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The Kottman Constant for α-Hölder Maps

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Abstract. We investigate the role of the Kottman constant of a Banach space X in the extension of α-Hölder continuous maps for every $\alpha \in (0,1]$.

1 Introduction

If X is an infinite-dimensional Banach space, then the Kottman constant [6] of X is defined as

$$\kappa(X) := \sup_{(x_n)\in B(X)} \operatorname{sep}(x_n),$$

where for any sequence we define $\operatorname{sep}(x_n) = \inf_{n\neq m} \|x_n - x_m\|$, and $B(X)$ denotes the unit ball of X. A well-known result of Elton and Odell [2] asserts that $\kappa(X) > 1$ for every infinite-dimensional Banach space X. Let us introduce a second parameter associated with a Banach space X. We define the constant $\lambda_1(X, c_0)$ as the infimum of all $\lambda > 0$ such that for every subset M of X every Lipschitz map $f: M \to c_0$ admits an extension $F: X \to c_0$ with $\operatorname{Lip}(F) \le \lambda \operatorname{Lip}(f)$. Kalton proved the following unexpected result [5, Proposition 5.8].

Proposition 1.1 *For every infinite-dimensional Banach space X,*

$$\kappa(X) = \lambda_1(X, c_0).$$

The aim of this note is to observe that the proof of [5, Proposition 5.8] contains the natural extension for α-Hölder maps; see Proposition 2.2. As far as we know, the first time that the Kottman constant was linked with the extension of α-Hölder maps was in the proof of a result of Lancien and Randrianantoanina [7, Theorem 2.2]. Although the Kottman constant is not mentioned, its role during the proof is quite evident. We present also a proof of their result where the Kottman constant appears explicitly; see Proposition 2.3.

Recall that given metric spaces (X, d) and (Y, ρ), we say a map $f: X \to Y$ is α-Hölder for $\alpha \in (0,1]$ if

$$\operatorname{Lip}_\alpha(f) = \sup\left\{ \frac{\rho(f(x), f(y))}{d(x,y)^\alpha} : x \neq y \right\} < \infty.$$

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Therefore for each $\alpha \in (0,1]$ we may define the constant $\lambda_\alpha(X,Y)$ as the infimum of all $\lambda > 0$ such that for every subset M of X, every α-Hölder map $f: M \to Y$ admits an extension $F: X \to Y$ with $\operatorname{Lip}_\alpha(F) \le \lambda \operatorname{Lip}_\alpha(f)$.

The paper is organized as follows. Section 2 contains the natural extension of [5, Proposition 5.8] to α-Hölder maps which is our main result. It also contains the quantitative version of the result of Lancien and Randrianantoanina. Section 3 deals with L_q instead of c_0 and gives a lower bound for $\lambda_\alpha(X, L_q)$.

2 The Estimate for c_0

To prove our main result (Proposition 2.2) we need the following easy lemma.

Lemma 2.1 $\lambda_\alpha(X,c_0) \ge \kappa(X)^\alpha, \quad \alpha \in (0,1]$.

Proof Let us denote by simplicity $\kappa = \kappa(X)$. Given $\varepsilon > 0$, find $(x_n) \in B(X)$ such that $\kappa - \varepsilon \le \|x_n - x_m\|$, where $n \ne m$. Define the function $f(x_n) = (\kappa - \varepsilon)^\alpha e_n$, for $n \in \mathbb{N}$, that is, α-Hölder with constant 1. Pick any extension F of f and denote $z = F(0) \in c_0$: $\|F(0) - F(x_n)\| \le \operatorname{Lip}_\alpha(F)$. Hence we have for each coordinate $n \in \mathbb{N}$

$$|z(n) - (\kappa - \varepsilon)^\alpha| \le \operatorname{Lip}_\alpha(F),$$

and taking the limit as n tends to infinity, we have $(\kappa - \varepsilon)^\alpha \le \operatorname{Lip}_\alpha(F)$. Since the extension F is arbitrary we have $(\kappa - \varepsilon)^\alpha \le \lambda_\alpha(X, c_0)$, and letting $\varepsilon \to 0$ we are done. ∎

The difficult part of the proof of our main result is proof of the reverse inequality, and this is due to Kalton. Thus we are ready to prove the natural extension of [5, Proposition 5.8].

