\xi \in \partial\Omega\colon\ g(\xi) > x\})\Phi\Big(\frac{rx}{8\varepsilon}\Big) \\ &\geqslant \omega(\{\xi \in \partial\Omega\colon\ g(\xi) > x\})\Big[\Phi\Big(\frac{rx}{8\alpha^2\varepsilon}\Big)\Big]^4. \end{aligned}$$

In the last inequality, we have twice used the condition Δ^2. For a given $B > 1$, we can find $\varepsilon > 0$ to have $B = r/(8\alpha^2\varepsilon^2)$. Then, for x big enough,

$$\omega(\{\xi \in \partial\Omega\colon\ \Phi(Bg(\xi)) > \Phi(Bx)\})[\Phi(Bx)]^4 \leqslant C'\Phi(x) \leqslant \Phi(Bx),$$

so, for large λ,

$$\omega(\{\xi \in \partial\Omega\colon\ \Phi(Bg(\xi)) > \lambda\}) \leqslant \frac{2\pi}{\lambda^3}.$$

Thus, the function $\Phi(Bg) \in L^{\Phi}(\partial\Omega, \omega)$ for all $B > 1$. Using Theorem 3.3 and the fact that $g \in M^{\Phi}(\partial\Omega)$, we deduce the order boundedness of $\widetilde{C_\varphi}\colon H^{\Phi}(\Omega) \to M^{\Phi}(\partial\Omega)$. □

Weak compactness and the Dunford–Pettis property of composition operators. In this section we will study weak compactness and the Dunford–Pettis property of composition operators. We start with the following criterion, which was proved in [7].

LEMMA 3.6 [7, Theorem 4]. *Assume that the Orlicz function Φ satisfies the Δ^0-condition and X is a subspace of M^{Φ}. Then the linear operator T mapping X into some Banach space Y is weakly compact if and only if for some $p \in [1, \infty)$ and all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\|Tf\| \leqslant C_\varepsilon\|f\|_p + \varepsilon\|f\|_{L^\Phi}$$

for every $f \in X$.

THEOREM 3.7. *Let Ω be a circular domain and $\varphi \in \Upsilon$. Assume that the Orlicz function Φ satisfies the Δ^0-condition. If the composition operator $C_\varphi\colon H^{\Phi}(\Omega) \to H^{\Phi}(\Omega)$ is weakly compact, then, for each $i \in \{0, \ldots, m\}$, the following condition is satisfied:*

$$\lim_{s\to 1^-}\sup_{p\in\partial\mathbb{D}}\Phi^{-1}\Big(\frac{1}{1-s}\Big)\|C_\varphi u^i_{p,s}\|_{H^\Phi(\Omega)} = 0. \tag{3.3}$$

Proof. We only need to use that the restriction of $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ to $HM^\Phi(\Omega)$ is weakly compact. From Lemma 3.6 and Theorem 2.4, one has that, for every $\varepsilon > 0$, there is a constant K_ε such that for each $f \in HM^\Phi(\Omega)$ the following inequality:

$$\|C_\varphi f\|_{H^\Phi(\Omega)} \leqslant K_\varepsilon \|f\|_{H^1(\Omega)} + \varepsilon \|f\|_{H^\Phi(\Omega)}$$

is satisfied. Using estimations (2.3) and (2.4) for $\varepsilon > 0$ and $f = u^i_{p,s}$,

$$\|C_\varphi u^i_{p,s}\|_\Phi \leqslant K_\varepsilon(1-s) + \varepsilon \frac{C_1}{\Phi^{-1}(\frac{1}{1-s})}.$$

Since $\lim_{x\to+\infty}(\Phi(x)/x) = +\infty$, $(1-s)\Phi^{-1}(1/(1-s)) \to 0$ for $s \to 1^-$. This easily implies (3.3). □

Applying Theorem 3.7 and Proposition 3.2, we obtain the following equivalence.

