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Abstract

The notion of Hochschild cochains induces an assignment from Aff, affine DG schemes, to monoidal DG categories. We show that this assignment extends, under appropriate finiteness conditions, to a functor $\mathbb{H} : \mathsf{Aff} \to \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$, where the latter denotes the category of monoidal DG categories and bimodules. Any functor $\mathbb{A} : \mathsf{Aff} \to \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$ gives rise, by taking modules, to a theory of sheaves of categories $\mathsf{ShvCat}^{\mathbb{A}}$. In this paper, we study $\mathsf{ShvCat}^{\mathbb{H}}$. Loosely speaking, this theory categorifies the theory of $\mathfrak{D}$-modules, in the same way as Gaitsgory's original ShvCat categorifies the theory of quasi-coherent sheaves. We develop the functoriality of $\mathsf{ShvCat}^{\mathbb{H}}$, its descent properties and the notion of $\mathbb{H}$-affineness. We then prove the $\mathbb{H}$-affineness of algebraic stacks: for $\mathcal{Y}$ a stack satisfying some mild conditions, the ∞-category $\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$ is equivalent to the ∞-category of modules for $\mathbb{H}(\mathcal{Y})$, the monoidal DG category of higher differential operators. The main consequence, for $\mathcal{Y}$ quasi-smooth, is the following: if $\mathcal{C}$ is a DG category acted on by $\mathbb{H}(\mathcal{Y})$, then $\mathcal{C}$ admits a theory of singular support in $\mathrm{Sing}(\mathcal{Y})$, where $\mathrm{Sing}(\mathcal{Y})$ is the space of singularities of $\mathcal{Y}$. As an application to the geometric Langlands programme, we indicate how derived Satake yields an action of $\mathbb{H}(\mathrm{LS}_{\check{G}})$ on $\mathfrak{D}(\mathrm{Bun}_G)$, thereby equipping objects of $\mathfrak{D}(\mathrm{Bun}_G)$ with singular support in $\mathrm{Sing}(\mathrm{LS}_{\check{G}})$.

1. Introduction

1.1 Overview

The present paper is a contribution to the field of categorical algebraic geometry. In this field one studies schemes and stacks via their categorical invariants, as opposed to their usual linear invariants. Among the usual invariants, typical examples are the coherent cohomology, the de Rham cohomology, the Picard group. An example of a categorical invariant is the symmetric monoidal category of quasi-coherent sheaves; other examples, including the invariant $\mathsf{ShvCat}^{\mathbb{H}}$ appearing in the title of this paper, will be given below.

The extra level of categorical abstraction might appear unjustified at first sight, but it turns out to be quite useful in several concrete situations. In this paper we will encounter a few, for instance in §§ 1.2.6, 1.4.1 and 1.11.

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The interplay between categorical and ordinary algebraic geometry is likely to be very fruitful. For more on the comparison between the two points of view, we recommend the discussion and the dictionary appearing in [Lur18, p. 720].

In the rest of this overview, after discussing some illuminating examples, we will roughly state the goals and the main results of this paper. These results and goals will be further clarified in the later sections of the introduction.

1.1.1 As mentioned earlier, given a scheme or an algebraic stack $\mathcal{Y}$, its most basic categorical invariant is the symmetric monoidal differential graded (DG) category $\mathrm{QCoh}(\mathcal{Y})$.

It turns out that there are strong analogies between the behaviour of $\mathrm{QCoh}(\mathcal{Y})$ for an algebraic stack $\mathcal{Y}$ and the behaviour of $H^*(Y, \mathcal{O}_Y)$ for an affine scheme[1] Y. In other words, categorical algebraic geometry has many more affine objects than ordinary algebraic geometry. Let us illustrate this principle with three examples.

1.1.2 *Tannaka duality.* For $\mathcal{Y}$ an algebraic stack satisfying mild conditions, Tannaka duality [Lur18, ch. 9] allows one to 'recover' $\mathcal{Y}$ from the symmetric monoidal DG category $\mathrm{QCoh}(\mathcal{Y})$. On the other hand, the DG algebra $H^*(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ does not recover $\mathcal{Y}$, unless $\mathcal{Y}$ is an affine DG scheme.

1.1.3 *Tensor products.* Given a diagram $X \to Z \leftarrow Y$ of (DG) affine schemes, one has

$$H^*(X \times_Z Y, \mathcal{O}_{X\times_Z Y}) \simeq H^*(X, \mathcal{O}_X) \underset{H^*(Z,\mathcal{O}_Z)}{\overset{\mathbf{L}}{\otimes}} H^*(Y, \mathcal{O}_Y).$$

Note that it is essential that the fibre product is taken in the derived sense. This formula obviously fails for very simple non-affine schemes and stacks. On the other hand, the categorical counterpart is the tensor product formula

$$\mathrm{QCoh}(\mathcal{X} \times_\mathcal{Z} \mathcal{Y}) \simeq \mathrm{QCoh}(\mathcal{X}) \underset{\mathrm{QCoh}(\mathcal{Z})}{\otimes} \mathrm{QCoh}(\mathcal{Y}), \tag{1.1}$$

which holds true for most algebraic stacks $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ that one encounters in practice; see, for instance, [BFN10].

The right-hand side of the above formula involves the tensor product of DG categories [Lur17], which plays a crucial role in the theory. Note that $\mathrm{QCoh}(\mathcal{Z})$ acts on $\mathrm{QCoh}(\mathcal{X})$ and on $\mathrm{QCoh}(\mathcal{Y})$ by pullback along the given maps $\mathcal{X} \xrightarrow{f} \mathcal{Z} \xleftarrow{g} \mathcal{Y}$.

1.1.4 1-*affineness.* In the categorical context, one considers *categorified quasi-coherent sheaves* over a scheme or a stack $\mathcal{Y}$. These categorified sheaves are defined in [Gai15b] under the name of 'sheaves of categories', and in [Lur18, ch. 10] under the name of 'quasi-coherent stacks'. They assemble into an ∞-category denoted $\mathsf{ShvCat}(\mathcal{Y})$. We will recall and generalize the notion of ShvCat in § 1.6.

In the above papers it is proven that most algebraic stacks, while far from being affine schemes, are nevertheless 1-*affine*: by definition, $\mathcal{Y}$ is 1-affine if the ∞-category $\mathsf{ShvCat}(\mathcal{Y})$ is equivalent to the ∞-category of modules DG categories for $\mathrm{QCoh}(\mathcal{Y})$. This categorifies the classical fact that, for Y an affine DG scheme, a quasi-coherent sheaf is the same as a module over $H^*(Y, \mathcal{O}_Y)$.

[1] We will soon be forced to consider DG schemes. By construction, the cohomology $H^*(Y, \mathcal{O}_Y)$ of an affine DG scheme is possibly non-zero in negative degrees: this explains the notation $H^*(Y, \mathcal{O}_Y)$ in place of the more tempting $H^0(Y, \mathcal{O}_Y)$.

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1.1.5 The above examples illustrate the point of view that $\mathrm{QCoh}(\mathcal{Y})$ is the categorical counterpart of the algebra of functions on an *affine* DG scheme.

In [Ber17b] we introduced another monoidal DG category, $\mathbb{H}(\mathcal{Y})$, which is the categorical counterpart of the algebra of differential operators on an affine DG scheme.

In a nutshell, the goal of the present paper is to develop the tensor product formula and the 1-affineness result with $\mathbb{H}(\mathcal{Y})$ in place of $\mathrm{QCoh}(\mathcal{Y})$.

1.1.6 *Tensor products for* $\mathbb{H}$. The tensor product formula in the $\mathbb{H}$ situation is by necessity slightly different from (1.1). Indeed, as explained in detail later, there is no natural action of $\mathbb{H}(\mathcal{Z})$ on $\mathbb{H}(\mathcal{X})$. Rather, these two monoidal DG categories are connected by a *transfer* bimodule category $\mathbb{H}_{\mathcal{X}\to\mathcal{Z}}$. (This is in perfect agreement with the situation of rings of differential operators, from which the notation is borrowed.) Under some conditions to be discussed later, the tensor product formula reads

$$\mathbb{H}_{\mathcal{X}\leftarrow\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}} \underset{\mathbb{H}(\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y})}{\otimes} \mathbb{H}_{\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}\to\mathcal{Y}} \simeq \mathbb{H}_{\mathcal{X}\to\mathcal{Z}} \underset{\mathbb{H}(\mathcal{Z})}{\otimes} \mathbb{H}_{\mathcal{Z}\leftarrow\mathcal{Y}}.$$

For some pleasing applications of this formula, the reader might look ahead at §§ 1.10 and 1.11.

1.1.7 1-*affineness for* $\mathbb{H}$ *(or* $\mathbb{H}$-*affineness).* The 1-affineness mentioned in § 1.1.4 corresponds, in the $\mathbb{H}$ setup, to our main Theorem 1.7.4, which establishes a tight link between modules categories for $\mathbb{H}(\mathcal{Y})$ and *categorified D-modules* on $\mathcal{Y}$. The latter are also called *sheaves of categories over* $\mathcal{Y}$ *with local actions of Hochschild cochains*, and denoted by $\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$. As we explain in the following sections, the objects of $\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$ are the sheaves of categories for which a notion of singular support is defined and well behaved.

1.2 Singular support via the $\mathbb{H}$-action

1.2.1 In [Ber17b] we introduced a monoidal DG category $\mathbb{H}(\mathcal{Y})$ attached to a quasi-smooth stack $\mathcal{Y}$. In contrast to $\mathrm{QCoh}(\mathcal{Y})$, which can be defined in vast generality, the construction of $\mathbb{H}(\mathcal{Y})$ requires some (mild) conditions on $\mathcal{Y}$. The definition of $\mathbb{H}(\mathcal{Y})$ and the necessary conditions on $\mathcal{Y}$ are recalled in § 1.3. For now, let us just say that any quasi-smooth stack $\mathcal{Y}$ satisfies those conditions.

1.2.2 As a brief reminder of the notion of quasi-smoothness: an algebraic stack $\mathcal{Y}$ is quasi-smooth if it is smooth locally a global complete intersection. It follows that, for any geometric point $y \in \mathcal{Y}$, the y-fibre $\mathbb{L}_{\mathcal{Y},y} := \mathbb{L}_{\mathcal{Y}}|_y$ of the contangent complex has cohomologies concentrated in degrees $[-1, 1]$.

Thus, to a quasi-smooth stack $\mathcal{Y}$ we associate the stack $\mathrm{Sing}(\mathcal{Y})$ that parametrizes pairs (y, ξ) with $y \in \mathcal{Y}$ and $\xi \in H^{-1}(\mathbb{L}_{\mathcal{Y},y})$. This is the space that controls the singularities of $\mathcal{Y}$ (see [AG15]), and it is equipped with a $\mathbb{G}_m$-action that rescales the fibres of the projection $\mathrm{Sing}(\mathcal{Y}) \to \mathcal{Y}$.

1.2.3 Suppose that a DG category $\mathcal{C}$ carries an action of $\mathbb{H}(\mathcal{Y})$. The goal of this paper is to explain how rich this structure is. As an example, let us informally state here the most important consequence of our main results.

Theorem 1.2.4. *Let* $\mathcal{Y}$ *be a quasi-smooth stack and* $\mathcal{C}$ *a left* $\mathbb{H}(\mathcal{Y})$-*module. Then* $\mathcal{C}$ *is equipped with a singular support theory relative to* $\mathrm{Sing}(\mathcal{Y})$.

1.2.5 To make sense of this, we need to explain what we mean by 'singular support theory'. First and foremost, this means that there is a map (the singular support map) from objects of $\mathcal{C}$ to closed conical subsets of $\mathrm{Sing}(\mathcal{Y})$. For each such subset $\mathcal{N} \subseteq \mathrm{Sing}(\mathcal{Y})$, we set $\mathcal{C}_{\mathcal{N}}$ to be the full subcategory of $\mathcal{C}$ spanned by those objects with singular support contained in $\mathcal{N}$.

The second feature of a singular support theory is that any inclusion $\mathcal{N} \subseteq \mathcal{N}'$ yields a colocalization (that is, an adjunction whose left adjoint is fully faithful) $\mathcal{C}_{\mathcal{N}} \rightleftarrows \mathcal{C}_{\mathcal{N}'}$.

1.2.6 Thus, the datum of an action of $\mathbb{H}(\mathcal{Y})$ on $\mathcal{C}$ immediately produces a multitude of semi-orthogonal decompositions of $\mathcal{C}$, one for each closed conical subset of $\mathrm{Sing}(\mathcal{Y})$. Obviously, these decompositions help compute Hom spaces between objects of $\mathcal{C}$.

More generally, the philosophy[2] is that, in the presence of an $\mathbb{H}(\mathcal{Y})$-action on $\mathcal{C}$, any decomposition of $\mathrm{Sing}(\mathcal{Y})$ into *atomic blocks* induces a decomposition of $\mathcal{C}$ into atomic blocks. By 'atomic blocks' we mean closed conical subsets of $\mathrm{Sing}(\mathcal{Y})$ that are of a particular significance or simplicity, such as the zero section, a particular fibre, or more generally the conormal bundle of a closed subset of $\mathcal{Y}$. See [AG18, Ber18] for applications of this principle.

1.2.7 It is also natural to require that singular support be functorial in $\mathcal{C}$. Namely, given an $\mathbb{H}(\mathcal{Y})$-linear functor $F : \mathcal{C} \to \mathcal{D}$ and $\mathcal{N} \subseteq \mathrm{Sing}(\mathcal{Y})$, we would like F to restrict to a functor $\mathcal{C}_{\mathcal{N}} \to \mathcal{D}_{\mathcal{N}}$. Fortunately, this is also guaranteed by our theory. Hence the informal statement of Theorem 1.2.4 could be improved as follows.

Theorem 1.2.8. *For $\mathcal{Y}$ a quasi-smooth stack, $\mathbb{H}(\mathcal{Y})$-module categories admit a singular support theory relative to* $\mathrm{Sing}(\mathcal{Y})$.

Remark 1.2.9. The proof of this theorem is an easy consequence of the construction of $\mathbb{H}(\mathcal{Y})$ (namely, the relation with Hochschild cochains as in § 1.5) and our $\mathbb{H}$-affineness theorem, Theorem 1.7.4.

Remark 1.2.10. Our expectation on possible usages of this theorem is the following. It is generally difficult to directly equip $\mathcal{C}$ with a singular support theory relative to $\mathrm{Sing}(\mathcal{Y})$; instead, one should try to exhibit an action of $\mathbb{H}(\mathcal{Y})$ on $\mathcal{C}$. In § 1.4 we will illustrate a concrete application of this point of view on the geometric Langlands programme.

1.2.11 There exists a monoidal functor $\mathrm{QCoh}(\mathcal{Y}) \to \mathbb{H}(\mathcal{Y})$; hence, an $\mathbb{H}(\mathcal{Y})$-action on $\mathcal{C}$ means in particular that $\mathcal{C}$ admits a $\mathrm{QCoh}(\mathcal{Y})$-action. Thus, our theorem above can be regarded as an improvement of the following one in the setting of quasi-smooth stacks.

Theorem 1.2.12. *Let $\mathcal{Y}$ be an algebraic stack (not necessarily quasi-smooth). Then left* $\mathrm{QCoh}(\mathcal{Y})$*-modules are equipped with a support theory relative to $\mathcal{Y}$.*

1.3 The monoidal category $\mathbb{H}(\mathcal{Y})$

Let us now recall the elements that go into the definition of $\mathbb{H}(\mathcal{Y})$, following [AG18] and [Ber17b]. Although the applications of this theory so far concern only $\mathcal{Y}$ quasi-smooth, the natural set-up for $\mathbb{H}(\mathcal{Y})$ is more general. Namely, we assume that $\mathcal{Y}$ is a quasi-compact algebraic stack which is perfect, bounded (eventually coconnective) and locally of finite presentation (lfp). See [BFN10] for the notion of 'perfect stack'.

[2] Strictly speaking, this is not a consequence of the results of this paper. We refer to the analysis of [Ber18].

1.3.1 The definition of $\mathbb{H}$ requires some familiarity with the theory of ind-coherent sheaves on formal completions. We refer to [GR17, ch. III] or [Ber17b] for a quick review.

Nevertheless, let us recall the most important concepts. First, $\mathcal{Y}_{\mathrm{dR}}$ denotes the *de Rham* prestack of $\mathcal{Y}$, whence $\mathcal{Y} \times_{\mathcal{Y}_{\mathrm{dR}}} \mathcal{Y}$ is the formal completion of the diagonal $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$. Second, we have the standard functor

$$\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) \longrightarrow \mathrm{IndCoh}(\mathcal{Y}),$$

which is the functor of acting on the dualizing sheaf $\omega_{\mathcal{Y}} \in \mathrm{IndCoh}(\mathcal{Y})$. The boundedness condition on $\mathcal{Y}$ is imposed so that $\Upsilon_{\mathcal{Y}}$ is fully faithful.

1.3.2 We define $\mathbb{H}(\mathcal{Y})$ to be the full subcategory of $\mathrm{IndCoh}(\mathcal{Y} \times_{\mathcal{Y}_{\mathrm{dR}}} \mathcal{Y})$ cut out by the requirement that the image of the pullback functor $\Delta^! : \mathrm{IndCoh}(\mathcal{Y} \times_{\mathcal{Y}_{\mathrm{dR}}} \mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y})$ be contained in the subcategory $\Upsilon_{\mathcal{Y}}(\mathrm{QCoh}(\mathcal{Y})) \subseteq \mathrm{IndCoh}(\mathcal{Y})$. Now, $\mathrm{IndCoh}(\mathcal{Y} \times_{\mathcal{Y}_{\mathrm{dR}}} \mathcal{Y})$ has a monoidal structure given by convolution, that is, pull-push along the correspondence

$$\mathcal{Y} \times_{\mathcal{Y}_{\mathrm{dR}}} \mathcal{Y} \times \mathcal{Y} \times_{\mathcal{Y}_{\mathrm{dR}}} \mathcal{Y} \xleftarrow{p_{12} \times p_{23}} \mathcal{Y} \times_{\mathcal{Y}_{\mathrm{dR}}} \mathcal{Y} \times_{\mathcal{Y}_{\mathrm{dR}}} \mathcal{Y} \xrightarrow{p_{13}} \mathcal{Y} \times_{\mathcal{Y}_{\mathrm{dR}}} \mathcal{Y}.$$

The lfp assumption on $\mathcal{Y}$ is crucial: it ensures that $\mathbb{H}(\mathcal{Y})$ is preserved by this multiplication, thereby inheriting a monoidal structure.

Example 1.3.3. Of course, $\mathbb{H}(\mathcal{Y})$ admits two obvious module categories: $\mathrm{IndCoh}(\mathcal{Y})$ and $\mathrm{QCoh}(\mathcal{Y})$. For $\mathrm{IndCoh}(\mathcal{Y})$, the theory of singular support of Theorem 1.2.4 reduces to the one developed by [AG15] and before by [BIK08].

Example 1.3.4. By [AG15], objects of $\mathrm{QCoh}(\mathcal{Y})$ have singular support contained in the zero section of $\mathrm{Sing}(\mathcal{Y})$: in our language, this is expressed by the fact that the action of $\mathbb{H}(\mathcal{Y})$ on $\mathrm{QCoh}(\mathcal{Y})$ factors through the monoidal localization

$$\mathbb{H}(\mathcal{Y}) \twoheadrightarrow \mathrm{QCoh}(\mathcal{Y} \times_{\mathcal{Y}_{\mathrm{dR}}} \mathcal{Y}).$$

The construction and study of this monoidal localization are deferred to another publication. For now, let us say that we will call $\mathcal{C} \in \mathbb{H}(\mathcal{Y})$-**mod** *tempered* if the $\mathbb{H}(\mathcal{Y})$-action factors through the above monoidal quotient.

1.4 $\mathbb{H}$ for Hecke

In this section, we anticipate a future application of Theorem 1.2.4. The reader not interested in geometric Langlands might skip ahead to § 1.5.

1.4.1 Let us recall the rough statement of the geometric Langlands conjecture (see [AG15]): there is a canonical equivalence $\mathfrak{D}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LS}_{\check{G}})$. This conjecture predicts in particular that any $\mathcal{F} \in \mathfrak{D}(\mathrm{Bun}_G)$ has a (nilpotent) singular support in $\mathrm{Sing}(\mathrm{LS}_{\check{G}})$. The question that prompted the writing of this paper and the study of $\mathbb{H}$ is the following: *is it possible to exhibit this structure on* $\mathfrak{D}(\mathrm{Bun}_G)$ *independently of the geometric Langlands conjecture?*

Having such a notion is evidently desirable, as it allows us to cut out $\mathfrak{D}(\mathrm{Bun}_G)$ into several subcategories by imposing singular support conditions. For instance, the zero section $O_{\mathrm{LS}_{\check{G}}} \subseteq \mathrm{Sing}(\mathrm{LS}_{\check{G}})$ ought to give rise to the DG category $\mathfrak{D}(\mathrm{Bun}_G)_{O_{\mathrm{LS}_{\check{G}}}}$ of *tempered* $\mathfrak{D}$-modules.