Proposition 2.2 *For every infinite-dimensional Banach space X*

$$\lambda_\alpha(X,c_0) = \kappa(X)^\alpha, \quad \alpha \in (0,1].$$

Proof Let us observe that

$$\lambda_\alpha(X,c_0) = \lambda_1(X^\alpha, c_0), \tag{2.1}$$

where as usual X^α denotes the snowflake of X, *i.e.*, the metric space $X^\alpha = (X, \|\cdot\|^\alpha)$. Therefore the result follows using the same proof of Kalton's result [5, Proposition 5.8]. Let us check it for the sake of completeness. Let us prove that the metric space X^α admits a Lipschitz extension with constant κ^α that by Lemma 2.1 will be enough. Suppose to the contrary that this last claim is false and we will reach a contradiction. Writing $\kappa = \kappa(X)$ for simplicity, [5, Theorem 5.1 (ii)] shows that there exists $a \in X^\alpha$, $\varepsilon > 0$, and a sequence $(x_n) \in X^\alpha$ such that

$$\kappa^\alpha \|x_k - a\|^\alpha + \varepsilon < \|x_j - x_k\|^\alpha, \quad j < k.$$

Since $\kappa^\alpha > 1$, we may suppose (x_n) is bounded. Hence, replacing x_n by $x_n - a$ and rescaling, we can find for some $\varepsilon' > 0$ a sequence $(x'_n) \in B(X^\alpha)$ (and thus

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$(x'_n) \in B(X)$) such that

$$\kappa^\alpha \|x'_k\|^\alpha + \varepsilon' < \|x'_j - x'_k\|^\alpha, \quad j < k. \tag{2.2}$$

Observe that the expression above gives that (x'_n) does not converge to 0 in X^α, otherwise $0 < \varepsilon' \leq \kappa^\alpha \|x'_k\|^\alpha + \varepsilon' < \|x'_j - x'_k\|^\alpha \to 0$, a contradiction. In particular, (x'_n) does not converge to 0 in X. Now observe that given $\varepsilon_1 > 0$, using the fact that a bounded sequence of real numbers has a convergent subsequence, it is not difficult to check that one can find a, b, and an infinite subset $\mathbb{N}_1$ of $\mathbb{N}$ such that $0 < a \leq \|x'_n\| \leq b \leq 1$ for all $n \in \mathbb{N}_1$ and $1 - \varepsilon_1 \leq \frac{a}{b}$. Indeed, if $r \neq 0$ is the limit point, then for the fixed $\varepsilon_1 > 0$ one can take $a = r - \theta$ and $b = r + \theta$ for $\theta > 0$ small enough. We need a special choice of $\varepsilon_1 > 0$. To this end, consider the function $f(t) = \kappa^\alpha (1-t)^\alpha$ whose limit as $t \to 0$ is κ^α. Therefore, by definition of limit, given $\frac{\varepsilon'}{2}$, there is some $t_0 > 0$ such that $f(t_0) \geq \kappa^\alpha - \frac{\varepsilon'}{2}$. Pick a, b, and $\mathbb{N}_1$ as above for the choice $\varepsilon_1 = t_0$. Now rescaling $1/b^\alpha$, the expression (2.2) for $j, k \in \mathbb{N}_1$, one has

$$\begin{aligned}\Big\| \frac{x'_j}{b} - \frac{x'_k}{b} \Big\|^\alpha &\geq \kappa^\alpha \frac{\|x'_k\|^\alpha}{b^\alpha} + \frac{\varepsilon'}{b^\alpha} \geq \kappa^\alpha \frac{\|x'_k\|^\alpha}{b^\alpha} + \varepsilon' \geq \kappa^\alpha \frac{a^\alpha}{b^\alpha} + \varepsilon' \geq \kappa^\alpha (1-\varepsilon_1)^\alpha + \varepsilon' \\ &\geq \kappa^\alpha + \frac{\varepsilon'}{2},\end{aligned}$$

where the last inequality follows by our choice of $\varepsilon_1 > 0$. That is, for $j, k \in \mathbb{N}_1$ with $j < k$ one has

$$\Big\| \frac{x'_j}{b} - \frac{x'_k}{b} \Big\|^\alpha \geq \kappa^\alpha + \frac{\varepsilon'}{2}. \tag{2.3}$$

To finish, observe that $(\kappa^\alpha + \frac{\varepsilon'}{2})^{1/\alpha} > \kappa$ and pick $\rho > 0$ such that $(\kappa^\alpha + \frac{\varepsilon'}{2})^{1/\alpha} - \kappa > \rho > 0$. For this $\rho > 0$, we have for $j, k \in \mathbb{N}_1$, and taking the α^{-1}-power in (2.3),

$$\Big\| \frac{x'_j}{b} - \frac{x'_k}{b} \Big\| \geq \kappa + \rho, \quad j < k.$$