Corollary 3.8. *Let Ω be a circular domain and $\varphi \in \Upsilon$. Assume that the Orlicz function Φ satisfies the Δ^0-condition. Then the composition operator $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is compact if and only if it is weakly compact.*

Notice that $\Phi \in \Delta^2$ implies $\Phi \in \Delta^1$, which in turn implies $\Phi \in \Delta^0$. Hence, by Theorems 3.7, 3.5, and Proposition 3.2, we get the following result.

Theorem 3.9. *Let Ω be a circular domain and $\varphi \in \Upsilon$. Assume that Φ is an Orlicz function and $\Phi \in \Delta^2$. The following assertions are equivalent:*

(1) $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ *is compact;*
(2) $\widetilde{C_\varphi}\colon H^\Phi(\Omega) \to M^\Phi(\partial\Omega)$ *is order bounded;*
(3) $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ *is weakly compact;*
(4) *for each $i \in \{0, \dots, m\}$,*

$$\lim_{s\to1^-} \sup_{p\in\partial\mathbb{D}} \Phi^{-1}\Big(\frac{1}{1-s}\Big)\|C_\varphi u^i_{p,s}\|_{H^\Phi(\Omega)} = 0.$$

The next theorem describes the connections between compactness and weak compactness of composition operators defined on Hardy and Hardy–Orlicz spaces on circular domains.

Theorem 3.10. *Let Ω be a circular domain and $\varphi \in \Upsilon$. Assume that Φ is an Orlicz function and $\Phi \in \nabla_2$. If one of the following conditions:*

(1) $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ *is compact;*
(2) $\Phi \in \Delta^0$ *and* $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ *is weakly compact*

is satisfied, then the composition operator $C_\varphi\colon H^2(\Omega) \to H^2(\Omega)$ is compact.

Proof. Assume that $C_\varphi\colon H^2(\Omega) \to H^2(\Omega)$ is not compact. Hence, by Theorem 3.1 (recall that if $\Psi(t) = t^2$, then $H^\Psi(\Omega) = H^2(\Omega)$), there exist $\beta \in (0,1)$ and sequences $\{\xi_n\} \subset \partial\Omega$ and $\{h_n\} \subset (0,1)$ such that $h_n \to 0$ and $\mu_\varphi(S(\xi_n, rh_n)) \geqslant \beta h_n$ for every $n \in \mathbb{N}$, where $r = \min_{0\leqslant i\leqslant m} r_i$.

For $i \in \{0 \cdots m\}$, from the sequence $\{\xi_n\}$ we take a subsequence $\{\xi_n^i\}_n$ such that $\{\xi_n^i\}_n \subset \Gamma_i$. For $i \in \{1 \cdots m\}$, put $p_n^i = r_i/(\xi_n^i - a_i)$ and $p_n^0 = \xi_n^0$. It is obvious that $p_n^i \in \partial\mathbb{D}$ for $i \in \{0 \cdots m\}$ and $n \in \mathbb{N}$. Define $v_n^i(z) := u^i_{p_n^i, h_n}$, $z \in \Omega$. By (2.4),

$$\|v_n^i\|_{H^\Phi(\Omega)} \approx \frac{1}{\Phi^{-1}(\frac{1}{h_n})}, \quad n \in \mathbb{N}.$$

This implies that the sequence $\{g_n^i\}_n$ defined by $g_n^i := \Phi^{-1}(1/h_n)v_n^i$, $n \in \mathbb{N}$, is bounded in $HM^\Phi(\Omega)$ for each $i \in \{0 \cdots m\}$. Since $\Phi \in \nabla_2$, we have also $\lim_{x\to+\infty}(\Phi^{-1}(x))/x = 0$ and

$$h_n^2\Phi^{-1}\Big(\frac{1}{h_n}\Big) \to 0,$$

so the sequence $\{g_n^i\}$ tends to zero uniformly on compact subsets of Ω and $\|g_n^i\|_{H^1(\Omega)} \to 0$ because $\|g_n^i\|_{H^1(\Omega)} \leqslant h_n\Phi^{-1}(1/h_n)$. Then, in both cases, we would have $\|C_\varphi g_n^i\|_{H^\Phi(\Omega)} \to 0$. In the case (1), this follows from Proposition 3.2, but in the case (2) this follows from Lemma 3.6. We are going to show that this is not true. Indeed,