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1.4.2 Our Theorem 1.2.4 gives a way to answer the above question. We make the following claim, which we plan to address elsewhere: *there is a canonical action of* $\mathbb{H}(\mathrm{LS}_{\check{G}})$ *on* $\mathfrak{D}(\mathrm{Bun}_G)$.

Modulo technical and foundational details, the construction of such action goes as follows.

– Consider the action of the *renormalized* spherical category $\mathrm{Sph}^{\mathrm{ren}}_{G,\mathsf{Ran}}$ on $\mathfrak{D}(\mathrm{Bun}_G)$.[3]
– *Derived geometric Satake* over Ran yields a monoidal equivalence between $\mathrm{Sph}^{\mathrm{ren}}_{G,\mathsf{Ran}}$ and the (not yet defined) convolution monoidal DG category

$$\mathrm{Sph}^{\mathrm{spec,ren}}_{\check{G},\mathsf{Ran}} := \mathrm{IndCoh}\big(\big(\mathrm{LS}_{\check{G}}(D) \times_{\mathrm{LS}_{\check{G}}(D^\times)} \mathrm{LS}_{\check{G}}(D)\big)^\wedge_{\mathrm{LS}_{\check{G}}(D)}\big)_{\mathsf{Ran}}.$$

– The argument of [Roz11] yields a monoidal localization

$$\mathrm{Sph}^{\mathrm{spec,ren}}_{\check{G},\mathsf{Ran}} \twoheadrightarrow \mathbb{H}(\mathrm{LS}_{\check{G}}),$$

with kernel denoted by $\mathcal{K}$.
– Now consider the *spherical category* $\mathrm{Sph}^{\mathrm{spec,naive}}_{G,\mathsf{Ran}}$, the monoidal localization

$$\mathrm{Sph}^{\mathrm{spec,naive}}_{G,\mathsf{Ran}} \twoheadrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$

with kernel denoted $\mathcal{K}^{\mathrm{naive}}$, and the monoidal functor

$$\mathrm{Sph}^{\mathrm{spec,naive}}_{G,\mathsf{Ran}} \longrightarrow \mathrm{Sph}^{\mathrm{spec,ren}}_{\check{G},\mathsf{Ran}}.$$

– By construction, the essential image of the resulting functor $\mathcal{K}^{\mathrm{naive}} \to \mathcal{K}$ generates the target under colimits.
– The *vanishing theorem* [Gai15a] states that objects of $\mathcal{K}^{\mathrm{naive}}$ act by zero on $\mathfrak{D}(\mathrm{Bun}_G)$, whence the same is true for objects of $\mathcal{K}$l in other words, the $\mathrm{Sph}^{\mathrm{ren}}_{G,\mathsf{Ran}}$-action on $\mathfrak{D}(\mathrm{Bun}_G)$ factors through an action of $\mathbb{H}(\mathrm{LS}_{\check{G}})$.

In particular, the construction implies that $\mathbb{H}(\mathrm{LS}_{\check{G}})$ acts on $\mathfrak{D}(\mathrm{Bun}_G)$ by Hecke functors.

1.5 $\mathbb{H}$ for Hochschild

To motivate the definition of $\mathbb{H}(\mathcal{Y})$ and to explain the connection with singular support, it is instructive to look at the case where $\mathcal{Y} = S$ is an affine DG scheme. Under our standing assumptions, S is of finite type, bounded and with perfect cotangent complex. (Hereafter, we denote by $\mathsf{Aff}^{<\infty}_{\mathrm{lfp}}$ the ∞-category of such affine schemes.) In this case, the monoidal category $\mathbb{H}(S)$ is very explicit: it is the monoidal DG category of right modules over the E_2-algebra

$$\mathrm{HC}(S) := \mathsf{End}_{\mathrm{QCoh}(S\times S)}(\Delta_*(\mathcal{O}_S))$$

of Hochschild cochains on S. Under the equivalence $\mathbb{H}(S) \simeq \mathrm{HC}(S)^{\mathrm{op}}\text{-}\mathsf{mod}$, the monoidal functor $\mathrm{QCoh}(S) \to \mathbb{H}(S)$ corresponds to induction along the E_2-algebra map $\Gamma(S, \mathcal{O}_S) \to \mathrm{HC}(S)^{\mathrm{op}}$.

1.5.1 From this description, one observes that Theorem 1.2.4 is obvious in the affine case. Indeed, as we have just seen, the datum of $\mathcal{C} \in \mathbb{H}(S)$-**mod** means that $\mathcal{C}$ is enriched over $\mathrm{HC}(S)^{\mathrm{op}}$. Now, the Hochschild–Kostant–Rosenberg theorem yields a graded algebra map

$$\mathrm{Sym}_{H^0(S,\mathcal{O}_S)}(H^1(\mathbb{T}_S)[-2]) \longrightarrow \mathrm{HH}^\bullet(S),$$

and, by definition, singular support for objects of $\mathcal{C}$ is computed just using the action of the left-hand side on $H^\bullet(\mathcal{C})$.

[3] See [AG15, § 12.2.3] for the pointwise (as opposed to Ran) version.

1.5.2 In summary, there is a hierarchy of structures that a DG category $\mathcal{C}$ might carry:

- an action of the E_2-algebra $\mathrm{HC}(S)^{\mathrm{op}}$;
- an action of the commutative graded algebra $\mathrm{Sym}_{H^0(S,\mathcal{O}_S)} H^1(S,\mathbb{T}_S)[-2]$ on $H^\bullet(\mathcal{C})$;
- an action of the commutative algebra $H^0(S,\mathcal{O}_S)$ on $H^\bullet(\mathcal{C})$.

The first two data endow objects of $\mathcal{C}$ with singular support, which is a closed conical subset of $\mathrm{Sing}(S)$; see [AG15]. The third datum only allows us to define ordinary support in S.

1.6 Sheaves of categories

Next, we would like to generalize the above constructions to non-affine schemes and then to algebraic stacks. The key hint is that singular support of quasi-coherent and ind-coherent sheaves can be computed smooth locally. Thus, we hope to be able to glue the local HC-actions as well.

1.6.1 The first step towards this goal is to understand the functoriality of $\mathbb{H}(S)$-**mod** along maps of affine schemes. This is not immediate, as $\mathrm{HC}(S)$ is not functorial in S. In particular, for $f : S \to T$ a morphism in $\mathsf{Aff}_{\mathrm{lfp}}^{<\infty}$, there is no natural monoidal functor between $\mathbb{H}(T)$ and $\mathbb{H}(S)$. However, these two monoidal categories are connected by a canonical bimodule

$$\mathbb{H}_{S\to T} := \mathrm{IndCoh}_0((S\times T)^\wedge_S).$$

Example 1.6.2. Observe that $\mathbb{H}_{S\to \mathrm{pt}} \simeq \mathrm{QCoh}(S)$ and $\mathbb{H}_{S\to S} = \mathbb{H}(S)$.

1.6.3 Moreover, for any string $S \to T \to U$ in $\mathsf{Aff}_{\mathrm{lfp}}^{<\infty}$, there is a natural functor

$$\mathbb{H}_{S\to T} \underset{\mathbb{H}(T)}{\otimes} \mathbb{H}_{T\to U} \longrightarrow \mathbb{H}_{S\to U}, \tag{1.2}$$

given by convolution along the obvious correspondence

$$(S\times T)^\wedge_S \times (T\times U)^\wedge_T \longleftarrow (S\times T\times U)^\wedge_S \longrightarrow (S\times U)^\wedge_S.$$

We will prove in Theorem 4.3.4 that (1.2) is an equivalence of $(\mathbb{H}(S),\mathbb{H}(U))$-bimodules. It follows that the assignment $[S\to T] \rightsquigarrow \mathbb{H}_{S\to T}$ upgrades to a functor

$$\mathbb{H} : \mathsf{Aff}_{\mathrm{lfp}}^{<\infty} \longrightarrow \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat}),$$

where $\mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$ is the ∞-category whose objects are monoidal DG categories and whose morphisms are bimodules.

1.6.4 A functor

$$\mathbb{A} : \mathsf{Aff} \to \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$$

(or a slight variation, for example the functor $\mathbb{H} : \mathsf{Aff}_{\mathrm{lfp}}^{<\infty} \to \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$) will be called a *coefficient system* in this paper. Informally, $\mathbb{A}$ consists of the following pieces of data:

- for an affine scheme S, a monoidal DG category $\mathbb{A}(S)$;
- for a map of affine schemes $f : S \to T$, an $(\mathbb{A}(S),\mathbb{A}(T))$-bimodule $\mathbb{A}_{S\to T}$;
- for any string of affine schemes $S\to T\to U$, an $(\mathbb{A}(S),\mathbb{A}(U))$-bilinear equivalence

$$\mathbb{A}_{S\to T} \underset{\mathbb{A}(T)}{\otimes} \mathbb{A}_{T\to U} \longrightarrow \mathbb{A}_{S\to U};$$

- a system of coherent compatibilities for higher compositions.

The reason for the terminology is that each $\mathbb{A}$ is the coefficient system for a sheaf of categories attached to it. More precisely, the datum of $\mathbb{A}$ as above allows us to define a functor

$$\mathsf{ShvCat}^{\mathbb{A}} : \mathsf{PreStk}^{\mathrm{op}} \longrightarrow \mathsf{Cat}_\infty$$

as follows:

- for S affine, we set $\mathsf{ShvCat}^{\mathbb{A}}(S) = \mathbb{A}(S)\text{-}\mathbf{mod}$;
- for $f : S \to T$ a map in Aff, we have a structure pullback functor

$$f^{*,\mathbb{A}} : \mathsf{ShvCat}^{\mathbb{A}}(T) = \mathbb{A}(T)\text{-}\mathbf{mod} \xrightarrow{\mathbb{A}_{S\to T} \underset{\mathbb{A}(T)}{\otimes} -} \mathsf{ShvCat}^{\mathbb{A}}(S) = \mathbb{A}(S)\text{-}\mathbf{mod};$$

- for $\mathcal{Y}$ a prestack, we define $\mathsf{ShvCat}^{\mathbb{A}}(\mathcal{Y})$ as a right Kan extension along the inclusion $\mathsf{Aff} \hookrightarrow \mathsf{PreStk}$, that is,

$$\mathsf{ShvCat}^{\mathbb{A}}(\mathcal{Y}) = \lim_{S \in (\mathsf{Aff}_{/\mathcal{Y}})^{\mathrm{op}}} \mathbb{A}(S)\text{-}\mathbf{mod}.$$

Thus, an object of $\mathsf{ShvCat}^{\mathbb{A}}(\mathcal{Y})$ is a collection of $\mathbb{A}(S)$-modules $\mathcal{C}_S$, one for each S mapping to $\mathcal{Y}$, together with compatible equivalences $\mathbb{A}_{S\to T} \otimes_{\mathbb{A}(T)} \mathcal{C}_T \simeq \mathcal{C}_S$.

Example 1.6.5. The easiest non-trivial example of coefficient system is arguably the one denoted by $\mathbb{Q}$ and defined as

$$\mathbb{Q}(S) := \mathrm{QCoh}(S), \quad \mathbb{Q}_{S\to T} := \mathrm{QCoh}(S) \in (\mathrm{QCoh}(S), \mathrm{QCoh}(T))\text{-}\mathbf{bimod}.$$

The theory of sheaves of categories associated to $\mathbb{Q}$ is the 'original one', developed by D. Gaitsgory in [Gai15b]. There such theory was denoted by ShvCat; in this paper, for the sake of uniformity, we will instead denote it by $\mathsf{ShvCat}^{\mathbb{Q}}$.

Example 1.6.6. Parallel to the above, consider the coefficient system $\mathbb{D} : \mathsf{Aff}_{\mathrm{aft}} \to \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$ defined by

$$\mathbb{D}(S) := \mathfrak{D}(S), \quad \mathbb{D}_{S\to T} := \mathfrak{D}(S) \in (\mathfrak{D}(S), \mathfrak{D}(T))\text{-}\mathbf{bimod}.$$

The theory $\mathsf{ShvCat}^{\mathbb{D}}$ is the theory of *crystals of categories*, also discussed in [Gai15b].

Remark 1.6.7. The following list of analogies is sometimes helpful: $\mathsf{ShvCat}^{\mathbb{Q}}$ categorifies quasi-coherent sheaves, $\mathsf{ShvCat}^{\mathbb{D}}$ categorifies locally constant sheaves, $\mathsf{ShvCat}^{\mathbb{H}}$ categorifies $\mathfrak{D}$-modules.

1.7 $\mathbb{H}$-affineness

In line with the first of the above analogies, the foundational paper [Gai15b] constructs an explicit adjunction

$$\mathbf{Loc}_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod} \rightleftarrows \mathsf{ShvCat}^{\mathbb{Q}}(\mathcal{Y}) : \mathbf{\Gamma}_{\mathcal{Y}}.$$

In line with the analogy again, a prestack $\mathcal{Y}$ is said to be 1-affine if these adjoints are mutually inverse equivalences. This is tautologically true in the case where $\mathcal{Y}$ is an affine scheme. However, there are several other examples: most notably many algebraic stacks (specifically, quasi-compact bounded algebraic stacks of finite type and with affine diagonal) are 1-affine; see [Gai15b, Theorem 2.2.6].

For the sake of uniformity, we take the liberty to rename '1-affineness' as '$\mathbb{Q}$-affineness'.

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1.7.1 One of our main constructions is the adjunction

$$\mathbf{Loc}^{\mathbb{H}}_{\mathcal{Y}} : \mathbb{H}(\mathcal{Y})\text{-}\mathbf{mod} \rightleftarrows \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y}) : \mathbf{\Gamma}^{\mathbb{H}}_{\mathcal{Y}}, \tag{1.3}$$

sketched below (and discussed thoroughly in § 6.2). Contrarily to the $\mathbb{Q}$-case, in the $\mathbb{H}$-case we do not allow $\mathcal{Y}$ to be an arbitrary prestack, but we need $\mathcal{Y}$ to be an algebraic stack satisfying the conditions that make $\mathbb{H}(\mathcal{Y})$ well defined; see § 1.3.

1.7.2 The definition of the left adjoint $\mathbf{Loc}^{\mathbb{H}}_{\mathcal{Y}}$ is easy. For a map $S \to \mathcal{Y}$ with $S \in \mathsf{Aff}^{<\infty}_{\mathrm{lfp}}$, look at the $(\mathbb{H}(S), \mathbb{H}(\mathcal{Y}))$-bimodule $\mathbb{H}_{S\to\mathcal{Y}} := \mathrm{IndCoh}_0((S \times \mathcal{Y})^{\wedge}_{S})$. Given $\mathcal{C} \in \mathbb{H}(\mathcal{Y})$-**mod**, we form the $\mathbb{H}$-sheaf of categories

$$\mathbf{Loc}^{\mathbb{H}}_{\mathcal{Y}}(\mathcal{C}) := \Big\{\mathbb{H}_{S\to\mathcal{Y}} \underset{\mathbb{H}(\mathcal{Y})}{\otimes} \mathcal{C}\Big\}_{S}.$$

To define the right adjoint $\mathbf{\Gamma}^{\mathbb{H}}_{\mathcal{Y}}$, we need to make sure that each bimodule $\mathbb{H}_{S\to\mathcal{Y}}$ admits a right dual. Such right dual exists and it is fortunately the obvious $(\mathbb{H}(\mathcal{Y}), \mathbb{H}(S))$-bimodule

$$\mathbb{H}_{\mathcal{Y}\leftarrow S} := \mathrm{IndCoh}_0((\mathcal{Y} \times S)^{\wedge}_{S}).$$

From this, it is straightforward to see that

$$\mathbf{\Gamma}^{\mathbb{H}}_{\mathcal{Y}}(\{\mathcal{E}_S\}_S) \simeq \lim_{S\in(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})^{\mathrm{op}}} \mathbb{H}_{\mathcal{Y}\leftarrow S} \underset{\mathbb{H}(S)}{\otimes} \mathcal{E}_S,$$

with its natural left $\mathbb{H}(\mathcal{Y})$-module structure.

1.7.3 We can now state our main theorem.

Theorem 1.7.4. *Any $\mathcal{Y} \in \mathsf{Stk}^{<\infty}_{\mathrm{lfp}}$ is $\mathbb{H}$-affine, that is, the adjoint functors in (1.3) are equivalences.*

In the rest of this introduction, we will explain our two applications of this theorem: the relation with singular support as in Theorem 1.2.4, and the functoriality of $\mathbb{H}$ for algebraic stacks.

1.8 Change of coefficients

Coefficient systems form an ∞-category. By definition, a morphism $\mathbb{A} \to \mathbb{B}$ consists of an $(\mathbb{A}(S), \mathbb{B}(S))$-bimodule $M(S)$ for any $S \in \mathsf{Aff}$, and of a system of compatible equivalences

$$\mathbb{A}_{S\to T} \underset{\mathbb{A}(T)}{\otimes} M(T) \simeq M(S) \underset{\mathbb{B}(S)}{\otimes} \mathbb{B}_{S\to T}. \tag{1.4}$$

Under mild conditions, a morphism of coefficient systems $\mathbb{A} \to \mathbb{B}$ gives rise to an adjunction

$$\mathsf{ind}^{\mathbb{A}\to\mathbb{B}}_{\mathcal{Y}} : \mathsf{ShvCat}^{\mathbb{A}}(\mathcal{Y}) \rightleftarrows \mathsf{ShvCat}^{\mathbb{B}}(\mathcal{Y}) : \mathsf{oblv}^{\mathbb{A}\to\mathbb{B}}_{\mathcal{Y}}, \tag{1.5}$$

which may be regarded as a categorified version of the usual 'extension/restriction of scalars' adjunction.

Example 1.8.1. For instance, QCoh yields a morphism $\mathbb{H} \to \mathbb{D}$; that is, $\mathrm{QCoh}(S)$ is naturally an $(\mathbb{H}(S), \mathfrak{D}(S))$-bimodule and there are natural equivalences

$$\mathbb{H}_{S\to T} \underset{\mathbb{H}(T)}{\otimes} \mathrm{QCoh}(T) \simeq \mathrm{QCoh}(S) \underset{\mathfrak{D}(S)}{\otimes} \mathbb{D}_{S\to T}$$

for any $S \to T$. In fact, both sides are obviously equivalent to $\mathrm{QCoh}(S)$.

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Example 1.8.2. Similarly, IndCoh gives rise to a morphism $\mathbb{D} \to \mathbb{H}$; indeed, both sides of

$$\mathbb{D}_{S\to T} \underset{\mathbb{D}(T)}{\otimes} \mathrm{IndCoh}(T) \simeq \mathrm{IndCoh}(S) \underset{\mathbb{H}(S)}{\otimes} \mathbb{H}_{S\to T}$$

are equivalent to $\mathrm{IndCoh}(T^{\wedge}_{S})$, as shown in the main body of the paper.

Remark 1.8.3. Continuing the analogies of Remark 1.6.7, one may think of $\mathrm{QCoh}(\mathcal{Y})$ as a categorification of the algebra $\mathcal{O}_{\mathcal{Y}}$ of functions on $\mathcal{Y}$ (a left $\mathfrak{D}$-module). Likewise, $\mathrm{IndCoh}(\mathcal{Y})$ categorifies the space of distributions on $\mathcal{Y}$ (a right $\mathfrak{D}$-module). Then the $\mathbb{H}$-affineness theorem states that $\mathbb{H}$ categorifies the algebra of differential operators on $\mathcal{Y}$. These observations help remember/explain the directions of the morphisms $\mathbb{H} \to \mathbb{D}$ and $\mathbb{D} \to \mathbb{H}$ in the two examples above: QCoh is naturally a left $\mathbb{H}$-module, while IndCoh is naturally a right $\mathbb{H}$-module.

Remark 1.8.4. Our Theorem 1.9.2 shows that the morphism $\mathrm{QCoh} : \mathbb{H} \to \mathbb{D}$ is 'optimal' in that the natural monoidal functor

$$\mathfrak{D}(Y) \longrightarrow \mathsf{Fun}_{\mathbb{H}(Y)}(\mathrm{QCoh}(Y), \mathrm{QCoh}(Y))$$

is an equivalence for any $Y \in \mathsf{Sch}^{<\infty}_{\mathrm{lfp}}$. On the other hand, the morphism $\mathrm{IndCoh} : \mathbb{D} \to \mathbb{H}$ is not optimal; in another work (see [Ber18] for more in this direction), we plan to show that

$$\mathsf{Fun}_{\mathbb{H}(Y)}(\mathrm{IndCoh}(Y), \mathrm{IndCoh}(Y)) \simeq \text{‘}\mathfrak{D}\text{’}(LY), \tag{1.6}$$

where '$\mathfrak{D}$'(LY) is the monoidal DG category introduced in [Ber17b]. For Y quasi-smooth, '$\mathfrak{D}$'(LY) is closely related to $\mathfrak{D}(\mathrm{Sing}(Y))$. We remark that the above equivalence (1.6) would provide an answer to the question 'What acts on IndCoh?' raised in [AG18, Remark 1.4.3].