Since the points $(b^{-1} x'_n)_{n \in \mathbb{N}_1} \in B(X)$, we have reached a contradiction with the Kottman constant of X. Thus by [5, Theorem 5.1], X^α has the Lipschitz (κ^α, c_0)-EP in the notation of [5]. That is, X^α admits a Lipschitz extension of c_0-valued Lipschitz maps with constant κ^α. Hence $\lambda_1(X^\alpha, c_0) \leq \kappa^\alpha$ and by Lemma 2.1 and (2.1) we are done. Recall that Kalton's argument shows also that the infimum defining $\lambda_\alpha(X, c_0)$ is attained. ∎

Recall that $\kappa(X) > 1$ (see [2]) and hence $\kappa(X)^\alpha > 1$ for every $\alpha \in (0,1]$. In other words, Lemma 2.1 shows there is no infinite-dimensional Banach space X for which $\lambda_\alpha(X, c_0) = 1$. This last was proved by Lancien and Randrianantoanina [7, Theorem 2.2] replacing c_0 by a separable Banach space Y containing an isomorphic copy of c_0. The next proposition can be read as a quantitative version of [7, Theorem 2.2] and the proof closely follows the original. Let us first introduce some basic notation [4]. We define a *gauge* to be a function $\omega\colon [0,\infty) \to [0,\infty)$ that is a continuous, increasing, subadditive function satisfying $\lim_{t\to 0} \omega(t) = 0$. For the rest of this subsection ω will always denote a gauge. Recall that given metric spaces (X,d) and (Y,ρ), a map

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$f: X \to Y$ is ω-Lipschitz if

$$\mathrm{Lip}_\omega(f) = \sup\Big\{ \frac{\rho(f(x), f(y))}{\omega(d(x,y))} : x \neq y \Big\} < \infty.$$

We may also define the constant $\lambda_\omega(X,Y)$ as the infimum of all $\lambda > 0$ such that for every subset M of X every ω-Lipschitz $f: M \to Y$ admits an extension $F: X \to Y$ with $\mathrm{Lip}_\omega(F) \leq \lambda \,\mathrm{Lip}_\omega(f)$. Since c_0 is a 2-absolute Lipschitz retract, it follows that c_0 is a 2-absolute ω-Lipschitz retract. In other words, the extension of ω-Lipschitz maps with values in c_0 is guaranteed with $\lambda_\omega(X, c_0) \leq 2$ for any Banach space X.

Proposition 2.3 *Let Y be a separable Banach space containing an isomorphic copy of c_0. For every infinite-dimensional Banach space X*

$$\frac{\omega(\kappa(X))}{\omega(1)} \leq \lambda_\omega(X,Y).$$

Proof Let us write $\kappa = \kappa(X)$ for simplicity. Fix $\varepsilon > 0$, and by James' distortion theorem [3] pick a $(1+\varepsilon)$-isomorphic copy of c_0 in Y through an isomorphism T into Y with $\|T\| \leq 1+\varepsilon$ and $\|T^{-1}\| \leq 1$. Denote by (e_n) the image of the canonical basis of c_0 by T and by (e_n^*) the Hahn–Banach extensions to Y of the corresponding coordinate functionals. By the separability assumption we can pick a subsequence $(e_{k_n}^*)$ of (e_n^*) such that $(e_{k_n}^*)$ w^*-converges to some point $y^* \in Y^*$. Given $\varepsilon_1 > 0$, pick $(x_n) \in B(X)$ for which $\kappa - \varepsilon_1 \leq \|x_n - x_m\|$, for $n \neq m$. Define the map $f(x_n) = (-1)^n \omega(\kappa - \varepsilon_1) e_{k_n}$ for each $n \in \mathbb{N}$. We trivially find that f is ω-Lipschitz with constant less than or equal to $(1+\varepsilon)$. Take any extension of f to X, namely F, and observe that

$$\|F(0) - F(x_n)\| \leq \mathrm{Lip}_\omega(F) \cdot \omega(\|x_n\|) \leq \mathrm{Lip}_\omega(F) \cdot \omega(1).$$

Therefore we find, for $y = F(0)$, that $|y(e_{k_n}^*) - (-1)^n \omega(\kappa - \varepsilon_1)| \leq \mathrm{Lip}_\omega(F) \cdot \omega(1)$, so that

$$-\mathrm{Lip}_\omega(F)\cdot\omega(1) - \omega(\kappa-\varepsilon_1) \leq y(e_{k_n}^*) \leq \mathrm{Lip}_\omega(F)\cdot\omega(1) - \omega(\kappa-\varepsilon_1), \quad \text{for } n \text{ odd},$$
$$-\mathrm{Lip}_\omega(F)\cdot\omega(1) + \omega(\kappa-\varepsilon_1) \leq y(e_{k_n}^*) \leq \mathrm{Lip}_\omega(F)\cdot\omega(1) + \omega(\kappa-\varepsilon_1), \quad \text{for } n \text{ even}.$$