$$\begin{aligned}\int_{\partial\Omega} \Phi\Big(\frac{4}{\beta}|g_n^i \circ \varphi^*|\Big)\,d\omega &\geqslant \int_\Omega \Phi\Big(\frac{4}{\beta}\Phi^{-1}\Big(\frac{1}{h_n}\Big)|v_n^i(z)|\Big)\,d\mu_\varphi \\ &\geqslant \int_{S(\xi_n, rh_n)} \Phi\Big(\frac{4}{\beta}\Phi^{-1}\Big(\frac{1}{h_n}\Big)|v_n^i(z)|\Big)\,d\mu_\varphi.\end{aligned}$$

Notice that for $z \in S(\xi_n, rh_n)$, we have $|v_n^i(z)| > \frac{1}{4}$. By convexity of Φ and the fact that $\beta \in (0,1)$,

$$\begin{aligned}\int_{\partial\Omega} \Phi\Big(\frac{4}{\beta}|g_n^i \circ \varphi^*|\Big)\,d\omega &\geqslant \int_\Omega \frac{4}{\beta}\Phi\Big(\Phi^{-1}\Big(\frac{1}{h_n}\Big)|v_n^i(z)|\Big)\,d\mu_\varphi \\ &\geqslant \frac{1}{\beta h_n}\mu_\varphi(S(\xi_n, rh_n)) \geqslant 1.\end{aligned}$$

This implies that $\|C_\varphi g_n^i\|_{H^\Phi(\Omega)} \geqslant \beta/4$ and proves the theorem. □

Finally, we show that for a certain class of Orlicz functions the classes of composition operators with the Dunford–Pettis property and weakly compact composition operators coincide.

THEOREM 3.11. *Let Ω be a circular domain, $\varphi \in \Upsilon$, and let Φ be an Orlicz function with $\Phi \in \nabla_2$. Then the composition operator $C_\varphi: H^\Phi(\Omega) \to H^\Phi(\Omega)$ which is a Dunford–Pettis operator satisfies (3.1), that is,*

$$\lim_{s\to 1^-}\sup_{p\in\partial\mathbb{D}} \Phi^{-1}\Big(\frac{1}{1-s}\Big)\|C_\varphi u^i_{p,s}\|_{H^\Phi(\Omega)} = 0$$

for every $i \in \{0, \ldots, m\}$.

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Proof. Put $g^i_{p,s} = \Phi^{-1}(1/(1-s))u^i_{p,s}$ for $i \in \{0, \dots, m\}$. If (3.1) were not satisfied, we could find a sequence $\{p_n\} \subset \partial\mathbb{D}$ and a sequence of numbers $\{s_n\} \subset (0,1)$ with $\lim_{n\to\infty} s_n = 1$ such that $\|C_\varphi g^i_{p_n,s_n}\|_{H^\Phi(\Omega)} \geqslant \delta > 0$ for every $n \geqslant 1$ and some $0 \leqslant i \leqslant m$. But we have $(1-s)^2\Phi^{-1}(1/(1-s)) \to 0$ when $s \to 1^-$. Therefore, $g^i_{p_n,s_n} \to 0$ tends to zero uniformly on compact subsets of Ω. Since on the unit ball of $H^\Phi(\Omega)$ the weak-star topology is a topology of uniform convergence on compact subsets of Ω, we have that $g^i_{p_n,s_n} \to 0$ weakly as $n \to \infty$. Thus, $\|C_\varphi g^i_{p_n,s_n}\|_{H^\Phi(\Omega)} \to 0$ and we get a contradiction. □

Corollary 3.12. *Let Ω be a circular domain, $\varphi \in \Upsilon$, and let Φ be an Orlicz function with $\Phi \in \nabla_2$. Then the composition operator $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is a Dunford–Pettis operator if and only if it is compact.*

Proof. By the previous theorem, we know that if $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is a Dunford–Pettis operator, then it satisfies (3.1). By Proposition 3.2, this is equivalent to the compactness of $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$. The reverse implication is obvious. □

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