Example 1.8.5. Another morphism of coefficient systems of interest in this paper is $\mathbb{Q} \to \mathbb{H}$, the one induced by the monoidal functor $\mathrm{QCoh}(S) \to \mathbb{H}(S)$. In this case, the adjunction (1.5) categorifies the induction/forgetful adjunction between quasi-coherent sheaves and left $\mathfrak{D}$-modules.

1.8.6 Here is how the $\mathbb{H}$-affineness theorem (Theorem 1.7.4) implies Theorem 1.2.4. The datum of a left $\mathbb{H}(\mathcal{Y})$-action $\mathcal{C}$ corresponds the datum of an object $\widetilde{\mathcal{C}} \in \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$. Now, on the one hand $\mathsf{ShvCat}^{\mathbb{H}}$ satisfies *smooth descent*; see Theorem 6.1.2. On the other hand, singular support is computed smooth locally. Hence, we are back to Theorem 1.2.4 for affine schemes, which has already been discussed.

1.9 Functoriality of $\mathbb{H}$ for algebraic stacks

The $\mathbb{H}$-affineness theorem has another consequence: it allows to extend the assignment $\mathcal{Y} \rightsquigarrow \mathbb{H}(\mathcal{Y})$ to a functor out of a certain ∞-category of correspondences of stacks.

1.9.1 Indeed, as we prove in this paper, $\mathsf{ShvCat}^{\mathbb{H}}$ enjoys a rich functoriality: besides the structure pullbacks $f^{*,\mathbb{H}} : \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Z}) \to \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$ associated to $f : \mathcal{Y} \to \mathcal{Z}$, there are also pushforward functors $f_{*,\mathbb{H}}$ (right adjoint to pullbacks) satisfying base-change along cartesian squares.

Now, Theorem 1.7.4 guarantees that the assignment $\mathcal{Y} \rightsquigarrow \mathbb{H}(\mathcal{Y})$ enjoys a parallel functoriality, as stated in the following theorem.

THEOREM 1.9.2. *There is a natural functor*

$$\mathsf{Corr}(\mathsf{Stk}_{\mathrm{lfp}}^{<\infty})_{\mathrm{bdd;all}} \longrightarrow \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$$

that sends

$$\mathcal{X} \rightsquigarrow \mathbb{H}(\mathcal{X}), \quad [\mathcal{X} \leftarrow \mathcal{W} \rightarrow \mathcal{Y}] \rightsquigarrow \mathbb{H}_{\mathcal{X}\leftarrow\mathcal{W}\rightarrow\mathcal{Y}} := \mathrm{IndCoh}_0((\mathcal{X}\times\mathcal{Y})^{\wedge}_{\mathcal{W}}).$$

Here $\mathsf{Corr}(\mathsf{Stk}_{\mathrm{lfp}}^{<\infty})_{\mathrm{bdd;all}}$ *is the* ∞*-category whose objects are objects of* $\mathsf{Stk}_{\mathrm{lfp}}^{<\infty}$ *and whose* 1*-morphisms are given by correspondences* $[\mathcal{X} \leftarrow \mathcal{W} \rightarrow \mathcal{Y}]$ *with bounded left leg.*

1.9.3 In the rest of this introduction, we exploit such functoriality in the case of classifying spaces of algebraic groups (§ 1.10) and in the case of local systems over a smooth complete curve (§ 1.11).

1.10 $\mathbb{H}$ for Harish-Chandra

For $\mathcal{Y}$ smooth, $\mathbb{H}(\mathcal{Y})$ is equivalent to $\mathrm{IndCoh}(\mathcal{Y}\times_{\mathcal{Y}_{\mathrm{dR}}}\mathcal{Y})$, with its natural convolution monoidal structure. For instance, if G is an affine algebraic group, we have

$$\mathbb{H}(BG) \simeq \mathrm{IndCoh}(G\backslash G_{\mathrm{dR}}/G).$$

This is the monoidal category of Harish-Chandra bimodules for the group G; see [Ber17a, § 2.3] for the connection with the theory of weak/strong actions on categories. Likewise,

$$\mathbb{H}_{\mathrm{pt}\rightarrow BG} \simeq \mathrm{IndCoh}(G_{\mathrm{dR}}/G)$$

is the DG category $\mathfrak{g}\text{-}\mathsf{mod}$ of modules for the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. More generally, for a group morphism $H \rightarrow G$, we have

$$\mathbb{H}_{BG\leftarrow BH} = \mathrm{IndCoh}((BG\times BH)^{\wedge}_{BH}) \simeq \mathrm{IndCoh}(G\backslash G_{\mathrm{dR}}/H) \simeq \mathfrak{g}\text{-}\mathsf{mod}^{H,w}.$$

This is the correct derived enhancement of the ordinary category of Harish-Chandra $(\mathfrak{g}, H)$-modules.

1.10.1 Theorem 1.9.2 yields the following equivalences:

$$\mathbb{H}_{BH\rightarrow BG} \underset{\mathbb{H}(BG)}{\otimes} \mathbb{H}_{BG\rightarrow\mathrm{pt}} \xrightarrow{\simeq} \mathbb{H}_{BH\rightarrow\mathrm{pt}} \simeq \mathrm{QCoh}(BH),$$
$$\mathbb{H}_{\mathrm{pt}\rightarrow BG} \underset{\mathbb{H}(BG)}{\otimes} \mathbb{H}_{BG\leftarrow\mathrm{pt}} \xrightarrow{\simeq} \mathfrak{D}(G),$$
$$\mathbb{H}_{\mathrm{pt}\leftarrow BG} \underset{\mathbb{H}(BG)}{\otimes} \mathbb{H}_{BG\rightarrow\mathrm{pt}} \xrightarrow{\simeq} \mathfrak{D}(BG).$$

1.10.2 Another way to prove these is via the theory of DG categories with G-action; see [Ber17a, § 2]. For instance, it was proven there that, for any category $\mathcal{C}$ equipped with a right strong action of G, there are natural equivalences

$$\mathcal{C}^{G,w} \underset{\mathbb{H}(BG)}{\otimes} \mathrm{Rep}(G) \simeq \mathcal{C}^{G}, \quad \mathcal{C}^{G,w} \underset{\mathbb{H}(BG)}{\otimes} \mathfrak{g}\text{-}\mathsf{mod} \simeq \mathcal{C}.$$

Now, let $\mathcal{C} = {}^{H,w}\mathfrak{D}(G)$, $\mathcal{C} = \mathfrak{D}(G)$ and $\mathcal{C} = \mathrm{Vect}$, respectively.

1.10.3 For generalizations of these computations to the topological setting, the reader may consult [Ber19b].

1.11 The gluing theorems in geometric Langlands

More interesting than $\mathbb{H}(BG)$ is the monoidal DG category $\mathbb{H}(\mathrm{LS}_G)$, to which we now turn our attention. Observe that, by construction, we have

$$\mathbb{H}_{\mathcal{Y}\leftarrow\mathcal{X}\to\mathrm{pt}} \simeq \mathrm{IndCoh}_0(\mathcal{Y}^{\wedge}_{\mathcal{X}}).$$

With this notation, the *spectral gluing theorem* of [AG18] may be rephrased as follows: there is an explicit $\mathbb{H}(\mathrm{LS}_{\check{G}})$-linear localization adjunction

$$(\gamma^{\mathrm{spec}})^L : \mathsf{Glue}_P\, \mathbb{H}_{\mathrm{LS}_{\check{G}}\leftarrow\mathrm{LS}_{\check{P}}\to\mathrm{pt}} \rightleftarrows \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LS}_{\check{G}}) : \gamma^{\mathrm{spec}}. \tag{1.7}$$

Here we have switched to the Langlands dual $\check{G}$ as we are going to discuss Langlands duality, and it is customary to have Langlands dual groups on the spectral side.

1.11.1 Let $\check{M}$ be the Levi quotient of a parabolic $\check{P}$. By Theorem 1.9.2, we can rewrite

$$\mathbb{H}_{\mathrm{LS}_{\check{G}}\leftarrow\mathrm{LS}_{\check{P}}\to\mathrm{pt}} \simeq \mathbb{H}_{\mathrm{LS}_{\check{G}}\leftarrow\mathrm{LS}_{\check{P}}\to\mathrm{LS}_{\check{M}}} \underset{\mathbb{H}(\mathrm{LS}_{\check{M}})}{\otimes} \mathbb{H}_{\mathrm{LS}_{\check{M}}\to\mathrm{pt}} \simeq \mathbb{H}_{\mathrm{LS}_{\check{G}}\leftarrow\mathrm{LS}_{\check{P}}\to\mathrm{LS}_{\check{M}}} \underset{\mathbb{H}(\mathrm{LS}_{\check{M}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}_{\check{M}}).$$

By the $\mathbb{H}$-affineness theorem, we reinterpret the bimodule $\mathbb{H}_{\mathrm{LS}_{\check{G}}\leftarrow\mathrm{LS}_{\check{P}}\to\mathrm{LS}_{\check{M}}}$, or better the functor

$$\mathbb{E}\mathrm{is}_{\check{P}} : \mathbb{H}(\mathrm{LS}_{\check{M}})\textbf{-mod} \longrightarrow \mathbb{H}(\mathrm{LS}_{\check{G}})\textbf{-mod}$$

attached to it, as an *Eisenstein series* functor in the setting of $\mathbb{H}$-sheaves of categories.

1.11.2 These considerations shed light on the left-hand side of (1.7). Coupled with the construction of § 1.4.2, they allow us to formulate a conjecture on the automorphic side of geometric Langlands. This conjecture explains how $\mathfrak{D}(\mathrm{Bun}_G)$ can be reconstructed algorithmically out of tempered $\mathfrak{D}$-modules for all the Levis of G, including G itself.

CONJECTURE 1.11.3 (Automorphic gluing). There is an explicit $\mathbb{H}(\mathrm{LS}_{\check{G}})$-linear localization adjunction

$$\gamma^L : \mathsf{Glue}_P\, \mathbb{E}\mathrm{is}_{\check{P}}(\mathfrak{D}(\mathrm{Bun}_M)^{\mathrm{temp}}) \rightleftarrows \mathfrak{D}(\mathrm{Bun}_G) : \gamma. \tag{1.8}$$

1.11.4 We make some comments on this conjecture and on some future research directions.

(i) We will construct the adjunction (1.8) in a follow-up paper; this will be relatively easy. The difficult part is to show that the right adjoint is fully faithful.

(ii) Actually, the conjecture can be pushed even further, as it is possible to guess what the essential image γ is. This follows from an explicit description of the essential image of γ^{spec}; see [Ber18].

(iii) Clearly, Conjecture 1.11.3 is related to the extended Whittaker conjecture; see [Gai15a, Ber19a]. The left-hand side of (1.8) is expected to be smaller than the extended Whittaker category.

1.12 Conventions

We refer to [GR17], [Gai15b] or [Ber17b] for a review of our conventions concerning category theory and algebraic geometry. In particular:

- we always work over an algebraically closed field $\Bbbk$ of characteristic 0;
- we denote by DGCat the (large) symmetric monoidal ∞-category of small *cocomplete* DG categories over $\Bbbk$ and continuous functors; see [Lur17] or [GR17].

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1.13 Structure of the paper

Section 2 is devoted to recalling some higher algebra: a few facts about rigid monoidal DG categories and their module categories, as well as several $(\infty, 2)$-categorical constructions (correspondences, lax $(\infty, 2)$-functors, algebras and bimodules).

The first part of § 3 is a reminder of the theory of IndCoh_0, as developed in [Ber17b]. In the second part of the same section, we discuss the $(\infty, 2)$-categorical functoriality of $\mathbb{H}$.

Section 4 introduces the notion of coefficient system, providing several examples of interest in present, as well as future, applications. In particular, we define the (*a priori* lax) coefficient system $\mathbb{H}$ and prove it is strict.

In § 5 we discuss the (left, right, ambidextrous) Beck–Chevalley conditions for coefficient systems. These conditions (which are satisfied in the examples of interest) guarantee that the resulting theory of sheaves of categories is very rich functorially; for example, it has pushforwards and base-change.

Finally, in § 6, we define $\mathsf{ShvCat}^{\mathbb{H}}$, the theory of *sheaves of categories with local actions of Hochschild cochains*, and prove the $\mathbb{H}$-affineness of algebraic stacks.

2. Some categorical algebra

In this section we recall some $(\infty, 1)$- and $(\infty, 2)$-categorical algebra needed later in the main sections of the paper. All the results we need concern the theory of algebras and bimodules. More specifically, we first need criteria for dualizability of bimodule categories; furthermore, we need some abstract constructions that relate 'algebras and bimodules' with $(\infty, 2)$-categories of correspondences.

We advise the reader to skip this material and refer to it only if necessary.

2.1 Dualizability of bimodule categories

Recall that DGCat admits colimits (as well as limits) and its tensor product preserves colimits in each variable [Lur17]. Hence, by [Lur17] again, we have a good theory of dualizability of algebras and bimodules in DGCat, whose main points we record below. We will need a criterion that relates the dualizability of a bimodule to the dualizability of its underlying DG category.

2.1.1 First, let us fix some terminology. Algebra objects in a symmetric monoidal ∞-category are always unital in this paper. In particular, monoidal DG categories are unital. Given A an algebra, denote by A^{rev} the algebra obtained by reversing the order of the multiplication. For a left A-module M and a right A-module N, we denote by $\mathsf{pr} : N \otimes M \to N \otimes_A M$ the tautological functor.

Our conventions regarding bimodules are as follows: an (A, B)-bimodule M is acted on on the left by A and on the right by B. Hence, endowing $C \in \mathsf{DGCat}$ with the structure of an (A, B)-bimodule amounts to endowing it with the structure of a left $A \otimes B^{\mathrm{rev}}$-module.

2.1.2 Let M be an (A, B)-bimodule. We say that M is *left dualizable* (as an (A, B)-bimodule) if there exists a (B, A)-bimodule M^L (called the *left dual* of M) realizing an adjunction

$$M^L \otimes_A - : A\text{-}\mathsf{mod} \rightleftarrows B\text{-}\mathsf{mod} : M \otimes_B -.$$

Similarly, M is *right dualizable* if there exists $M^R \in (B, A)\text{-}\mathsf{bimod}$ (the *right dual* of M) realizing an adjunction

$$M \otimes_B - : B\text{-}\mathsf{mod} \rightleftarrows A\text{-}\mathsf{mod} : M^R \otimes_A -.$$

We say that an (A,B)-bimodule M is *ambidextrous* if both M^L and M^R exist and are equivalent as (B,A)-bimodules.

Remark 2.1.3. Being (left or right) dualizable as a $(\mathrm{Vect},\mathrm{Vect})$-bimodule is equivalent to being dualizable as a DG category. By definition, being 'left (or right) dualizable as a right A-module' means being 'left (or right) dualizable as a (Vect,A)-module'. Similarly for left A-modules.

2.1.4 Let M be an (A,B)-bimodule which is dualizable as a DG category. Then we can contemplate three (B,A)-bimodules: M^L, M^R (if they exist) as well as M^*, the dual of $\mathsf{oblv}_{A,B}(M)$ equipped with the dual actions.

In particular, a monoidal DG category A is called *proper* if it is dualizable as a plain DG category. In this case, we denote by $S_A := A^*$ its dual, equipped with the tautological (A,A)-bimodule structure.

2.1.5 Recall the notion of *rigid* monoidal DG category; see [Gai15b, Appendix D]. Any rigid A is automatically proper. Furthermore, its dual $S_A := A^*$ comes equipped with the canonical object $1_A^{\mathrm{fake}} := (u^R)^\vee(\Bbbk)$, where u^R is the (continuous) right adjoint to the unit functor $u : \mathrm{Vect} \to A$. The left A-linear functor

$$\sigma_A : A \longrightarrow S_A, \quad a \rightsquigarrow a \star 1_A^{\mathrm{fake}}$$

is an equivalence; in particular, any rigid monoidal category is self-dual. We say that A is *very rigid* if the canonical equivalence $\sigma_A : A \to S_A$ admits a lift to an equivalence of (A,A)-bimodules.[4]

PROPOSITION 2.1.6. *Let A, B be rigid monoidal DG categories and M an (A,B)-bimodule which is dualizable as a DG category. Then M is right dualizable as an (A,B)-bimodule and $M^R \simeq M^* \otimes_A S_A$. Likewise, M is left dualizable and $M^L \simeq S_B \otimes_B M^*$.*

Proof. The formula for M^R is proven as in the 'if' direction of [Gai15b, Proposition D.5.4], which in turn is a consequence of [Gai15b, Corollary D.4.5]. In the notation there, the twist $(-)_{\psi_A}$ corresponds to our $- \otimes_A S_A$. The formula for M^L follows similarly. □

COROLLARY 2.1.7. *Let A, B be very rigid and M an (A,B)-bimodule which is dualizable as a DG category. Then we have canonical (B,A)-linear equivalences $M^R \simeq M^* \simeq M^L$.*

2.2 Some $(\infty, 2)$-categorical algebra

In this section we recall some abstract $(\infty,2)$-categorical nonsense and provide some examples of $(\infty,2)$-categories and of lax $(\infty,2)$-functors between them. All the statements below look obvious enough and no proof will be given.

2.2.1 We assume familiarity with the notion of $(\infty,2)$-*category* and with the notion of (lax) $(\infty,2)$-functor between $(\infty,2)$-categories; see, for example, [GR17, Appendix A]. For an $(\infty,2)$-category $\mathbf{C}$, we denote by $\mathbf{C}^{1-\mathrm{op}}$ the $(\infty,2)$-category obtained from $\mathbf{C}$ by flipping the 1-arrows. Similarly, we denote by $\mathbf{C}^{2-\mathrm{op}}$ the $(\infty,2)$-category obtained by flipping the directions of the 2-arrows.

[4] Compare this notion with the more general notion of 'symmetric Frobenius algebra object', discussed in [Lur17, Remark 4.6.5.7].

2.2.2 *Correspondences.* Let $\mathcal{C}$ be an ∞-category equipped with fibre products. We refer to [GR17, ch. V.1] for the construction of the ∞-category of correspondences associated to $\mathcal{C}$. In particular, for vert and horiz two subsets of the space morphisms of $\mathcal{C}$ satisfying some natural requirements, one considers the ∞-category $\mathsf{Corr}(\mathcal{C})_{\mathrm{vert;horiz}}$, defined in the usual way: objects of $\mathsf{Corr}(\mathcal{C})_{\mathrm{vert;horiz}}$ coincide with the objects of $\mathcal{C}$, while 1-morphisms in $\mathsf{Corr}(\mathcal{C})_{\mathrm{vert;horiz}}$ are given by correspondences

$$[c \leftarrow h \rightarrow d]$$

with left leg in vert and right leg in horiz.

To enhance $\mathsf{Corr}(\mathcal{C})_{\mathrm{vert;horiz}}$ to an $(\infty, 2)$-category, we must further choose a subset adm $\subset$ vert $\cap$ horiz of *admissible arrows*, closed under composition. Then, following [GR17, ch. V.1], we define the $(\infty, 2)$-category

$$\mathsf{Corr}(\mathcal{C})^{\mathrm{adm}}_{\mathrm{vert;horiz}}.$$

This is one of the most important $(\infty, 2)$-categories of the present paper.

To fix notation, recall that a 2-arrow

$$[c \leftarrow h \rightarrow d] \implies [c \leftarrow h' \rightarrow d]$$

in $\mathsf{Corr}(\mathcal{C})^{\mathrm{adm}}_{\mathrm{vert;horiz}}$ is by definition an admissible arrow $h \to h'$ compatible with the maps to $c \times d$.

As explained in [GR17, ch. V.3], $\mathsf{Corr}(\mathcal{C})^{\mathrm{adm}}_{\mathrm{vert;horiz}}$ is symmetric monoidal with tensor product induced by the cartesian symmetric monoidal product on $\mathcal{C}$.

2.2.3 *Algebras and bimodules.* The other important $(\infty, 2)$-category of this paper is $\mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$, the $(\infty, 2)$-category of monoidal DG categories, bimodules, and natural transformations. We refer to [Hau17] for a rigorous construction. More generally, that paper gives a construction of $\mathsf{ALG}^{\mathrm{bimod}}(\mathcal{S})$ for any (nice enough) symmetric monoidal $(\infty, 2)$-category $\mathcal{S}$.