If $\mathrm{Lip}_\omega(F)\cdot\omega(1) < \omega(\kappa - \varepsilon_1)$, write $\eta = \omega(\kappa-\varepsilon_1) - \mathrm{Lip}_\omega(F)\cdot\omega(1) > 0$. From above we find that for n even $y(e_{k_n}^*) \geq \eta$ while for n odd one has $y(e_{k_n}^*) \leq -\eta$. Hence the sequence $(e_{k_n}^*)$ is not w^*-convergent, which is absurd. Therefore $\omega(\kappa - \varepsilon_1) \leq \mathrm{Lip}_\omega(F)\cdot\omega(1)$. Since this last must hold for every extension F of f, we find that $\omega(\kappa-\varepsilon_1) \leq (1+\varepsilon)\lambda_\omega(X,Y)\cdot\omega(1)$ and letting $\varepsilon_1 \to 0$ and $\varepsilon \to 0$, we are done. ∎

The result of Lancien and Randrianantoanina [7, Theorem 2.2] for α-Hölder maps can be recovered by taking the gauges $\omega(t) = t^\alpha$ with $\alpha \in (0,1]$. Recall that ℓ_∞ is a 1-absolute ω-Lipschitz retract (see [1, Lemma 1.1.]). Thus $\lambda_\omega(X, \ell_\infty) = 1$ for every infinite-dimensional Banach space X while $\omega(\kappa(X)) > \omega(1)$ since ω is increasing. In particular, the separability assumption of Proposition 2.3 cannot be removed.

Let us give one last application of our results. We introduce $\mathcal{B}_C(X, c_0)$ as the set of those α such that any α-Hölder function f from a subset of X with values in c_0 and $\mathrm{Lip}_\alpha(f) = K$ can be extended to a function F on the whole X with $\mathrm{Lip}_\alpha(F) \leq C\cdot K$. Then Proposition 2.2 and a routine argument immediately gives the following.

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Corollary 2.4 *For every infinite-dimensional Banach space X*

$$\mathcal{B}_C(X, c_0) = \Big(0, \frac{\log C}{\log \kappa(X)}\Big], \quad 1 \le C \le \kappa(X).$$

Or symmetrically,

$$\mathcal{B}_{\kappa(X)^\alpha}(X, c_0) = (0, \alpha],$$

for $\alpha \in (0,1]$.

3 An Estimate for L_q

Let us note that the main idea of the proof of Lemma 2.1 still works if c_0 is replaced by other classic sequence spaces, such as ℓ_q-spaces. In general, the extension of α-Hölder maps with values in ℓ_q is not guaranteed. However, for those cases in which there is an extension, *i.e.*, $\lambda_\alpha(X, \ell_q) < \infty$, the following bound could be useful.

Corollary 3.1 *For every infinite-dimensional Banach space X*

$$2^{-\frac{1}{q}} \kappa(X)^\alpha \le \lambda_\alpha(X, \ell_q), \qquad \alpha \in (0,1].$$

The situation for L_q seems to be different; so let us study a lower bound for $\lambda_\alpha(X, L_q)$. Since in many cases $\lambda_\alpha(X, L_q) = \infty$, it is clear that our lower bound only makes sense for those Banach spaces X for which $\lambda_\alpha(X, L_q) < \infty$. Recall that Naor proved [8, Theorem 1] that $\lambda_\alpha(L_p, L_q) < \infty$ for $\alpha \le \frac{p}{2}$ and $1 < p, q \le 2$. Therefore our main motivation is the case of L_q with $1 < q \le 2$. The main technical obstruction to giving a lower bound using the argument in Lemma 2.1 is that there are no natural coordinates in L_q. To dodge this obstacle, we use a technical lemma due to Naor [8, Lemma 2]. For every $N \in \mathbb{N}$, put $\Omega = \{-1, +1\}^N$ endowed with the uniform probability measure on Ω and denote by $r_1, \dots, r_N$ the Rademacher functions on Ω.

Lemma 3.2 (Naor) *For all* $1 < q < \infty$ *and* $Z \in L_q(\Omega)$

$$\frac{1}{N} \sum_{n=1}^{N} \mathbb{E}|Z - r_n|^q \ge 1 - C\sqrt{\log N / N},$$

where C depends only on q.

We are ready to present the main result of this section.