We denote by $\mathsf{Alg}^{\mathrm{bimod}}(\mathcal{S})$ the $(\infty, 1)$-category underlying $\mathsf{ALG}^{\mathrm{bimod}}(\mathcal{S})$: that is, the former is obtained from the latter by discarding non-invertible 2-morphisms.

2.2.4 There is an obvious functor

$$\iota_{\mathsf{Alg} \to \mathsf{Bimod}} : \mathsf{Alg}(\mathsf{DGCat})^{\mathrm{op}} \longrightarrow \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat}) \tag{2.1}$$

that is the identity on objects and that sends a monoidal functor $A \to B$ to the (B, A)-bimodule B.

The tautological functor

$$\mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})^{\mathrm{op}} \xrightarrow{\textbf{-mod}} \mathsf{Cat}_\infty$$

upgrades to a (strict) $(\infty, 2)$-functor

$$\mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})^{1-\mathrm{op}} \xrightarrow{\textbf{-mod}} \mathsf{Cat}_\infty,$$

where now Cat_∞ is considered as an $(\infty, 2)$-category.

2.2.5 Let $\mathcal{C}$ denote an $(\infty, 1)$-category admitting fibre products and equipped with the cartesian symmetric monoidal structure. Let $F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{DGCat}$ be a lax-monoidal functor. (The example we have in mind is $\mathcal{C} = \mathsf{PreStk}$ and $F = \mathrm{QCoh}$.)

These data give rise to a lax $(\infty, 2)$-functor

$$\widetilde{F} : \left(\mathsf{Corr}(\mathcal{C})^{\mathrm{all}}_{\mathrm{all;all}}\right)^{2-\mathrm{op}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat}),$$

described informally as follows:

- an object $c \in \mathcal{C}$ gets sent to $F(c)$, with its natural monoidal structure;
- a correspondence $[c \leftarrow h \rightarrow d]$ gets sent to the $(F(c), F(d))$-bimodule $F(h)$;
- a map between correspondences, given by an arrow $h' \rightarrow h$ over $c \times d$, gets sent to the associated $(F(c), F(d))$-linear arrow $F(h) \rightarrow F(h')$;
- for two correspondences $[c \leftarrow h \rightarrow d]$ and $[d \leftarrow k \rightarrow e]$, the lax composition is encoded by the natural $(F(c), F(e))$-linear arrow

$$F(h) \underset{F(d)}{\otimes} F(k) \longrightarrow F(h \times_d k).$$

2.2.6 Here is another example of the interaction between lax-monoidal functors and lax $(\infty, 2)$-functors. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a lax-monoidal functor between 'well-behaved' monoidal $(\infty, 1)$-categories. Then F induces a lax $(\infty, 2)$-functor

$$\widetilde{F} : \mathsf{ALG}^{\mathrm{bimod}}(\mathcal{C}) \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathcal{D}).$$

To define it, it suffices to recall that, since F is lax monoidal, it preserves algebra and bimodule objects. The fact that $\widetilde{F}$ is a lax $(\infty, 2)$-functor comes from the natural map (not necessarily an isomorphism)

$$F(c') \otimes_{F(c)} F(c'') \longrightarrow F(c' \otimes_c c'').$$

2.2.7 Recall the ∞-category $\mathrm{Mod}(\mathsf{DGCat})$ whose objects are pairs (A, M) with A a monoidal DG category and M an A-module. Morphisms $(A, M) \rightarrow (B, N)$ consist of pairs (ϕ, f) where $\phi : A \rightarrow B$ is a monoidal functor and $f : M \rightarrow N$ an A-linear functor.

There is a lax $(\infty, 2)$-functor

$$\mathsf{LOOP}_{\mathrm{Mod}} : \mathrm{Mod}(\mathsf{DGCat})^{\mathrm{op}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat}), \tag{2.2}$$

described informally as follows:

- an object $(A, M) \in \mathrm{Mod}(\mathsf{DGCat})$ goes to the monoidal DG category $\mathsf{End}_A(M) := \mathsf{Fun}_A(M, M)$;
- a morphism $(A, M) \xrightarrow{(\phi, f)} (B, N)$ gets sent to the $(\mathsf{End}_B(N), \mathsf{End}_A(M))$-bimodule $\mathsf{Fun}_A(M, N)$;
- a composition $(A, M) \xrightarrow{(\phi, f)} (B, N) \xrightarrow{(\psi, g)} (C, P)$ goes over to the $(\mathsf{End}_C(P), \mathsf{End}_A(M))$-bimodule

$$\mathsf{Fun}_B(N, P) \underset{\mathsf{End}_B(N)}{\otimes} \mathsf{Fun}_A(M, N);$$

- the lax structure comes from the tautological morphism (not invertible, in general)

$$\mathsf{Fun}_B(N, P) \underset{\mathsf{End}_B(N)}{\otimes} \mathsf{Fun}_A(M, N) \longrightarrow \mathsf{Fun}_A(M, P) \tag{2.3}$$

induced by composition.

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2.2.8 For later use, we record here the following tautological observation. Let $\mathcal{I}$ be an $(\infty,1)$-category and $\mathbb{A} : \mathcal{I} \to \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$ be a lax $(\infty,2)$-functor. Assume given the following data:

- for each $i \in \mathcal{I}$, a monoidal subcategory $\mathbb{A}'(i) \hookrightarrow \mathbb{A}(i)$;
- for each $i \to j$, a full subcategory $\mathbb{A}'_{i\to j} \hookrightarrow \mathbb{A}_{i\to j}$ preserved by the $(\mathbb{A}'(i), \mathbb{A}'(j))$-action.

Assume also that, for each string $i \to j \to k$, the functor

$$\mathbb{A}'_{i\to j} \otimes \mathbb{A}'_{j\to k} \hookrightarrow \mathbb{A}_{i\to j} \otimes \mathbb{A}_{j\to k} \xrightarrow{\mathsf{pr}} \mathbb{A}_{i\to j} \otimes_{\mathbb{A}(j)} \mathbb{A}_{j\to k} \xrightarrow{\eta_{i\to j\to k}} \mathbb{A}_{i\to k}$$

lands in $\mathbb{A}'_{i\to k} \subseteq \mathbb{A}_{i\to k}$. Then the assignment

$$i \rightsquigarrow \mathbb{A}'(i), \quad (i \to j) \rightsquigarrow \mathbb{A}'_{i\to j}$$

naturally upgrades to a *lax* $(\infty,2)$-functor $\mathbb{A}' : \mathcal{I} \to \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$.

3. IndCoh_0 on formal moduli problems

In the section we study the sheaf theory IndCoh_0 from which $\mathbb{H}$ originates. As mentioned in the introduction to [Ber17b], IndCoh_0 enjoys $(\infty,1)$-categorical functoriality as well as $(\infty,2)$-categorical functoriality. The former was developed in [Ber17b], and is recalled here in Theorem 3.1.6. The latter is one of the main subjects of the present paper: it consists of an extension of the assignment $\mathcal{Y} \rightsquigarrow \mathbb{H}(\mathcal{Y})$ to a lax $(\infty,2)$-functor from a certain $(\infty,2)$-category of correspondences to $\mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$.

3.1 $(\infty,1)$-categorical functoriality

In this section we review the definition of the assignment IndCoh_0 and its basic functoriality. We follow [Ber17b] closely.

3.1.1 Let Stk denote the ∞-category of perfect quasi-compact algebraic stacks of finite type and with affine diagonal; see, for example, [BFN10]. Inside Stk, we single out the subcategory $\mathsf{Stk}^{<\infty}_{\mathrm{lfp}}$ consisting of those stacks that are bounded and with perfect cotangent complex (both properties can be checked on an atlas).

3.1.2 For $\mathcal{C}$ an ∞-category, denote by $\mathsf{Arr}(\mathcal{C}) := \mathcal{C}^{\Delta^1}$ the ∞-category whose objects are arrows in $\mathcal{C}$ and whose 1-morphisms are commutative squares. We will be interested in the ∞-category $\mathsf{Arr}(\mathsf{Stk}^{<\infty}_{\mathrm{lfp}})$ and in the functor

$$\mathrm{IndCoh}_0 : \mathsf{Arr}(\mathsf{Stk}^{<\infty}_{\mathrm{lfp}})^{\mathrm{op}} \longrightarrow \mathsf{DGCat} \tag{3.1}$$

defined by

$$[\mathcal{Y} \to \mathcal{Z}] \rightsquigarrow \mathrm{IndCoh}_0(\mathcal{Z}^\wedge_{\mathcal{Y}}).$$

Recall from [AG18] or [Ber17b] that $\mathrm{IndCoh}_0(\mathcal{Z}^\wedge_{\mathcal{Y}})$ is defined by the pullback square

$$\begin{array}{ccc} \mathrm{IndCoh}_0(\mathcal{Z}^\wedge_{\mathcal{Y}}) & \xrightarrow{('f)^{!,0}} & \mathrm{QCoh}(\mathcal{Y}) \\ \big\downarrow \iota & & \big\downarrow \Upsilon_{\mathcal{Y}} \\ \mathrm{IndCoh}(\mathcal{Z}^\wedge_{\mathcal{Y}}) & \xrightarrow{('f)^{!}} & \mathrm{IndCoh}(\mathcal{Y}) \end{array} \tag{3.2}$$

In particular, when writing $\mathrm{IndCoh}_0(\mathcal{Z}^\wedge_{\mathcal{Y}})$ we are committing a potentially dangerous abuse of notation: it would be better to write $\mathrm{IndCoh}_0(\mathcal{Y} \to \mathcal{Z}^\wedge_{\mathcal{Y}})$, as the latter category depends on the formal moduli problem $\mathcal{Y} \to \mathcal{Z}^\wedge_{\mathcal{Y}}$ and in particular on the derived structure of $\mathcal{Y}$.

3.1.3 For two objects $[\mathcal{Y}_1 \to \mathcal{Z}_1]$ and $[\mathcal{Y}_2 \to \mathcal{Z}_2]$ in $\mathsf{Arr}(\mathsf{Stk}_{\mathrm{lfp}}^{<\infty})$, a morphism ξ from the former to the latter is given by a commutative square

$$\begin{array}{ccc} \mathcal{Y}_1 & \xrightarrow{\xi_{\mathrm{top}}} & \mathcal{Y}_2 \\ \downarrow{}'f_1 & & \downarrow{}'f_2 \\ \mathcal{Z}_1 & \xrightarrow{\xi_{\mathrm{bottom}}} & \mathcal{Z}_2 \end{array} \tag{3.3}$$

The structure pullback functor

$$\xi^{!,0} : \mathrm{IndCoh}_0((\mathcal{Z}_2)^{\wedge}_{\mathcal{Y}_2}) \longrightarrow \mathrm{IndCoh}_0((\mathcal{Z}_1)^{\wedge}_{\mathcal{Y}_1})$$

is the obvious one induced by the pullback functor $\xi^! : \mathrm{IndCoh}((\mathcal{Z}_2)^{\wedge}_{\mathcal{Y}_2}) \to \mathrm{IndCoh}((\mathcal{Z}_1)^{\wedge}_{\mathcal{Y}_1})$, where we are abusing notation again by confusing ξ with the map $(\mathcal{Z}_1)^{\wedge}_{\mathcal{Y}_1} \to (\mathcal{Z}_2)^{\wedge}_{\mathcal{Y}_2}$. We will do this throughout the paper, and hope it will not be too unpleasant for the reader.

3.1.4 Let us now recall the extension of (3.1) to a functor out of a category of correspondences. Notice that $\mathsf{Arr}(\mathsf{PreStk})$ admits fibre products, computed objectwise; its subcategory $\mathsf{Arr}(\mathsf{Stk}_{\mathrm{lfp}}^{<\infty})$ is closed under products, but not under fibre products. Thus, to have a well-defined category of correspondences, we must choose appropriate classes of horizontal and vertical arrows.

We say that a commutative diagram (3.1.3), thought of as a morphism in $\mathsf{Arr}(\mathsf{Stk}_{\mathrm{lfp}}^{<\infty})$, is schematic (or bounded, or proper) if so is the top horizontal map. It is clear that

$$\mathsf{Corr}\big(\mathsf{Arr}(\mathsf{Stk}_{\mathrm{lfp}}^{<\infty})\big)_{\mathrm{schem\&bdd;all}} \tag{3.4}$$

is well defined.

For the theorem below, we will need to further upgrade (3.4) to an $(\infty, 2)$-category by allowing as admissible arrows (see § 2.2.2 for the terminology) those ξ that are schematic, bounded and proper. We denote by

$$\mathsf{Corr}\big(\mathsf{Arr}(\mathsf{Sch}_{\mathrm{lfp}}^{<\infty})\big)_{\mathrm{schem\&bdd;all}}^{\mathrm{schem\&bdd\&proper}}$$

the resulting $(\infty, 2)$-category.

3.1.5 If ξ is bounded and schematic in the above sense, then the pushforward $\xi_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}((\mathcal{Z}_1)^{\wedge}_{\mathcal{Y}_1}) \to \mathrm{IndCoh}((\mathcal{Z}_2)^{\wedge}_{\mathcal{Y}_2})$ is continuous and preserves the IndCoh_0-subcategories, thereby descending to a functor $\xi_{*,0}$. For the proof, see [Ber17b].

Theorem 3.1.6. *The above pushforward functors upgrade the functor* $\mathrm{IndCoh}_0^!$ *of (3.1) to an* $(\infty, 2)$*-functor*

$$\mathrm{IndCoh}_0 : \mathsf{Corr}\big(\mathsf{Arr}(\mathsf{Sch}_{\mathrm{lfp}}^{<\infty})\big)_{\mathrm{schem\&bdd;all}}^{\mathrm{schem\&bdd\&proper}} \longrightarrow \mathsf{DGCat},$$

where DGCat *is viewed as an* $(\infty, 2)$*-category in the obvious way.*

Remark 3.1.7. The existence of the above $(\infty, 2)$-functor is deduced (essentially formally) by the $(\infty, 2)$-functor

$$\mathrm{IndCoh} : \mathsf{Corr}(\mathsf{PreStk}_{\mathrm{laft}})_{\text{ind-inf-schem; all}}^{\text{ind-inf-sch \& ind-proper}} \longrightarrow \mathsf{DGCat} \tag{3.5}$$

constructed in [GR17, ch. III.3]. For later use, we will also need another fact from the same book: the above $(\infty, 2)$-category of correspondences possesses a symmetric monoidal structure, and (3.5) is naturally symmetric monoidal; see [GR17, ch. V.3]. It follows that the $(\infty, 2)$-functor on Theorem 3.1.6 is symmetric monoidal, too.

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3.1.8 *Example.* For $f : \mathcal{Y} \to \mathcal{Z}$, the admissible arrow $\mathcal{Y} \to \mathcal{Z}^\wedge_\mathcal{Y}$ yields an adjuction

$$\mathrm{QCoh}(\mathcal{Y}) \underset{('f)^{!,0} := \Phi_\mathcal{Y} \circ ('f)^!}{\overset{('f)_{*,0} \simeq ('f)_*^{\mathrm{IndCoh}} \circ \Upsilon_\mathcal{Y}}{\rightleftarrows}} \mathrm{IndCoh}_0(\mathcal{Z}^\wedge_\mathcal{Y}). \tag{3.6}$$

Let us also recall that $\mathrm{IndCoh}_0(\mathcal{Z}^\wedge_\mathcal{Y})$ is self-dual and that these two adjoints $('f)_{*,0}$ and $('f)^{!,0}$ are dual to each other.

3.2 $(\infty, 2)$-categorical functoriality

In this section we enhance the assignment

$$\mathcal{X} \rightsquigarrow \mathrm{IndCoh}_0((\mathcal{X} \times \mathcal{X})^\wedge_\mathcal{X}) =: \mathbb{H}(\mathcal{X}),$$
$$[\mathcal{X} \leftarrow \mathcal{W} \to \mathcal{Y}] \rightsquigarrow \mathrm{IndCoh}_0((\mathcal{X} \times \mathcal{Y})^\wedge_\mathcal{W}) =: \mathbb{H}^{\mathrm{geom}}_{\mathcal{X} \leftarrow \mathcal{W} \to \mathcal{Y}}$$

to a lax $(\infty, 2)$-functor

$$\mathbb{H}^{\mathrm{geom}} : \mathsf{Corr}\big(\mathsf{Stk}^{<\infty}_{\mathrm{lfp}}\big)^{\mathrm{schem\&bdd\&proper}}_{\mathrm{bdd;all}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat}),$$

which we will prove is strict towards the end of the paper (Theorem 6.5.3). Here we have used the notation $\mathbb{H}^{\mathrm{geom}}$ for emphasis, as later we will encounter a categorical construction producing a lax $(\infty, 2)$-functor $\mathbb{H}^{\mathrm{cat}}$. We will eventually show that these two lax $(\infty, 2)$-functors are identified and then denoted simply by $\mathbb{H}$.

Remark 3.2.1. The condition of boundedness of the horizontal arrows is necessary to have a well-defined ∞-category of correspondences.

3.2.2 We begin by observing that, for any $\mathcal{X} \in \mathsf{Stk}$, the DG category

$$\mathbb{I}^{\wedge,\mathrm{geom}}(\mathcal{X}) := \mathrm{IndCoh}(\mathcal{X} \times_{\mathcal{X}_{\mathrm{dR}}} \mathcal{X})$$

possesses a convolution monoidal structure and that, for any correspondence $[\mathcal{Y} \leftarrow \mathcal{W} \to \mathcal{Z}]$ in Stk, the DG category

$$\mathbb{I}^{\wedge,\mathrm{geom}}_{\mathcal{Y} \leftarrow \mathcal{W} \to \mathcal{Z}} := \mathrm{IndCoh}((\mathcal{X} \times \mathcal{Y})^\wedge_\mathcal{W}) \simeq \mathrm{IndCoh}(\mathcal{Y} \times_{\mathcal{Y}_{\mathrm{dR}}} \mathcal{W}_{\mathrm{dR}} \times_{\mathcal{Z}_{\mathrm{dR}}} \mathcal{Z})$$

admits the structure of an $(\mathbb{I}^{\wedge,\mathrm{geom}}(\mathcal{Y}), \mathbb{I}^{\wedge,\mathrm{geom}}(\mathcal{Z}))$-bimodule.

3.2.3 Let us now enhance the assignment

$$\mathcal{X} \rightsquigarrow \mathrm{IndCoh}((\mathcal{X} \times \mathcal{X})^\wedge_\mathcal{X}) =: \mathbb{I}^{\wedge,\mathrm{geom}}(\mathcal{X}),$$
$$[\mathcal{X} \leftarrow \mathcal{W} \to \mathcal{Y}] \rightsquigarrow \mathrm{IndCoh}((\mathcal{X} \times \mathcal{Y})^\wedge_\mathcal{W}) =: \mathbb{I}^{\wedge,\mathrm{geom}}_{\mathcal{X} \leftarrow \mathcal{W} \to \mathcal{Y}}$$

to a *lax* $(\infty, 2)$-functor

$$\mathbb{I}^{\wedge,\mathrm{geom}} : \mathsf{Corr}(\mathsf{Stk})^{\mathrm{schem\&proper}}_{\mathrm{all;all}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat}). \tag{3.7}$$

To construct this, we first appeal to the lax symmetric monoidal structure on (3.5): § 2.2.6 yields a lax $(\infty, 2)$-functor

$$\mathrm{IndCoh} : \mathsf{ALG}^{\mathrm{bimod}}\big(\mathsf{Corr}(\mathsf{PreStk}_{\mathrm{laft}})^{\text{ind-inf-sch \& ind-proper}}_{\text{ind-inf-schem; all}}\big) \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat}).$$

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All that remains is to precompose with the lax $(\infty,2)$-functor

$$\mathsf{Corr}(\mathsf{Stk})^{\text{schem\&proper}}_{\text{all;all}} \longrightarrow \mathsf{ALG}^{\text{bimod}}(\mathsf{Corr}(\mathsf{PreStk}_{\text{laft}}))^{\text{ind-inf-sch \& ind-proper}}_{\text{ind-inf-schem; all}} \tag{3.8}$$

that sends

$$\begin{gathered} \mathcal{Y} \rightsquigarrow \mathcal{Y} \times_{\mathcal{Y}_{\text{dR}}} \mathcal{Y}; \\ [\mathcal{Y} \leftarrow \mathcal{W} \rightarrow \mathcal{Z}] \rightsquigarrow \mathcal{Y}^{\wedge}_{\mathcal{W}} \underset{\mathcal{W}_{\text{dR}}}{\times} \mathcal{Z}^{\wedge}_{\mathcal{W}} \simeq \mathcal{Y} \times_{\mathcal{Y}_{\text{dR}}} \mathcal{W}_{\text{dR}} \times_{\mathcal{Z}_{\text{dR}}} \mathcal{Z}; \\ \left[\mathcal{Y} \begin{array}{c} \swarrow \mathcal{U} \searrow \\ \downarrow f \\ \nwarrow \mathcal{W} \nearrow \end{array} \mathcal{Z}\right] \rightsquigarrow \left(\mathcal{Y} \times_{\mathcal{Y}_{\text{dR}}} \mathcal{U}_{\text{dR}} \times_{\mathcal{Z}_{\text{dR}}} \mathcal{Z} \xrightarrow{\widetilde{f_{\text{dR}}}} \mathcal{Y} \times_{\mathcal{Y}_{\text{dR}}} \mathcal{W}_{\text{dR}} \times_{\mathcal{Z}_{\text{dR}}} \mathcal{Z}\right). \end{gathered}$$

Observe that the requirement that f be schematic and proper implies that f_{dR}, and hence $\widetilde{f_{\text{dR}}}$, is inf-schematic and ind-proper.

Remark 3.2.4. The lax $(\infty,2)$-functor (3.8) is a geometric version of the formally similar lax $(\infty,2)$-functor (2.2).

3.2.5 Let us now turn to the construction of $\mathbb{H}^{\text{geom}}$. For $\mathcal{Y} \in \mathsf{Stk}^{<\infty}_{\text{lfp}}$, the canonical inclusion

$$\iota : \text{IndCoh}_0(\mathcal{Y} \times_{\mathcal{Y}_{\text{dR}}} \mathcal{Y}) \hookrightarrow \text{IndCoh}(\mathcal{Y} \times_{\mathcal{Y}_{\text{dR}}} \mathcal{Y})$$

is monoidal. Moreover, the left action of $\text{IndCoh}_0(\mathcal{Y} \times_{\mathcal{Y}_{\text{dR}}} \mathcal{Y})$ on $\text{IndCoh}((\mathcal{Y} \times \mathcal{Z})^{\wedge}_{\mathcal{W}})$ preserves the subcategory $\text{IndCoh}_0((\mathcal{Y} \times \mathcal{Z})^{\wedge}_{\mathcal{W}})$. This is an easy diagram chase left to the reader.

Thus, we are in a position to apply the paradigm of § 2.2.8 to obtain a lax $(\infty,2)$-functor

$$\mathbb{H}^{\text{geom}} : \mathsf{Corr}\big(\mathsf{Stk}^{<\infty}_{\text{lfp}}\big)^{\text{schem\&bdd\&proper}}_{\text{bdd;all}} \longrightarrow \mathsf{ALG}^{\text{bimod}}(\mathsf{DGCat}), \tag{3.9}$$

as desired. We repeat here that one of the aims of this paper is to show that such lax $(\infty,2)$-functor is actually strict: this is accomplished in Theorem 6.5.3. In the next section, we give an overview of the strategy of the proof of such theorem. This could serve as a guide through the constructions of the remainder of the present paper.

3.3 Outline of the proof of Theorem 6.5.3

It suffices to prove that the lax $(\infty,2)$-functor $\mathbb{H}^{\text{geom}} : \mathsf{Corr}\big(\mathsf{Stk}^{<\infty}_{\text{lfp}}\big)_{\text{bdd;all}} \to \mathsf{ALG}^{\text{bimod}}(\mathsf{DGCat})$ is strict. We will proceed in stages.

3.3.1 First, we look at the restriction of $\mathbb{H}^{\text{geom}}$ along the functor

$$\mathsf{Aff}^{<\infty}_{\text{lfp}} \to \mathsf{Corr}\big(\mathsf{Stk}^{<\infty}_{\text{lfp}}\big)_{\text{bdd;all}}$$

which is the natural inclusion on objects, and $[S \to T] \rightsquigarrow [S \xleftarrow{=} S \to T]$ on 1-morphisms.

Using results from the theory of ind-coherent sheaves, we show in Theorem 4.3.4 that such lax $(\infty,2)$-functor is strict. By definition, this is simply the functor $\mathbb{H} : \mathsf{Aff}^{<\infty}_{\text{lfp}} \to \mathsf{Alg}^{\text{bimod}}(\mathsf{DGCat})$ discussed in § 1.6.3.

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3.3.2 Next, we show that the restriction of $\mathbb{H}^{\mathrm{geom}}$ to $\mathsf{Corr}(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{\mathrm{bdd;all}}$ is strict (Corollary 5.2.13). We do so in an indirect way, by establishing some important duality properties of $\mathbb{H}$. Namely, we show that, for each map $U \to T$ in $\mathsf{Aff}_{\mathrm{lfp}}^{<\infty}$, the bimodule $\mathbb{H}_{U\to T}$ admits a right dual (which happens to be a left dual as well), denoted by $\mathbb{H}_{T\leftarrow U}$. Having such right duals allows us to form the bimodules

$$\mathbb{H}_{S\leftarrow U\to T} := \mathbb{H}_{S\leftarrow U} \underset{\mathbb{H}(U)}{\otimes} \mathbb{H}_{U\to T}, \quad \mathbb{H}_{S\to V\leftarrow T} := \mathbb{H}_{S\to V} \underset{\mathbb{H}(V)}{\otimes} \mathbb{H}_{V\leftarrow T}.$$

We also show that $\mathbb{H}_{S\to V\leftarrow T} \simeq \mathbb{H}_{S\leftarrow S\times_V T\to T}$ naturally, provided that at least one arrow between $S \to V$ and $T \to V$ is bounded. This is enough to extend $\mathbb{H}$ to a strict functor

$$\mathbb{H}^{\mathsf{Corr}} : \mathsf{Corr}(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{\mathrm{bdd;all}} \longrightarrow \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat}).$$

By inspection, such functor coincides with the restriction of $\mathbb{H}^{\mathrm{geom}}$ to $\mathsf{Corr}(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{\mathrm{bdd;all}}$, whence the latter is also strict.

Remark 3.3.3. The fact that left and right duals coincide implies that we could also have defined $\mathbb{H}^{\mathsf{Corr}}$ on $\mathsf{Corr}(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{\mathrm{all;bdd}}$. These two versions of $\mathbb{H}^{\mathsf{Corr}}$, exchanged by duality, agree on $\mathsf{Corr}(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{\mathrm{bdd;bdd}}$.

3.3.4 To study $\mathbb{H}^{\mathrm{geom}}$ on stacks, we introduce the sheaf theory $\mathsf{ShvCat}^{\mathbb{H}}$, which is the right Kan extension of the functor

$$(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})^{\mathrm{op}} \to \mathsf{Cat}_\infty, \quad S \rightsquigarrow \mathbb{H}(S)\text{-}\mathbf{mod}.$$

Note that Theorem 4.3.4 is essential to make this well defined.

In principle, $\mathsf{ShvCat}^{\mathbb{H}}$ comes equipped only with pullback functors. However, thanks to the existence of the right duals $\mathbb{H}_{T\leftarrow S}$, there are also $*$-pushforward functors (right adjoints to pullbacks), which turn out to satisfy base-change against pullbacks. Symmetrically, the existence of the left duals provides !-pushforward functors (left adjoints to pullbacks), also satisfying base-change against pullbacks.[5]

3.3.5 In Theorem 6.5.1, we prove the $\mathbb{H}$-affineness theorem, which states that, for any $\mathcal{Y} \in \mathsf{Stk}_{\mathrm{lfp}}^{<\infty}$, the ∞-category $\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$ is equivalent to $\mathbb{H}^{\mathrm{geom}}(\mathcal{Y})\text{-}\mathbf{mod}$. This theorem, together with the above base-change properties, automatically upgrades the assignment $\mathcal{Y} \rightsquigarrow \mathbb{H}^{\mathrm{geom}}(\mathcal{Y})$ to a strict $(\infty, 2)$-functor out of $\mathsf{Corr}(\mathsf{Stk}_{\mathrm{lfp}}^{<\infty})_{\mathrm{bdd;all}}$. Fortunately, such functor is easily seen to match with $\mathbb{H}^{\mathrm{geom}}$, thereby proving that the latter is strict, too.

3.3.6 An important technical result, which we use frequently, is the smooth descent property for $\mathsf{ShvCat}^{\mathbb{H}}$, proven in § 6.1: any object $\mathcal{C} \in \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$ is determined by its restrictions along *smooth* maps $S \to \mathcal{Y}$, with S affine. This is a very convenient simplification. For instance, let $\mathrm{IndCoh}_{/\mathcal{Y}} \in \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$ be the sheaf corresponding to $\mathrm{IndCoh}(\mathcal{Y}) \in \mathbb{H}^{\mathrm{geom}}(\mathcal{Y})\text{-}\mathbf{mod}$ via $\mathbb{H}$-affineness. In § 6.6 we will show that the restriction of $\mathrm{IndCoh}_{/\mathcal{Y}}$ along a smooth map $S \to \mathcal{Y}$ is the $\mathbb{H}(S)$-module $\mathrm{IndCoh}(S)$, whereas the restriction along a non-smooth map does not admit such a simple description.

[5] We will eventually show that pullbacks in $\mathsf{ShvCat}^{\mathbb{H}}$ are ambidextrous (i.e. $*$-pushforwards coincide with !-pushforwards), but this requires the $\mathbb{H}$-affineness theorem first.

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4. Coefficient systems for sheaves of categories

In this section we introduce one of the central notions of this paper, the notion of *coefficient system*, together with its companion notion of *lax coefficient system*.

We present a list of examples, and, in particular, we define the coefficient system $\mathbb{H}$ related to Hochschild cochains. Let us anticipate that $\mathbb{H}$ arises naturally as a lax coefficient system and some work is needed in order to prove that it is actually strict. (Here and later, the adjective 'strict' is used to emphasize that a certain coefficient system is a genuine one, not a lax one.)

4.1 Definition and examples

Consider the $(\infty, 2)$-category $\mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$, whose objects are monoidal DG categories, whose 1-morphisms are bimodule categories, and whose 2-morphisms are functors of bimodules. Recall that the $(\infty, 1)$-category underlying $\mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$ will be denoted by $\mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$.

A coefficient system is a functor

$$\mathbb{A} : \mathsf{Aff} \longrightarrow \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat}).$$

A lax coefficient system is a lax $(\infty, 2)$-functor

$$\mathbb{A} : \mathsf{Aff} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat}).$$

4.1.1 Thus, a lax coefficient system $\mathbb{A}$ consists of:

- a monoidal category $\mathbb{A}(S)$, for each affine scheme S;
- an $(\mathbb{A}(S), \mathbb{A}(T))$-bimodule $\mathbb{A}_{S \to T}$ for any map of affine schemes $S \to T$;
- an $(\mathbb{A}(S), \mathbb{A}(U))$-linear functor

  $$\eta_{S \to T \to U} : \mathbb{A}_{S \to T} \underset{\mathbb{A}(T)}{\otimes} \mathbb{A}_{T \to U} \longrightarrow \mathbb{A}_{S \to U}$$

  for any string $S \to T \to U$ of affine schemes;
- natural compatibilities for higher compositions.

Clearly, such $\mathbb{A}$ is a strict (i.e. non-lax) coefficient system if and only if all functors $\eta_{S \to T \to U}$ are equivalences.

4.1.2 One obtains variants of the above definitions by replacing the source ∞-category Aff with a subcategory $\mathsf{Aff}_{\mathrm{type}}$, where 'type' is a property of affine schemes. For instance, we will often consider $\mathsf{Aff}_{\mathrm{aft}}$ (the full subcategory of affine schemes almost of finite type) or $\mathsf{Aff}_{\mathrm{lfp}}^{<\infty}$ (affine schemes that are bounded and locally of finite presentation).

We now give a list of examples of (lax) coefficient systems, in decreasing order of simplicity.

4.1.3 *Example 1.* Any monoidal DG category $\mathcal{A}$ yields a 'constant' coefficient system $\underline{\mathcal{A}}$ whose value on $S \to T$ is $\mathcal{A}$, considered as a bimodule over itself.

4.1.4 *Example 2.* Slightly less trivial: coefficient systems induced by a functor $\mathsf{Aff} \to \mathsf{Alg}(\mathsf{DGCat})^{\mathrm{op}}$ via the functor $\iota_{\mathsf{Alg} \to \mathsf{Bimod}}$ defined in (2.1). These coefficient systems are automatically strict.

For instance, we have the coefficient system $\mathbb{Q}$ which sends

$$S \rightsquigarrow \mathrm{QCoh}(S), \qquad [S \to T] \rightsquigarrow \mathrm{QCoh}(S) \in (\mathrm{QCoh}(S), \mathrm{QCoh}(T))\text{-}\mathbf{bimod}.$$

Similarly, we have $\mathbb{D}$, obtained as above using $\mathfrak{D}$-modules rather than quasi-coherent sheaves. This coefficient system is defined only out of $\mathsf{Aff}_{\mathrm{aft}} \subset \mathsf{Aff}$.

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4.1.5 *Example 3.* Let us precompose the lax $(\infty, 2)$-functor

$$\mathsf{LOOP}_{\mathrm{Mod}} : \mathrm{Mod}(\mathsf{DGCat})^{\mathrm{op}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$$

of § 2.2.7 with the functor

$$\mathsf{Aff}_{\mathrm{aft}} \longrightarrow \mathrm{Mod}(\mathsf{DGCat})^{\mathrm{op}}, \quad S \rightsquigarrow (\mathfrak{D}(S) \circlearrowright \mathrm{IndCoh}(S))$$

that encodes the action of $\mathfrak{D}$-modules on ind-coherent sheaves. Since $\mathrm{IndCoh}(S)$ is self-dual as a $\mathfrak{D}(S)$-module (Corollary 4.2.2), we obtain a lax coefficient system

$$\mathbb{I}^{\wedge} : \mathsf{Aff}_{\mathrm{aft}} \longrightarrow \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$$

described informally by

$$\begin{aligned} &S \rightsquigarrow \mathrm{IndCoh}(S \times_{S_{\mathrm{dR}}} S), \\ &[S \to T] \rightsquigarrow \mathrm{IndCoh}(S \times_{T_{\mathrm{dR}}} T) \in (\mathrm{IndCoh}(S \times_{S_{\mathrm{dR}}} S), \mathrm{IndCoh}(T \times_{T_{\mathrm{dR}}} T))\textbf{-bimod}, \\ &[S \to T \to U] \rightsquigarrow \mathrm{IndCoh}(S \times_{T_{\mathrm{dR}}} T) \underset{\mathrm{IndCoh}(T \times_{T_{\mathrm{dR}}} T)}{\otimes} \mathrm{IndCoh}(T \times_{U_{\mathrm{dR}}} U) \longrightarrow \mathrm{IndCoh}(S \times_{U_{\mathrm{dR}}} U). \end{aligned}$$

In other words, $\mathbb{I}^{\wedge}$ is obtained by restricting the very general $\mathbb{I}^{\wedge,\mathrm{geom}}$ defined in § 3.2.3 to $\mathsf{Aff}_{\mathrm{aft}}$. We will prove that $\mathbb{I}^{\wedge}$ is strict in Proposition 4.2.5.

4.1.6 *Example 4.* As a variation on the above example, let $\mathbb{H}$ be the lax coefficient system

$$\mathbb{H} : \mathsf{Aff}_{\mathrm{lfp}}^{<\infty} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$$

defined by

$$\begin{aligned} &S \rightsquigarrow \mathbb{H}(S) := \mathrm{IndCoh}_0(S \times_{S_{\mathrm{dR}}} S), \\ &[S \to T] \rightsquigarrow \mathbb{H}_{S \to T} := \mathrm{IndCoh}_0(S \times_{T_{\mathrm{dR}}} T) \in (\mathbb{H}(S), \mathbb{H}(T))\textbf{-bimod}, \\ &[S \to T \to U] \rightsquigarrow \mathbb{H}_{S \to T} \underset{\mathbb{H}(T)}{\otimes} \mathbb{H}_{T \to U} \longrightarrow \mathbb{H}_{S \to U}. \end{aligned}$$

Similarly to $\mathbb{I}^{\wedge}$, this is the restriction of (3.9) to affine schemes. We will show that $\mathbb{H}$ is strict too.

The importance of $\mathbb{H}$ comes from the monoidal equivalence

$$\mathbb{H}(S) \simeq \mathrm{HC}(S)^{\mathrm{op}}\text{-}\mathsf{mod}.$$

To be precise, we have the following. First, the equivalence $\mathbb{H}(S) \simeq \mathrm{HC}(\mathrm{IndCoh}(S))^{\mathrm{op}}\text{-}\mathsf{mod}$ is obvious. Second, [AG15, Proposition F.1.5] provides a natural isomorphism $\mathrm{HC}(\mathrm{IndCoh}(S)) \simeq \mathrm{HC}(\mathrm{QCoh}(S)) =: \mathrm{HC}(S)$ of E_2-algebras.

4.1.7 *Example 5.* One last example arising in a geometric fashion. Let $\mathcal{Y} : \mathsf{Aff} \to \mathsf{Corr}(\mathsf{PreStk})_{\mathrm{all;all}}^{\mathrm{all}}$ be an arbitrary lax $(\infty, 2)$-functor, described informally by the assignments

$$S \rightsquigarrow \mathcal{Y}_S, \quad [S \to T] \rightsquigarrow \mathcal{Y}_S \leftarrow \mathcal{Y}_{S \to T} \to \mathcal{Y}_T.$$

The lax structure amounts to the data of maps

$$\mathcal{Y}_{S \to T} \underset{\mathcal{Y}_T}{\times} \mathcal{Y}_{T \to U} \longrightarrow \mathcal{Y}_{S \to U} \tag{4.1}$$

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over $\mathcal{Y}_S \times \mathcal{Y}_U$, for any string $S \to T \to U$. Recalling now the paradigm of § 2.2.5, we obtain a lax $(\infty, 2)$-functor

$$\mathsf{Corr}(\mathsf{PreStk})^{\mathrm{all}}_{\mathrm{all;all}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$$

defined by sending

$$\mathcal{Y}_S \rightsquigarrow \mathrm{QCoh}(\mathcal{Y}_S), \quad [\mathcal{Y}_S \leftarrow \mathcal{Y}_{S\to T} \to \mathcal{Y}_T] \rightsquigarrow \mathrm{QCoh}(\mathcal{Y}_{S\to T}).$$

The combination of this with $\mathcal{Y}$ yields a lax coefficient system, which is strict if the maps (4.1) are isomorphisms and the prestacks $\mathcal{Y}_{S\to T}$ are nice enough.[6]

4.1.8 *Sub-example: singular support.* The theory of singular support provides an important example of the above construction: the assignment

$$[S \to T] \rightsquigarrow \mathrm{Sing}(S)/\mathbb{G}_m \leftarrow S \times_T \mathrm{Sing}(T)/\mathbb{G}_m \to \mathrm{Sing}(T)/\mathbb{G}_m,$$

where $\mathrm{Sing}(U) := \mathrm{Spec}(\mathrm{Sym}_{H^0(U,\mathcal{O}_U)} H^1(U, \mathbb{T}_U))$ is equipped with the obvious weight-2 dilation action.

We obtain a coefficient system $\mathbb{S}' : \mathsf{Aff}_{\text{q-smooth}} \longrightarrow \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$ defined on quasi-smooth affine schemes. By construction, if $\mathcal{C}$ is a module category over $\mathbb{S}'(U)$, then objects of $\mathcal{C}$ are equipped with a notion of support in $\mathrm{Sing}(U)$; see [AG15] for more details.

4.2 The coefficient system $\mathbb{I}^\wedge$

Let us prove that $\mathbb{I}^\wedge$ and $\mathbb{H}$ are strict coefficient systems. We will need to use the following fact.

Lemma 4.2.1. *For any diagram $Y \to W \leftarrow Z$ in $\mathsf{Sch}_{\mathrm{aft}}$, the exterior tensor product yields the equivalence*

$$\mathrm{IndCoh}(Y) \underset{\mathfrak{D}(W)}{\otimes} \mathrm{IndCoh}(Z) \xrightarrow{\simeq} \mathrm{IndCoh}(Y \times_{W_{\mathrm{dR}}} Z). \tag{4.2}$$

Proof. Note that $Y \times_{W_{\mathrm{dR}}} Z \simeq (Y \times Z)^\wedge_{Y \times_W Z}$. Hence, by [AG18, Proposition 3.1.2], the right-hand side is equivalent to

$$\mathrm{QCoh}(Y \times_{W_{\mathrm{dR}}} Z) \underset{\mathrm{QCoh}(Y\times Z)}{\otimes} \mathrm{IndCoh}(Y \times Z),$$

while the left-hand side is obviously equivalent to

$$\big(\mathrm{QCoh}(Y) \underset{\mathfrak{D}(W)}{\otimes} \mathrm{QCoh}(Z)\big) \underset{\mathrm{QCoh}(Y\times Z)}{\otimes} \mathrm{IndCoh}(Y \times Z).$$

Now the statement reduces to the analogous statement with IndCoh replaced by QCoh, which is well known. □

Corollary 4.2.2. *For $Y \in \mathsf{Sch}_{\mathrm{aft}}$, the DG category $\mathrm{IndCoh}(Y)$ is self-dual as a $\mathfrak{D}(Y)$-module.*

Proof. One uses the equivalence of the above lemma to write the evaluation and coevaluation as standard pull-push formulas. □

Corollary 4.2.3. *For any map $Y \to Z$ in $\mathsf{Sch}_{\mathrm{aft}}$, we obtain a natural equivalence*

$$\mathrm{IndCoh}(Y \times_{Z_{\mathrm{dR}}} Z) \simeq \mathsf{Fun}_{\mathfrak{D}(Z)}(\mathrm{IndCoh}(Y), \mathrm{IndCoh}(Z)).$$

In the special case $Y = Z$, the 'composition' monoidal structure on the right-hand side corresponds to the 'convolution' monoidal structure on the left-hand side.