Proposition 3.3 *For every infinite-dimensional Banach space X and* $1 < q \le 2$

$$2^{-\frac{1}{q^*}} \kappa(X)^\alpha \le \lambda_\alpha(X, L_q), \qquad \alpha \in (0,1].$$

Proof Let κ, λ_α be as in Proposition 2.2 and for given $\varepsilon > 0$, pick $(x_n) \in B(X)$ for which $\kappa - \varepsilon \le \|x_n - x_m\|$, for $n \ne m$. Fix $N \in \mathbb{N}$ and define the map $f_N(x_n) = (\kappa - \varepsilon)^\alpha r_n$ for each $n \in \{1, \dots, N\}$. It turns out that f_N is α-Hölder with constant less than or equal to $2^{\frac{1}{q^*}}$:

$$\|f_N(x_n) - f_N(x_m)\|_{L_q} = (\kappa - \varepsilon)^\alpha \|r_n - r_m\|_{L_q} \le 2^{1-\frac{1}{q}} \|x_n - x_m\|^\alpha.$$

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Take any extension of f_N to X, namely F, and observe that

$$\|F(0) - F(x_n)\|_{L_q} \leq \mathrm{Lip}_\alpha(F)\|r_n\|^\alpha \leq \mathrm{Lip}_\alpha(F).$$

Let us denote $F(0) = Z \in L_q(\Omega)$. Then we have that

$$\mathrm{Lip}_\alpha(F)^q \geq \frac{1}{N}\sum_{n=1}^{N} \mathbb{E}|Z - (\kappa - \varepsilon)^\alpha r_n|^q = (\kappa - \varepsilon)^{\alpha q}\frac{1}{N}\sum_{n=1}^{N} \mathbb{E}|(\kappa - \varepsilon)^{-\alpha} Z - r_n|^q$$
$$\geq (\kappa - \varepsilon)^{\alpha q}\Big(1 - C\sqrt{\log N/N}\Big),$$

where the last inequality follows from Lemma 3.2. Since this must hold for every extension F of f_N and every $N \in \mathbb{N}$, we find that $(\kappa - \varepsilon)^\alpha \leq \lambda_\alpha 2^{\frac{1}{q^*}}$, and letting $\varepsilon \to 0$, we are done. ∎

Since it is well known that $\kappa(L_p) = 2^{\frac{1}{p}}$ for $1 < p \leq 2$, Proposition 3.3 yields the following.

Corollary 3.4 *For* $1 < p, q \leq 2$,

$$2^{\frac{\alpha}{p} - \frac{1}{q^*}} \leq \lambda_\alpha(L_p, L_q).$$

Recall that Naor also proved [8, Theorem 1] that there is no isometric extension for $\alpha > \frac{p}{q^*}$ and $1 < p, q \leq 2$. As Corollary 3.4 shows, the Kottman constant explains geometrically why there is no isometric extension for these values. To finish, Corollary 3.4 gives $1 \leq \lambda_\alpha(L_p, L_q)$ for $\alpha = \frac{p}{q^*}$ and $1 < p, q \leq 2$, while [8, Theorem 1] provides us with $\lambda_\alpha(L_p, L_q) = 1$. So it is sharp.

References

[1] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*. American Mathematical Society Colloquium Publications, 48. American Mathematical Society, Providence, RI, 2000.

[2] J. Elton and E. Odell, *The unit ball of every infinite-dimensional normed linear space contains a* $(1+\varepsilon)$*-separated sequence*. Colloq. Math. **44**(1981), 105–109.

[3] R. C. James, *Uniformly non square Banach spaces*. Ann. of Math. **80**(1964), 542–550. http://dx.doi.org/10.2307/1970663

[4] N. J. Kalton, *Spaces of Lipschitz and Hölder functions and their applications*. Collect. Math. **55**(2004), 171–217.

[5] ———, *Extending lipschitz maps into* $C(K)$*-spaces*, Israel J. Math. **162**(2007), 275–315. http://dx.doi.org/10.1007/s11856-007-0099-2

[6] C. A. Kottman, *Packing and reflexivity in Banach spaces*, Tran. Amer. Math. Soc. **150**(1970), 565–576. http://dx.doi.org/10.1090/S0002-9947-1970-0265918-7

[7] G. Lancien and B. Randrianantoanina, *On the extension of Hölder maps with values in spaces of continuous functions*. Israel J. Math. **147**(2005), 75–92. http://dx.doi.org/10.1007/BF02785360

[8] A. Naor, *A phase transition phenomenon between the isometric and isomorphic extension problems for Hölder functions between* L_p *spaces*. Mathematika **48**(2001), 253–271. http://dx.doi.org/10.1112/S0025579300014480

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