[6] That is, nice enough so that QCoh interchanges fibre products among these prestacks with tensor products of categories. For instance, 1-affine algebraic stacks are nice enough.

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4.2.4 The lax-coefficient system $\mathbb{I}^\wedge$ is the restriction of the lax $(\infty,2)$-functor $\mathrm{IndCoh}^{\wedge,\mathrm{geom}}$ to $\mathsf{Aff}_{\mathrm{aft}}$. Consider now the intermediate lax $(\infty,2)$-functor $\mathsf{Sch}_{\mathrm{aft}} \to \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$, also denoted by $\mathbb{I}^\wedge$ by an abuse of notation. Our present aim is to prove the following result.

Proposition 4.2.5. *The lax $(\infty,2)$-functor*

$$\mathbb{I}^\wedge : \mathsf{Sch}_{\mathrm{aft}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$$

is strict.

The proof of the above proposition will be given after some preparation.

4.2.6 For $Y \in \mathsf{Sch}_{\mathrm{aft}}$, Corollary 4.2.3 shows that $\mathrm{IndCoh}(Y)$ admits the structure of an $(\mathrm{IndCoh}(Y \times_{Y_{\mathrm{dR}}} Y), \mathfrak{D}(Y))$-bimodule, as well as the structure of a $(\mathfrak{D}(Y), \mathrm{IndCoh}(Y \times_{Y_{\mathrm{dR}}} Y))$-bimodule. Now, one verifies directly that the latter bimodule is left dual to the former, that is, there is an adjunction

$$\mathrm{IndCoh}(Y \times_{Y_{\mathrm{dR}}} Y)\text{-}\mathbf{mod} \underset{\mathrm{IndCoh}(Y) \underset{\mathfrak{D}(Y)}{\otimes} -}{\overset{\mathrm{IndCoh}(Y) \underset{\mathrm{IndCoh}(Y \times_{Y_{\mathrm{dR}}} Y)}{\otimes} -}{\rightleftarrows}} \mathfrak{D}(Y)\text{-}\mathbf{mod}. \tag{4.3}$$

Lemma 4.2.7. *These two adjoint functors form a pair of mutually inverse equivalences. In particular, we also have an adjunction in the other direction:*

$$\mathfrak{D}(Y)\text{-}\mathbf{mod} \underset{\mathrm{IndCoh}(Y) \underset{\mathrm{IndCoh}(Y \times_{Y_{\mathrm{dR}}} Y)}{\otimes} -}{\overset{\mathrm{IndCoh}(Y) \underset{\mathfrak{D}(Y)}{\otimes} -}{\rightleftarrows}} \mathrm{IndCoh}(Y \times_{Y_{\mathrm{dR}}} Y)\text{-}\mathbf{mod}. \tag{4.4}$$

Proof. The left adjoint in (4.3) is fully faithful by (4.2) and the right adjoint is colimit-preserving. By the Barr–Beck theorem, it suffices to show that the right adjoint in (4.3) is conservative, a statement which is the content of the next lemma. □

Lemma 4.2.8. *For $Y \in \mathsf{Sch}_{\mathrm{aft}}$, the functor*

$$\mathrm{IndCoh}(Y) \underset{\mathfrak{D}(Y)}{\otimes} - : \mathfrak{D}(Y)\text{-}\mathbf{mod} \longrightarrow \mathsf{DGCat}$$

is conservative.

Proof. Let $f : \mathcal{M} \to \mathcal{N}$ be a $\mathfrak{D}(Y)$-linear functor with the property that

$$\mathrm{id} \otimes f : \mathrm{IndCoh}(Y) \underset{\mathfrak{D}(Y)}{\otimes} \mathcal{M} \longrightarrow \mathrm{IndCoh}(Y) \underset{\mathfrak{D}(Y)}{\otimes} \mathcal{N}$$

is an equivalence. We need to show that f itself is an equivalence.

Denote by $\widehat{Y}_\bullet$ the Čech nerve of $q : Y \to Y_{\mathrm{dR}}$. Recall that the natural arrow

$$\mathfrak{D}(Y) := \mathrm{IndCoh}(Y_{\mathrm{dR}}) \longrightarrow \mathrm{IndCoh}(|\widehat{Y}_\bullet|) \simeq \lim_{[n] \in \mathbf{\Delta}} \mathrm{IndCoh}(\widehat{Y}_n)$$

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is an equivalence and that each of the structure functors composing the above cosimplicial category admits a left adjoint (indeed, each structure map $\widehat{Y}_m \to \widehat{Y}_n$ is a nil-isomorphism between inf-schemes). Consequently, the tautological functor

$$\mathcal{C} \longrightarrow \lim_{[n]\in\mathbf{\Delta}} \Big(\mathrm{IndCoh}(\widehat{Y}_n) \underset{\mathfrak{D}(Y)}{\otimes} \mathcal{C}\Big)$$

is an equivalence for any $\mathcal{C} \in \mathfrak{D}(S)$-**mod**. Under these identifications, our functor $f : \mathcal{M} \to \mathcal{N}$ is the limit of the equivalences

$$\mathrm{id} \otimes f : \mathrm{IndCoh}(\widehat{Y}_n) \underset{\mathfrak{D}(Y)}{\otimes} \mathcal{M} \longrightarrow \mathrm{IndCoh}(\widehat{Y}_n) \underset{\mathfrak{D}(Y)}{\otimes} \mathcal{N},$$

whence it is itself an equivalence. □

4.2.9 We are now ready for the proof of the proposition left open above.

Proof of Proposition 4.2.5. Thanks to (4.2), it suffices to prove that, for any $Y \in \mathsf{Sch}_{\mathrm{aft}}$, the obvious functor $q_*^{\mathrm{IndCoh}} \circ \Delta^! : \mathrm{IndCoh}(Y) \otimes \mathrm{IndCoh}(Y) \to \mathfrak{D}(Y)$ induces an equivalence

$$\mathrm{IndCoh}(Y) \underset{\mathrm{IndCoh}(Y \times_{Y_{\mathrm{dR}}} Y)}{\otimes} \mathrm{IndCoh}(Y) \xrightarrow{\simeq} \mathfrak{D}(Y). \tag{4.5}$$

The latter is precisely the counit of the adjunction (4.3), which we have shown to be an equivalence. □

4.3 The coefficient system $\mathbb{H}$

Our present aim is to prove Theorem 4.3.4, which states that the lax coefficient system

$$\mathbb{H} : \mathsf{Aff}_{\mathrm{lfp}}^{<\infty} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$$

is *strict*. Actually, the theorem proves something slightly stronger, namely, the parallel statement for schemes that are not necessarily affine.

4.3.1 We need a preliminary result, which is of interest in its own right.

PROPOSITION 4.3.2. *Let $f : X \to Y$ be a map in $\mathsf{Sch}_{\mathrm{lfp}}^{<\infty}$. Then the $(\mathfrak{D}(X), \mathbb{H}(Y))$-linear functor*

$$\mathrm{IndCoh}(X) \underset{\mathbb{H}(X)}{\otimes} \mathbb{H}_{X\to Y} \longrightarrow \mathrm{IndCoh}(Y_X^\wedge), \tag{4.6}$$

obtained as the composition

$$\mathrm{IndCoh}(X) \underset{\mathbb{H}(X)}{\otimes} \mathbb{H}_{X\to Y} \longrightarrow \mathrm{IndCoh}(X) \underset{\mathbb{I}^\wedge(X)}{\otimes} \mathbb{I}^\wedge_{X\to Y} \xrightarrow{\simeq} \mathrm{IndCoh}(Y_X^\wedge),$$

is an equivalence of categories.

Proof. The source category is compactly generated by objects of the form $[C_X, ('f)_*^{\mathrm{IndCoh}}(\omega_X)]$ for $C_X \in \mathrm{Coh}(X)$. Hence, it is clear that the functor in question (let us denote it by ϕ) admits a continuous and conservative right adjoint: indeed, ϕ sends

$$[C_X, ('f)_*^{\mathrm{IndCoh}}(\omega_X)] \rightsquigarrow ('f)_*^{\mathrm{IndCoh}}(C_X),$$

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whence it preserves compactness and generates the target under colimits. It remains to show that ϕ is fully faithful on objects of the form $[C_X, ('f)_*^{\mathrm{IndCoh}}(\omega_X)]$. The nil-isomorphism $\beta : (X \times X)^\wedge_X \to (X \times Y)^\wedge_X$ induces the adjunction

$$\beta_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}((X \times X)^\wedge_X) \rightleftarrows \mathrm{IndCoh}((X \times Y)^\wedge_X) : \beta^!.$$

Observe that both functors are $\mathrm{IndCoh}((X \times X)^\wedge_X)$-linear and preserve the IndCoh_0-subcategories. To conclude the proof, just note that $('f)_*^{\mathrm{IndCoh}}(\omega_X)$ is the image of the unit of $\mathbb{H}(X)$ under $\beta_*^{\mathrm{IndCoh}}$, and use the above adjunction. $\square$

Corollary 4.3.3. *For $f : X \to Y$ as above and $\mathcal{C}$ a right $\mathbb{I}^\wedge(X)$-module, the natural functor*

$$\mathcal{C} \underset{\mathbb{H}(X)}{\otimes} \mathbb{H}_{X \to Y} \longrightarrow \mathcal{C} \underset{\mathbb{I}^\wedge(X)}{\otimes} \mathbb{I}^\wedge_{X \to Y}$$

is an equivalence.

Proof. It suffices to prove the assertion for $\mathcal{C} = \mathbb{I}^\wedge(X)$, viewed as a right module over itself. Thanks to the right $\mathbb{I}^\wedge(X)$-linear equivalence

$$\mathbb{I}^\wedge(X) \simeq \mathrm{IndCoh}(X) \underset{\mathfrak{D}(X)}{\otimes} \mathrm{IndCoh}(X),$$

the assertion reduces to the proposition above. $\square$

Theorem 4.3.4. *The lax $(\infty, 2)$-functor*

$$\mathbb{H} : \mathsf{Sch}_{\mathrm{lfp}}^{<\infty} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat}),$$

obtained by restricting $\mathbb{H}^{\mathrm{geom}}$ to schemes, is strict.

Proof. Let $U \to X \to Y$ be a string in $\mathsf{Sch}_{\mathrm{lfp}}^{<\infty}$. We need to prove that the convolution functor

$$\mathbb{H}_{U \to X} \underset{\mathbb{H}(X)}{\otimes} \mathbb{H}_{X \to Y} \longrightarrow \mathbb{I}^\wedge_{U \to Y} \tag{4.7}$$

is an equivalence onto the subcategory $\mathbb{H}_{U \to Y} \subseteq \mathbb{I}^\wedge_{U \to Y}$. One easily checks that the essential image of the functor is indeed $\mathbb{H}_{U \to Y}$, whence it remains to prove fully faithfulness. By construction, (4.7) factors as the composition

$$\mathbb{H}_{U \to X} \underset{\mathbb{H}(X)}{\otimes} \mathbb{H}_{X \to Y} \longrightarrow \mathbb{I}^\wedge_{U \to X} \underset{\mathbb{H}(X)}{\otimes} \mathbb{H}_{X \to Y} \longrightarrow \mathbb{I}^\wedge_{U \to Y}.$$

Now the first arrow is obviously fully faithful, while the second one is an equivalence by the above corollary. $\square$

4.4 Morphisms between coefficient systems

Coefficient systems assemble into an ∞-category:

$$\mathsf{CoeffSys} := \mathsf{Fun}(\mathsf{Aff}, \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})).$$

Hence, it makes sense to consider *morphisms* of coefficient systems. This notion has already been discussed in § 1.8, where some examples have been given. Here we just recall the only morphism of interest in this paper, the arrow $\mathbb{Q} \to \mathbb{H}$.

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4.4.1 Let $\mathbb{A}$ and $\mathbb{B}$ be two coefficient systems. Consider the following pieces of data:

- for each $S \in \mathsf{Aff}$, a monoidal functor $\mathbb{A}(S) \to \mathbb{B}(S)$;
- for each $S \to T$, an $(\mathbb{A}(S), \mathbb{A}(T))$-linear functor

$$\eta_{S\to T} : \mathbb{A}_{S\to T} \longrightarrow \mathbb{B}_{S\to T} \tag{4.8}$$

  that induces an $(\mathbb{A}(S), \mathbb{B}(T))$-equivalence $\mathbb{A}_{S\to T} \otimes_{\mathbb{A}(T)} \mathbb{B}(T) \to \mathbb{B}_{S\to T}$;
- natural higher compatibilities with respect to strings of affine schemes.

These data give rise to a morphism $\mathbb{A} \to \mathbb{B}$.

4.4.2 It is easy to see that the morphism $\mathbb{Q} \to \mathbb{H}$ (defined on $\mathsf{Aff}^{<\infty}_{\mathrm{lfp}}$) falls under this rubric. Indeed, we just need to verify that the tautological $(\mathrm{QCoh}(S), \mathbb{H}(T))$-linear functor

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(T)}{\otimes} \mathbb{H}(T) \longrightarrow \mathbb{H}_{S\to T} \tag{4.9}$$

is an equivalence, for any $S \to T$ in $\mathsf{Aff}^{<\infty}_{\mathrm{lfp}}$. This has been proven in [Ber17b] in greater generality.

5. Coefficient systems: dualizability and base-change

As mentioned in the introduction, a coefficient system $\mathbb{A} : \mathsf{Aff}_{\mathrm{type}} \longrightarrow \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$ yields a functor

$$\mathsf{ShvCat}^{\mathbb{A}} := \textbf{-mod} \circ \mathbb{A}^{\mathrm{op}} : (\mathsf{Aff}_{\mathrm{type}})^{\mathrm{op}} \longrightarrow \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})^{\mathrm{op}} \xrightarrow{\textbf{-mod}} \mathsf{Cat}_\infty$$

and then, by right Kan extension, a functor

$$\mathsf{ShvCat}^{\mathbb{A}} : (\mathsf{Stk}_{\mathrm{type}})^{\mathrm{op}} \longrightarrow \mathsf{Cat}_\infty,$$

where $\mathsf{Stk}_{\mathrm{type}}$ denotes the ∞-category of algebraic stacks with affine diagonal and with an atlas in $\mathsf{Aff}_{\mathrm{type}}$.

This is only half of what we need to accomplish though: it is not enough to just have pullbacks functors in $\mathsf{ShvCat}^{\mathbb{A}}$, we want pushforwards too. To put it more formally, we wish to extend $\mathsf{ShvCat}^{\mathbb{A}}$ to a functor out of

$$\mathsf{Corr}(\mathsf{Stk}_{\mathrm{type}})_{\mathrm{vert;horiz}},$$

for an appropriate choice of vertical and horizontal arrows. In this section we examine this possibility for affine schemes. Actually, we will look for something stronger: we check under what conditions the coefficient system $\mathbb{A}$ itself admits an extension to a functor

$$\mathsf{Corr}(\mathsf{Aff}_{\mathrm{type}})_{\mathrm{vert;horiz}} \longrightarrow \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat}), \tag{5.1}$$

or even better to an $(\infty, 2)$-functor

$$\mathsf{Corr}(\mathsf{Aff}_{\mathrm{type}})^{\mathrm{adm}}_{\mathrm{vert;horiz}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat}). \tag{5.2}$$

5.1 The Beck–Chevalley conditions

As we now explain, the *(left or right) Beck–Chevalley conditions* are conditions on a coefficient system $\mathbb{A}$ that automatically guarantee the existence of an $(\infty, 2)$-functor $\mathbb{A}^{\mathsf{Corr}}$ extending $\mathbb{A}$.

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5.1.1 We say that $\mathbb{A}$ satisfies the *right Beck–Chevalley condition* if the two requirements of §§ 5.1.2 and 5.1.5 are met.

5.1.2 *The first requirement.* We ask that, for *any* arrow $S \to T$ in $\mathsf{Aff}_{\text{type}}$, the $(\mathbb{A}(S), \mathbb{A}(T))$-bimodule $\mathbb{A}_{S\to T}$ be *right dualizable*; see § 2.1.2 for our conventions. Let us denote by $\mathbb{A}_{T\leftarrow S}$ such right dual.

5.1.3 Assume now that $\mathbb{A}$ satisfies the above requirement, so that the bimodules $\mathbb{A}_{?\leftarrow ?}$ are defined. Before formulating the second requirement, we need to fix some notation. For a commutative (but not necessarily cartesian) diagram

$$\begin{array}{ccc} U & \xrightarrow{F} & T \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle g} \\ S & \xrightarrow{f} & V \end{array} \tag{5.3}$$

in $\mathsf{Aff}_{\text{type}}$, define

$$\mathbb{A}_{S\leftarrow U\to T} := \mathbb{A}_{S\leftarrow U} \underset{\mathbb{A}(U)}{\otimes} \mathbb{A}_{U\to T}; \quad \mathbb{A}_{S\to V\leftarrow T} := \mathbb{A}_{S\to V} \underset{\mathbb{A}(V)}{\otimes} \mathbb{A}_{V\leftarrow T}.$$

Denote by u-type the largest class of arrows in $\mathsf{Aff}_{\text{type}}$ that makes $\mathsf{Corr}(\mathsf{Aff}_{\text{type}})_{\text{all};u\text{-type}}$ well defined.[7] Namely, an arrow $S \to T$ in $\mathsf{Aff}_{\text{type}}$ belongs to u-type if, for any $T' \to T$ in $\mathsf{Aff}_{\text{type}}$, the scheme $S \times_T T'$ belongs to $\mathsf{Aff}_{\text{type}}$.

5.1.4 Consider a commutative diagram like (5.3). The resulting commutative diagram

$$\begin{array}{ccc} \mathbb{A}(U) & \xleftarrow{\mathbb{A}_{U\to T}} & \mathbb{A}(T) \\ {\scriptstyle \mathbb{A}_{U\to S}}\uparrow & & \uparrow{\scriptstyle \mathbb{A}_{T\to V}} \\ \mathbb{A}(S) & \xleftarrow{\mathbb{A}_{S\to V}} & \mathbb{A}(V) \end{array}$$

in $\mathsf{Alg}^{\text{bimod}}(\mathsf{DGCat})$ gives rise, by changing the vertical arrows with their right duals, to a *lax* commutative diagram

$$\begin{array}{ccc} \mathbb{A}(U) & \xleftarrow{\mathbb{A}_{U\to T}} & \mathbb{A}(T) \\ {\scriptstyle \mathbb{A}_{S\leftarrow U}}\downarrow & \nwarrow & \downarrow{\scriptstyle \mathbb{A}_{V\leftarrow T}} \\ \mathbb{A}(S) & \xleftarrow{\mathbb{A}_{S\to V}} & \mathbb{A}(V). \end{array}$$

In other words, any commutative diagram (5.3) yields a canonical $(\mathbb{A}(S), \mathbb{A}(T))$-linear functor

$$\mathbb{A}_{S\to V\leftarrow T} \longrightarrow \mathbb{A}_{S\leftarrow U\to T}. \tag{5.4}$$

5.1.5 *The second requirement.* In particular, for $S \to V \in u$-type and $T \to V$ arbitrary, we have

$$\mathbb{A}_{S\to V\leftarrow T} \longrightarrow \mathbb{A}_{S\leftarrow S\times_V T\to T} \tag{5.5}$$

and we require that such functor be an equivalence.

[7] The letter 'u' in the notation u-type stands for the word 'universal'.

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5.1.6 Let us now explain what the right Beck–Chevalley condition is good for. Tautologically, if $\mathbb{A}$ satisfies the right Beck–Chevalley condition, the assignment

$$S \rightsquigarrow \mathbb{A}(S), \quad [S \leftarrow U \to T] \rightsquigarrow \mathbb{A}_{S\leftarrow U\to T} \tag{5.6}$$

extends to a functor

$$\mathsf{Corr}(\mathsf{Aff}_{\text{type}})_{\text{all};u\text{-type}} \longrightarrow \mathsf{Alg}^{\text{bimod}}(\mathsf{DGCat}).$$

Further, thanks to [GR17, ch. V.1, Theorem 3.2.2], the latter automatically extends further to an $(\infty, 2)$-functor

$$\mathbb{A}^{\text{R-BC}} : \mathsf{Corr}(\mathsf{Aff}_{\text{type}})^{u\text{-type},2-\text{op}}_{\text{all};u\text{-type}} \longrightarrow \mathsf{ALG}^{\text{bimod}}(\mathsf{DGCat}).$$

Thus, for $\mathbb{A}$ satisfying the right Beck–Chevalley condition, the corresponding sheaf theory $\mathsf{ShvCat}^{\mathbb{A}}|_{\mathsf{Aff}^{\text{op}}_{\text{type}}}$ admits $*$-pushforwards (defined to be right adjoint to pullbacks). Moreover, these pushforwards satisfy base-change against pullbacks along the appropriate fibre squares.

5.1.7 The definition of *left Beck–Chevalley condition* for $\mathbb{A}$ is totally symmetric: each $\mathbb{A}_{S\to T}$ must admit a left dual $\mathbb{A}^L_{T\leftarrow S}$ and, for any cartesian diagram (5.3) with $T \to V$ in u-type, the structure functor

$$\mathbb{A}^L_{S\leftarrow U} \underset{\mathbb{A}(U)}{\otimes} \mathbb{A}_{U\to T} \longrightarrow \mathbb{A}_{S\to V} \underset{\mathbb{A}(V)}{\otimes} \mathbb{A}^L_{V\leftarrow T}$$

must be an equivalence. Thus, if $\mathbb{A}$ satisfies the left Beck–Chevalley condition, the sheaf theory $\mathsf{ShvCat}^{\mathbb{A}}|_{\mathsf{Aff}^{\text{op}}_{\text{type}}}$ admits !-pushforwards (defined to be left adjoint to pullbacks), again, satisfying base-change against pullbacks along the appropriate fibre squares.

5.1.8 A coefficient system $\mathbb{A}$ is said to be *ambidextrous* if it satisfies the right Beck–Chevalley condition and, for any $S \to T \in \mathsf{Aff}_{\text{type}}$, the $(\mathbb{A}(T), \mathbb{A}(S))$-bimodule $\mathbb{A}_{S\to T}$ is ambidextrous (see § 2.1.2 for the definition). Any ambidextrous $\mathbb{A}$ automatically satisfies the left Beck–Chevalley condition as well. Thus, for $\mathbb{A}$ ambidextrous, we obtain two extensions of $\mathbb{A}$,

$$\mathbb{A}^{\text{R-BC}} : \mathsf{Corr}(\mathsf{Aff}_{\text{type}})^{u\text{-type},2-\text{op}}_{u\text{-type;all}} \longrightarrow \mathsf{ALG}^{\text{bimod}}(\mathsf{DGCat}),$$
$$\mathbb{A}^{\text{L-BC}} : \mathsf{Corr}(\mathsf{Aff}_{\text{type}})^{u\text{-type}}_{\text{all};u\text{-type}} \longrightarrow \mathsf{ALG}^{\text{bimod}}(\mathsf{DGCat}),$$

that are exchanged by duality.

5.1.9 Let us spell out these pieces of structure in more detail. First, up to switching vertical and horizontal arrows in $\mathbb{A}^{\text{R-BC}}$ (see Remark 3.3.3), the two $(\infty, 2)$-functors $\mathbb{A}^{\text{R-BC}}, \mathbb{A}^{\text{L-BC}}$ have a common underlying $(\infty, 1)$-functor

$$\mathbb{A}^{\mathsf{Corr}} : \mathsf{Corr}(\mathsf{Aff}_{\text{type}})_{\text{all};u\text{-type}} \longrightarrow \mathsf{Alg}^{\text{bimod}}(\mathsf{DGCat})$$
$$[S \leftarrow U \to T] \rightsquigarrow \mathbb{A}_{S\leftarrow U\to T} := \mathbb{A}_{S\leftarrow U} \underset{\mathbb{A}(U)}{\otimes} \mathbb{A}_{U\to T}.$$

Secondly, the two enhancements of $\mathbb{A}^{\mathsf{Corr}}$ to $\mathbb{A}^{\text{L-BC}}$ and $\mathbb{A}^{\text{R-BC}}$ amount to the following data: for $U' \to U$ of u-type over $S \times T$, there are two mutually dual structure functors $\mathbb{A}_{S\leftarrow U'\to T} \rightleftarrows \mathbb{A}_{S\leftarrow U\to T}$, compatible in U in the natural way. Such enhancements will be used in §§ 6.3.1 and 6.3.2 to construct the two kinds of pushforwards in the setting of $\mathsf{ShvCat}^{\mathbb{H}}$ on stacks.

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5.1.10 *Easy examples.* It is obvious that $\mathbb{Q}$ and $\mathbb{D}$ are ambidextrous. For instance, for the former,

$$\mathbb{Q}^{\mathsf{Corr}} : \mathsf{Corr}(\mathsf{Aff})_{\mathrm{all;all}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$$

is defined on 1-arrows by $\mathbb{Q}_{S \leftarrow U \to T} \simeq \mathrm{QCoh}(U)$, the latter equipped with its obvious $(\mathrm{QCoh}(S), \mathrm{QCoh}(T))$-bimodule structure. The two mutually dual structure functors $\mathbb{Q}_{S \leftarrow U' \to T} \rightleftarrows \mathbb{Q}_{S \leftarrow U \to T}$ are simply the pullback and pushforward functors along $U' \to U$.

We leave it as an exercise to show that the coefficient system $\mathbb{S}'$ responsible for singular support is ambidextrous: it extends to a functor out of $\mathsf{Corr}(\mathsf{Aff}_{\text{q-smooth}})^{\mathrm{smooth}}_{\mathrm{all;smooth}}$.

5.1.11 Let us now turn to $\mathbb{I}^{\wedge}$. We have the following result, which will later help us understand base-change for $\mathbb{H}$.

Proposition 5.1.12. *The functor $\mathbb{I}^{\wedge} : \mathsf{Sch}_{\mathrm{aft}} \to \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$ satisfies the right Beck–Chevalley condition, so that it extends to an $(\infty, 2)$-functor*

$$(\mathbb{I}^{\wedge})^{\text{R-BC}} : \mathsf{Corr}(\mathsf{Sch}_{\mathrm{aft}})^{\mathrm{all},2-\mathrm{op}}_{\mathrm{all;all}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat}). \tag{5.7}$$

Proof. We start by setting up some notation. For $X \to Y$ in $\mathsf{Sch}_{\mathrm{aft}}$, consider the maps

$$\zeta : (X \times X)^{\wedge}_{X} \simeq X \times_{X_{\mathrm{dR}}} X \longrightarrow (X \times X)^{\wedge}_{X \times_Y X} \simeq X \times_{Y_{\mathrm{dR}}} X,$$

$$\eta : (Y \times Y)^{\wedge}_{X} \simeq Y \times_{Y_{\mathrm{dR}}} X_{\mathrm{dR}} \times_{Y_{\mathrm{dR}}} Y \longrightarrow (Y \times Y)^{\wedge}_{Y} \simeq Y \times_{Y_{\mathrm{dR}}} Y, \tag{5.8}$$

where ζ is induced by $\Delta_{X/Y} : X \to X \times_Y X$. With the help of Lemma 4.2.1, one can easily check that the functors

$$\zeta^! : \mathbb{I}^{\wedge}_{X \to Y} \underset{\mathbb{I}^{\wedge}(Y)}{\otimes} \mathbb{I}^{\wedge}_{Y \leftarrow X} \longrightarrow \mathbb{I}^{\wedge}(X), \quad \eta^! : \mathbb{I}^{\wedge}(Y) \longrightarrow \mathbb{I}^{\wedge}_{Y \leftarrow X} \underset{\mathbb{I}^{\wedge}(X)}{\otimes} \mathbb{I}^{\wedge}_{X \to Y}$$

exhibit $\mathbb{I}^{\wedge}_{Y \leftarrow X} := \mathrm{IndCoh}(Y \times_{Y_{\mathrm{dR}}} X)$ as the right dual of the $(\mathbb{I}^{\wedge}(X), \mathbb{I}^{\wedge}(Y))$-bimodule $\mathbb{I}^{\wedge}_{X \to Y}$.

Now let

$$\begin{array}{ccc} U & \xrightarrow{F} & S \\ \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle g} \\ T & \xrightarrow{f} & V \end{array} \tag{5.9}$$

be a commutative square in $\mathsf{Sch}_{\mathrm{aft}}$. By Lemma 4.2.1, one easily gets equivalences

$$\mathbb{I}^{\wedge}_{S \leftarrow U \to T} \simeq \mathrm{IndCoh}(S \times_{S_{\mathrm{dR}}} U_{\mathrm{dR}} \times_{T_{\mathrm{dR}}} T), \quad \mathbb{I}^{\wedge}_{S \to V \leftarrow T} \simeq \mathrm{IndCoh}(S \times_{V_{\mathrm{dR}}} T),$$

compatible with the natural $(\mathbb{I}^{\wedge}(S), \mathbb{I}^{\wedge}(T))$-bimodule structures on both sides. Further, the structure arrow induced by the right Beck–Chevalley condition

$$\mathbb{I}^{\wedge}_{S \to V \leftarrow T} \longrightarrow \mathbb{I}^{\wedge}_{S \leftarrow U \to T}$$

is the !-pullback functor along the natural map $U_{\mathrm{dR}} \to (S \times_V T)_{\mathrm{dR}}$, whence it is an equivalence whenever the square is nil-cartesian (i.e. cartesian at the level of reduced schemes). □

Remark 5.1.13. The same argument with the functors $\zeta_*^{\mathrm{IndCoh}}$ and η_*^{IndCoh} shows that $\mathbb{I}^{\wedge}$ satisfies the left Beck–Chevalley condition, too. It follows that $\mathbb{I}^{\wedge}$ is ambidextrous.

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5.2 Base-change for $\mathbb{H}$

The aim of this section is to show that $\mathbb{H}$ is ambidextrous (Theorem 5.2.10). After this is proven, we will summarize the important consequences of this result.

5.2.1 Observe that, for any $S \in \mathsf{Aff}_{\mathrm{lfp}}^{<\infty}$, the monoidal category $\mathbb{H}(S)$ is rigid and compactly generated. Recall now the definition of $1^{\mathrm{fake}}_{\mathbb{H}(S)} \in \mathbb{H}(S)^*$ and the notion of very rigid monoidal category; see § 2.1.5.

Proposition 5.2.2. *For any $S \in \mathsf{Aff}_{\mathrm{lfp}}^{<\infty}$, the monoidal DG category $\mathbb{H}(S)$ is very rigid.*

Proof. It suffices to show that $1^{\mathrm{fake}}_{\mathbb{H}(S)} \in \mathbb{H}(S)^*$ admits a lift through the forgetful functor

$$\mathsf{Fun}_{\mathbb{H}(S)\otimes\mathbb{H}(S)^{\mathrm{rev}}}\big(\mathbb{H}(S), \mathbb{H}(S)^*\big) \longrightarrow \mathbb{H}(S)^*.$$

Recall from [Ber17b] that the functor

$$\mathfrak{D}(S) \xrightarrow{\mathsf{oblv}_L} \mathrm{QCoh}(S) \xrightarrow{\Upsilon_S} \mathrm{IndCoh}(S) \xrightarrow{'\Delta_*^{\mathrm{IndCoh}}} \mathbb{H}(S)$$

factors as the composition

$$\mathfrak{D}(S) \longrightarrow \mathsf{Fun}_{\mathbb{H}(S)\otimes\mathbb{H}(S)^{\mathrm{rev}}}\big(\mathbb{H}(S), \mathbb{H}(S)\big) \longrightarrow \mathbb{H}(S),$$

where the DG category in the middle is by definition the Drinfeld centre of $\mathbb{H}(S)$. A variation of the argument there shows that

$$\mathfrak{D}(S) \xrightarrow{\mathsf{oblv}_L} \mathrm{QCoh}(S) \xrightarrow{\Xi_S} \mathrm{IndCoh}(S) \xrightarrow{'\Delta_*^{\mathrm{IndCoh}}} \mathbb{H}(S)^*$$

factors as the composition

$$\mathfrak{D}(S) \longrightarrow \mathsf{Fun}_{\mathbb{H}(S)\otimes\mathbb{H}(S)^{\mathrm{rev}}}\big(\mathbb{H}(S), \mathbb{H}(S)^*\big) \longrightarrow \mathbb{H}(S)^*.$$

Finally, one computes $1^{\mathrm{fake}}_{\mathbb{H}(S)} \in \mathbb{H}(S)^*$ explicitly: it is readily checked that

$$1^{\mathrm{fake}}_{\mathbb{H}(S)} \simeq \Delta_*^{\mathrm{IndCoh}}(\Xi_S(\mathcal{O}_S)),$$

a fact that concludes the proof. □

5.2.3 Coupling this with Corollary 2.1.7, we obtain that each bimodule $\mathbb{H}_{S\to T}$ is ambidextrous: moreover, its left and right duals are canonically identified with $(\mathbb{H}_{S\to T})^*$.

Let us now determine the right dual to $\mathbb{H}_{S\to T}$ explicitly. By the above, we already know what the DG category underlying $\mathbb{H}_{T\leftarrow S} := (\mathbb{H}_{S\to T})^R$ must be: it is the dual of the DG category $\mathrm{IndCoh}_0((S\times T)^\wedge_S)$. The latter is self-dual as a plain DG category, so we are just searching for the correct $(\mathbb{H}(T), \mathbb{H}(S))$-bimodule structure on $\mathrm{IndCoh}_0((S\times T)^\wedge_S)$.

We claim that $\mathbb{H}_{T\leftarrow S}$ is equivalent to $\mathrm{IndCoh}_0((T\times S)^\wedge_S)$, equipped with the obvious $(\mathbb{H}(T), \mathbb{H}(S))$-bimodule structure. We will establish this fact directly, by constructing the evaluation and coevaluation that make $\mathrm{IndCoh}_0((T\times S)^\wedge_S)$ right dual to $\mathbb{H}_{S\to T}$.

Lemma 5.2.4. *For $S\to T$ a map in $\mathsf{Aff}_{\mathrm{lfp}}^{<\infty}$, the natural functor*

$$\mathbb{H}_{S\to T} \underset{\mathbb{H}(T)}{\otimes} \mathrm{IndCoh}_0((T\times S)^\wedge_S) \longrightarrow \mathbb{I}^\wedge_{S\to T} \underset{\mathbb{I}^\wedge(T)}{\otimes} \mathbb{I}^\wedge_{T\leftarrow S} \simeq \mathrm{IndCoh}(S\times_{T_{\mathrm{dR}}} S) \xrightarrow{\zeta^!} \mathbb{I}^\wedge(S)$$

lands into the full subcategory $\mathbb{H}(S) \subseteq \mathbb{I}^\wedge(S)$.

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Proof. We will use the commutative diagram

$$\begin{array}{ccccc} (S \times_{S\times T} S)^{\wedge}_{S} & \xrightarrow{\pi} & S & \xrightarrow{'\Delta} & (S \times S)^{\wedge}_{S} \\ \downarrow & & \downarrow{\scriptstyle \widetilde{\Delta}_{S/T}} & \swarrow{\scriptstyle \zeta} & \\ S \times_T S & \xrightarrow{\xi} & (S \times S)^{\wedge}_{S\times_T S} & & \end{array}$$

with cartesian square. The DG category

$$\mathbb{H}_{S\to T} \underset{\mathbb{H}(T)}{\otimes} \mathrm{IndCoh}_0((T \times S)^{\wedge}_{S})$$

is generated by a single canonical compact object, which is sent by our functor to $\zeta^! \circ \xi_*^{\mathrm{IndCoh}}(\omega_{S\times_T S}) \in \mathbb{I}^{\wedge}(S)$. Hence, it suffices to show that the object

$$(\widetilde{\Delta}_{S/T})^! \circ \xi_*^{\mathrm{IndCoh}}(\omega_{S\times_T S}) \simeq \pi_*^{\mathrm{IndCoh}} \circ \pi^!(\omega_S)$$

belongs to the image of $\Upsilon_S : \mathrm{QCoh}(S) \hookrightarrow \mathrm{IndCoh}(S)$. This is clear: $\pi_*^{\mathrm{IndCoh}}\pi^!$ is equivalent as a functor to the universal envelope of the Lie algebroid $\mathbb{T}_{S/S\times T} \to \mathbb{T}_S$, and by assumption $\mathbb{T}_{S/S\times T}$ belongs to $\Upsilon_S(\mathsf{Perf}(S))$. We conclude as in [AG18, Proposition 3.2.3]. □

5.2.5 Hence, we have constructed an $(\mathbb{H}(S), \mathbb{H}(S))$-linear functor

$$\mathbb{H}_{S\to T} \underset{\mathbb{H}(T)}{\otimes} \mathrm{IndCoh}_0((T \times S)^{\wedge}_{S}) \longrightarrow \mathbb{H}(S), \tag{5.10}$$

which will be our evaluation. To construct the coevaluation, we need another lemma.

LEMMA 5.2.6. *For a diagram $S \leftarrow U \to T$ in $\mathsf{Aff}^{<\infty}_{\mathrm{lfp}}$, the functor*

$$\mathrm{IndCoh}_0((S \times U)^{\wedge}_{U}) \underset{\mathbb{H}(U)}{\otimes} \mathbb{H}_{U\to T} \to \mathbb{I}^{\wedge}_{S\leftarrow U} \underset{\mathbb{I}^{\wedge}(U)}{\otimes} \mathbb{I}^{\wedge}_{U\to T}$$
$$\xrightarrow{\simeq} \mathrm{IndCoh}(S \times_{S_{\mathrm{dR}}} U_{\mathrm{dR}} \times_{T_{\mathrm{dR}}} T) \simeq \mathrm{IndCoh}((S \times T)^{\wedge}_{U})$$

is an equivalence onto the subcategory $\mathrm{IndCoh}_0((S \times T)^{\wedge}_{U}) \subseteq \mathrm{IndCoh}((S \times T)^{\wedge}_{U})$.

Proof. Denote by $\phi : U \to S \times T$ and by $'\phi : U \to (S \times T)^{\wedge}_{U}$ the obvious maps. The source DG category is compactly generated by a single canonical object. Base-change along the pullback square

$$\begin{array}{ccc} U & \longrightarrow & S \times_{S_{\mathrm{dR}}} U \times_{T_{\mathrm{dR}}} T \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\ U \times U & \longrightarrow & S \times_{U_{\mathrm{dR}}} U \times U \times_{T_{\mathrm{dR}}} T \end{array}$$

shows that such object is sent to $'\phi_*^{\mathrm{IndCoh}}(\omega_U) \in \mathrm{IndCoh}((S\times T)^{\wedge}_{U})$, which is a compact generator of $\mathrm{IndCoh}_0((S \times T)^{\wedge}_{U})$. All that remains is to show that the functor

$$\mathrm{IndCoh}_0((S \times U)^{\wedge}_{U}) \underset{\mathbb{H}(U)}{\otimes} \mathbb{H}_{U\to T} \longrightarrow \mathbb{I}^{\wedge}_{S\leftarrow U} \underset{\mathbb{I}^{\wedge}(U)}{\otimes} \mathbb{I}^{\wedge}_{U\to T}$$

is fully faithful. This is evident: the functor in question arises as the composition

$$\mathrm{IndCoh}_0((S \times U)^{\wedge}_{U}) \underset{\mathbb{H}(U)}{\otimes} \mathbb{H}_{U\to T} \hookrightarrow \mathbb{I}^{\wedge}_{S\leftarrow U} \underset{\mathbb{H}(U)}{\otimes} \mathbb{H}_{U\to T} \xrightarrow{\simeq} \mathbb{I}^{\wedge}_{S\leftarrow U} \underset{\mathbb{I}^{\wedge}(U)}{\otimes} \mathbb{I}^{\wedge}_{U\to T},$$

where the second arrow is an equivalence thanks to Corollary 4.3.3. □

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5.2.7 We now use $\eta^! : \mathrm{IndCoh}((T\times T)^\wedge_T) \longrightarrow \mathrm{IndCoh}((T\times T)^\wedge_S)$ as in (5.8), together with the equivalence

$$\theta : \mathrm{IndCoh}_0((T\times S)^\wedge_S) \underset{\mathbb{H}(S)}{\otimes} \mathbb{H}_{S\to T} \to \mathrm{IndCoh}_0((T\times T)^\wedge_S)$$

of the above lemma, to construct the functor

$$\mathbb{H}(T) \xrightarrow{\eta^!} \mathrm{IndCoh}_0((T\times T)^\wedge_S) \xrightarrow{\theta^{-1}} \mathrm{IndCoh}_0((T\times S)^\wedge_S) \underset{\mathbb{H}(S)}{\otimes} \mathbb{H}_{S\to T}. \tag{5.11}$$

As the next proposition shows, this is the coevaluation we were looking for.

Proposition 5.2.8. *Let $f : S \to T$ be a map in $\mathsf{Aff}^{<\infty}_{\mathrm{lfp}}$. Then the functors (5.10) and (5.11) exhibit $\mathrm{IndCoh}_0((T\times S)^\wedge_S)$, with its natural $(\mathbb{H}(T),\mathbb{H}(S))$-bimodule structure, as the right dual of $\mathbb{H}_{S\to T}$.*

Proof. This follows formally from the analogous statement for $\mathbb{I}^\wedge_{S\to T}$. □

5.2.9 Henceforth, we will freely use the $(\mathbb{H}(T),\mathbb{H}(S))$-linear equivalence $\mathbb{H}_{T\leftarrow S} \simeq \mathrm{IndCoh}_0((T\times S)^\wedge_S)$. We are finally ready to settle the ambidexterity of the coefficient system $\mathbb{H}$.

Theorem 5.2.10. *The coefficient system $\mathbb{H} : \mathsf{Aff}^{<\infty}_{\mathrm{lfp}} \longrightarrow \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat})$ is ambidextrous.*

Half of the proof of this theorem has been done in Lemma 5.2.6. All that remains is to add the following statement.

Lemma 5.2.11. *Let $S \to V \leftarrow T$ be a diagram in $\mathsf{Aff}^{<\infty}_{\mathrm{lfp}}$, with either $S\to V$ or $T \to V$ bounded.*[8] *Then the functor*

$$\mathbb{H}_{S\to V} \underset{\mathbb{H}(V)}{\otimes} \mathbb{H}_{V\leftarrow T} \longrightarrow \mathbb{I}^\wedge_{S\to V} \underset{\mathbb{I}^\wedge(V)}{\otimes} \mathbb{I}^\wedge_{V\leftarrow T} \xrightarrow{\simeq} \mathrm{IndCoh}(S\times_{V_{\mathrm{dR}}} T)$$

is an equivalence onto the subcategory

$$\mathrm{IndCoh}_0((S\times T)^\wedge_{S\times_V T}) \subseteq \mathrm{IndCoh}(S\times_{V_{\mathrm{dR}}} T).$$

Proof. Let $\xi : S\times_V T \to (S\times T)^\wedge_{S\times_V T} \simeq S\times_{V_{\mathrm{dR}}} T$ be the canonical map. As before, $\mathbb{H}_{S\to V} \underset{\mathbb{H}(V)}{\otimes} \mathbb{H}_{V\leftarrow T}$ is compactly generated by its canonical object. Now, the functor in question sends such object to $\xi^{\mathrm{IndCoh}}_*(\omega_{S\times_V T})$, which is a compact generator of $\mathrm{IndCoh}_0((S\times T)^\wedge_{S\times_V T})$. Hence, all that remains is to verify that the functor

$$\mathbb{H}_{S\to V} \underset{\mathbb{H}(V)}{\otimes} \mathbb{H}_{V\leftarrow T} \longrightarrow \mathbb{I}^\wedge_{S\to V} \underset{\mathbb{I}^\wedge(V)}{\otimes} \mathbb{I}^\wedge_{V\leftarrow T}$$

is fully faithful. Assume that $S\to V$ is bounded; the argument for the other case is symmetric. We have the following sequence of left $\mathrm{QCoh}(S)$-linear fully faithful functors:

$$\begin{aligned}
\mathbb{H}_{S\to V} \underset{\mathbb{H}(V)}{\otimes} \mathbb{H}_{V\leftarrow T} &\simeq \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(V)}{\otimes} \mathbb{H}_{V\leftarrow T}\\
&\hookrightarrow \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(V)}{\otimes} \mathbb{I}^\wedge_{V\leftarrow T}\\
&\simeq \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(V)}{\otimes} \mathrm{IndCoh}(V) \underset{\mathfrak{D}(V)}{\otimes} \mathrm{IndCoh}(T).
\end{aligned}$$

To conclude, recall [Gai13, Proposition 4.4.2] that the tautological functor $\mathrm{QCoh}(S) \otimes_{\mathrm{QCoh}(V)} \mathrm{IndCoh}(V) \to \mathrm{IndCoh}(S)$ is fully faithful whenever $S\to V$ is bounded. □

[8] This ensures that $S\times_V T$ is bounded, so that $\mathrm{IndCoh}_0((S\times T)^\wedge_{S\times_V T})$ is well-defined.

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5.2.12 Following the template of § 5.1.9, let us summarize the consequences of the ambidexterity of $\mathbb{H}$. First, we obtain that $\mathbb{H}$ extends to a functor

$$\mathbb{H}^{\mathsf{Corr}} : \mathsf{Corr}(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{\mathrm{all;bdd}} \longrightarrow \mathsf{Alg}^{\mathrm{bimod}}(\mathsf{DGCat}),$$

which has been shown to send

$$[S \leftarrow U \to T] \rightsquigarrow \mathbb{H}_{S\leftarrow U\to T} := \mathbb{H}_{S\leftarrow U} \underset{\mathbb{H}(U)}{\otimes} \mathbb{H}_{U\to T} \simeq \mathrm{IndCoh}_0((S\times T)^{\wedge}_{U}).$$

In other words, $\mathbb{H}^{\mathsf{Corr}}$ coincides with the restriction of $\mathbb{H}^{\mathrm{geom}}$ on $\mathsf{Corr}(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{\mathrm{all;bdd}}$. Therefore, we have the following result.

Corollary 5.2.13. *The lax $(\infty,2)$-functor $\mathbb{H}^{\mathrm{geom}}$ is strict when restricted to* $\mathsf{Corr}(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{\mathrm{all;bdd}}$.

5.2.14 Secondly, $\mathbb{H}^{\mathsf{Corr}}$ admits two extensions to $(\infty,2)$-functors,

$$\mathbb{H}^{\text{R-BC}} : \mathsf{Corr}(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})^{\mathrm{bdd},2-\mathrm{op}}_{\mathrm{all;bdd}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$$

and

$$\mathbb{H}^{\text{L-BC}} : \mathsf{Corr}(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})^{\mathrm{bdd}}_{\mathrm{all;bdd}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat}),$$

described as follows. To a 2-morphism

$$[S \leftarrow U' \to T] \to [S \leftarrow U \to T],$$

induced by $U' \to U$ bounded, $\mathbb{H}^{\text{R-BC}}$ assigns the !-pullback

$$\mathrm{IndCoh}_0((S\times T)^{\wedge}_{U}) \longrightarrow \mathrm{IndCoh}_0((S\times T)^{\wedge}_{U'}),$$

while the $\mathbb{H}^{\text{L-BC}}$ assigns the dual $(*,0)$-pushforward

$$\mathrm{IndCoh}_0((S\times T)^{\wedge}_{U'}) \longrightarrow \mathrm{IndCoh}_0((S\times T)^{\wedge}_{U}),$$

which is well defined thanks to boundedness; see Theorem 3.1.6.

6. Sheaves of categories relative to $\mathbb{H}$

The coefficient system $\mathbb{H}$ allows us to define the ∞-category $\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{X})$, for any prestack $\mathcal{X} \in \mathsf{Fun}((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})^{\mathrm{op}}, \mathsf{Grpd}_{\infty})$. As we are only interested in studying $\mathsf{ShvCat}^{\mathbb{H}}$ on algebraic stacks, we only consider the functor

$$\mathsf{ShvCat}^{\mathbb{H}} : (\mathsf{Stk}^{<\infty}_{\mathrm{lfp}})^{\mathrm{op}} \longrightarrow \mathsf{Cat}_{\infty},$$

where $\mathsf{Stk}^{<\infty}_{\mathrm{lfp}}$ consists of those bounded algebraic stacks that have affine diagonal and perfect cotangent complex.

In this section we explain several constructions regarding $\mathsf{ShvCat}^{\mathbb{H}}$, which we then use to prove our main theorems. We first show that $\mathsf{ShvCat}^{\mathbb{H}}$ satisfies smooth descent. Secondly, we discuss pushforwards and base-change as follows: by Theorem 5.2.10, $\mathbb{H}$ is ambidextrous; accordingly, $\mathsf{ShvCat}^{\mathbb{H}}$ will admit extensions to categories of correspondences in two mutually dual ways. Next, we discuss the notion of $\mathbb{H}$-affineness of objects of $\mathsf{Stk}^{<\infty}_{\mathrm{lfp}}$: we show that $\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$ is the ∞-category of modules over the monoidal DG category $\mathbb{H}^{\mathrm{geom}}(\mathcal{Y})$. Finally, we deduce that the lax $(\infty,2)$-functor $\mathbb{H}^{\mathrm{geom}}$ is actually strict.

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6.1 Descent

Define

$$\mathsf{ShvCat}^{\mathbb{H}} : (\mathsf{Stk}_{\mathrm{lfp}}^{<\infty})^{\mathrm{op}} \longrightarrow \mathsf{Cat}_\infty$$

to be the right Kan extension of

$$\mathsf{ShvCat}^{\mathbb{H}} = \textbf{-mod} \circ \mathbb{H} : (\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})^{\mathrm{op}} \longrightarrow \mathsf{Cat}_\infty$$

along the inclusion $\mathsf{Aff}_{\mathrm{lfp}}^{<\infty} \hookrightarrow \mathsf{Stk}_{\mathrm{lfp}}^{<\infty}$. The purpose of this section is to show that the functor $\mathsf{ShvCat}^{\mathbb{H}}$ satisfies smooth descent.

6.1.1 Objects of

$$\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y}) \simeq \lim_{S \in (\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/\mathcal{Y}}} \mathbb{H}(S)\textbf{-mod}$$

will be often represented simply by $\mathcal{C} \simeq \{\mathcal{C}_S\}_{S \in (\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/\mathcal{Y}}}$, leaving the coherent system of compatibilities $\mathbb{H}_{S \to T} \otimes_{\mathbb{H}(T)} \mathcal{C}_T \simeq \mathcal{C}_S$ implicit. For any $f : \mathcal{X} \to \mathcal{Y}$ in $\mathsf{Stk}_{\mathrm{lfp}}^{<\infty}$, denote by $f^{*,\mathbb{H}}$ the structure functor. Explicitly (and tautologically), $f^{*,\mathbb{H}}$ sends

$$\{\mathcal{C}_S\}_{S \in (\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/\mathcal{Z}}} \rightsquigarrow \{\mathcal{C}_S\}_{S \in (\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/\mathcal{Y}}}.$$

In what follows, elements of $S \in (\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/\mathcal{Y}}$ will be denoted by $\phi_{S \to \mathcal{Y}} : S \to \mathcal{Y}$. It is obvious that $(\phi_{S \to \mathcal{Y}})^{*,\mathbb{H}}(\mathcal{C}) = \mathcal{C}_S$.

Theorem 6.1.2. *The functor $\mathsf{ShvCat}^{\mathbb{H}} : (\mathsf{Stk}_{\mathrm{lfp}}^{<\infty})^{\mathrm{op}} \to \mathsf{Cat}_\infty$ satisfies smooth descent. In particular, for any $\mathcal{Y}$, the restriction functor*

$$\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y}) \longrightarrow \lim_{S \in (\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/\mathcal{Y},\mathrm{smooth}}} \mathbb{H}(S)\textbf{-mod}$$

is an equivalence. Here, $(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/\mathcal{Y},\mathrm{smooth}}$ is the subcategory of $(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/\mathcal{Y}}$ whose objects are smooth maps $S \to \mathcal{Y}$ and whose morphisms are triangles $S \to T \to \mathcal{Y}$ with all maps smooth.

6.1.3 We will need a few preliminary results that will be stated and proven after having fixed some notation.

Let $\phi : U \to S$ be a smooth cover in $\mathsf{Aff}_{\mathrm{lfp}}^{<\infty}$ and let $U_\bullet$ be its associated Čech simplicial scheme. For any arrow $[m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{op}}$, denote by $\phi_{[m]\to[n]} : U_m \to U_n$ and $\phi_n : U_n \to S$ the induced (smooth) maps.

Now let $\mathcal{Y} \in \mathsf{Stk}_{\mathrm{lfp}}^{<\infty}$ be a stack under S. The above maps induce functors

$$(\Phi_{[m]\to[n]})_{*,0} : \mathrm{IndCoh}_0(\mathcal{Y}^\wedge_{U_n}) \longrightarrow \mathrm{IndCoh}_0(\mathcal{Y}^\wedge_S),$$
$$(\Phi_n)_{*,0} : \mathrm{IndCoh}_0(\mathcal{Y}^\wedge_{U_n}) \longrightarrow \mathrm{IndCoh}_0(\mathcal{Y}^\wedge_S).$$

We obtain a functor

$$\varepsilon : \operatorname*{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{IndCoh}_0(\mathcal{Y}^\wedge_{U_n}) \longrightarrow \mathrm{IndCoh}_0(\mathcal{Y}^\wedge_S). \tag{6.1}$$

Lemma 6.1.4. *The functor (6.1) is an equivalence.*

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Proof. Denote by

$$\mathrm{IndCoh}_0(\mathcal{Y}_S^\wedge)_{[U,*]}$$

the colimit category appearing in the left-hand side of (6.1). We will proceed in several steps.

Step 1. We need to introduce an auxiliary category. Denote by $(\Phi_n)^?$ and $(\Phi_{[m]\to[n]})^?$ the possibly discontinuous right adjoints to $(\Phi_n)_{*,0}$ and $(\Phi_{[m]\to[n]})_{*,0}$. Consider the cosimplicial DG category

$$\big(\mathrm{IndCoh}_0(\mathcal{Y}^\wedge_{U_\bullet}), (\Phi_{[m]\to[n]})^?\big) \tag{6.2}$$

and define $\mathrm{IndCoh}_0(\mathcal{Y}_S^\wedge)^{[U,?]}$ to be its totalization. Of course,

$$\mathrm{IndCoh}_0(\mathcal{Y}_S^\wedge)^{[U,?]} \simeq \mathrm{IndCoh}_0(\mathcal{Y}_S^\wedge)_{[U,*]}$$

via the usual limit–colimit procedure. However, the former interpretation allows us to write ε^R as the functor

$$\varepsilon^R : \mathrm{IndCoh}_0(\mathcal{Y}_S^\wedge) \longrightarrow \mathrm{IndCoh}_0(\mathcal{Y}_S^\wedge)^{[U,?]}$$

given by the limit of the $(\Phi_n)^?$.

Step 2. We will prove the lemma by showing that ε^R is an equivalence. By a standard argument, it suffices to check two facts:

– the (discontinuous) forgetful functor

$$(\Phi_0)^? : \mathrm{IndCoh}_0(\mathcal{Y}_S^\wedge)^{[U,?]} \longrightarrow \mathrm{IndCoh}_0(\mathcal{Y}_U^\wedge)$$

is monadic;

– the cosimplicial category (6.2) satisfies the monadic Beck–Chevalley condition.

Step 3. In this step, we will prove the first item above. To this end, we define

$$\mathrm{QCoh}(S)_{[U,*]} := \operatorname*{colim}_{[n],\phi_*} \mathrm{QCoh}(U_n), \quad \mathrm{QCoh}(S)^{[U,?]} := \lim_{[n],\phi^?} \mathrm{QCoh}(U_n),$$

where $(\phi_{[m]\to[n]})^?$ is the discontinuous right adjoint to $(\phi_{[m]\to[n]})_*$. It is easy to see that there is a commutative square

$$\begin{array}{ccc}
\mathrm{IndCoh}_0(\mathcal{Y}_S^\wedge)^{[U,?]} & \xrightarrow{\quad (\Phi_0)^? \quad} & \mathrm{IndCoh}_0(\mathcal{Y}_U^\wedge) \\
\downarrow & & \downarrow \\
\mathrm{QCoh}(S)^{[U,?]} & \xrightarrow{(\phi_0)^? := ((\phi_0)_*)^R} & \mathrm{QCoh}(U)
\end{array}$$

where the vertical arrows are the structure (conservative) functors induced by the morphism $\mathbb{Q} \to \mathbb{H}$. Hence, it suffices to show that the bottom horizontal arrow is monadic, and the latter has been established in [Gai15b, § 8.1].

Step 4. All that remains is to verify the second item of Step 2 above. This is a particular case of the lemma below. ☐

Lemma 6.1.5. *Consider a diagram*

$$\begin{array}{ccccc} U' & \xrightarrow{h'} & V' & & \\ \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} & & \\ U & \xrightarrow{h} & V & \longrightarrow & Z \end{array}$$

in $\mathsf{Aff}_{\mathrm{lfp}}^{<\infty}$, *where the square is cartesian with all maps smooth. We do not require that* $V \to Z$ *be smooth. Then the natural lax commutative diagram*

$$\begin{array}{ccc} \mathrm{IndCoh}_0(Z^{\wedge}_{V'}) & \xleftarrow{(\Phi_{h'})_{*,0}} & \mathrm{IndCoh}_0(Z^{\wedge}_{U'}) \\ \uparrow{\scriptstyle (\Phi_v)^?} & & \uparrow{\scriptstyle (\Phi_{v'})^?} \\ \mathrm{IndCoh}_0(Z^{\wedge}_{V}) & \xleftarrow{(\Phi_{h})_{*,0}} & \mathrm{IndCoh}_0(Z^{\wedge}_{U}) \end{array} \tag{6.3}$$

is commutative.[9]

Proof. We proceed in steps here as well.

Step 1. For $f : X \to V$ a map in $\mathsf{Sch}_{\mathrm{aft}}$, denote the induced functor by $\Phi_f : Z^{\wedge}_X \to Z^{\wedge}_V$. Recall the equivalence

$$\mathrm{IndCoh}(Z^{\wedge}_V) \underset{\mathfrak{D}(V)}{\otimes} \mathfrak{D}(X) \xrightarrow{\simeq} \mathrm{IndCoh}(Z^{\wedge}_X) \tag{6.4}$$

given by exterior tensor product (Lemma 4.2.1). One immediately checks that, under such equivalence, $(\Phi_f)_*^{\mathrm{IndCoh}}$ goes over to the functor

$$\mathrm{IndCoh}(Z^{\wedge}_V) \underset{\mathfrak{D}(V)}{\otimes} \mathfrak{D}(X) \xrightarrow{\mathrm{id}\otimes f_{*,\mathrm{dR}}} \mathrm{IndCoh}(Z^{\wedge}_V) \underset{\mathfrak{D}(V)}{\otimes} \mathfrak{D}(V) \simeq \mathrm{IndCoh}(Z^{\wedge}_V).$$

Thus, whenever f is smooth, $(\Phi_f)_*^{\mathrm{IndCoh}}$ admits a left adjoint which we denote by $(\Phi_f)^{*,\mathrm{IndCoh}}$; this is obtained from the $\mathfrak{D}$-module $*$-pullback $f^{*,\mathrm{dR}} \simeq f^{!,\mathrm{dR}}[-2\dim_f]$ by tensoring up. Hence, for f smooth, we have an equivalence

$$(\Phi_f)^{*,\mathrm{IndCoh}} \simeq (\Phi_f)^![-2\dim_f]. \tag{6.5}$$

Step 2. Applying the above to h and h', we see that the functors $(\Phi_h)^{*,\mathrm{IndCoh}}$ and $(\Phi_{h'})^{*,\mathrm{IndCoh}}$ preserve the IndCoh_0-subcategories. We thus have a diagram

$$\begin{array}{ccc} \mathrm{IndCoh}_0(Z^{\wedge}_{V'}) & \xrightarrow{(\phi_{h'})^{*,\mathrm{IndCoh}}} & \mathrm{IndCoh}_0(Z^{\wedge}_{U'}) \\ \downarrow{\scriptstyle (\Phi_v)_{*,0}} & & \downarrow{\scriptstyle (\Phi_{v'})_{*,0}} \\ \mathrm{IndCoh}_0(Z^{\wedge}_{V}) & \xrightarrow{(\phi_{h})^{*,\mathrm{IndCoh}}} & \mathrm{IndCoh}_0(Z^{\wedge}_{U}), \end{array} \tag{6.6}$$

which is immediately seen to be commutative thanks to (6.5) and base-change for IndCoh_0.

Step 3. We leave it to the reader to check that the horizontal arrows in the commutative diagram (6.6) are left adjoint to the horizontal arrows of (6.3). Hence, we obtain the desired assertion by passing to the diagram right adjoint to (6.6). □

[9] As usual, for $f : X \to V$ one of the above maps, we have denoted the induced functors by $(\Phi_f)_{*,0}$ and $(\Phi_f)^?$.

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6.1.6 Let us finally prove Theorem 6.1.2.

Proof of Theorem 6.1.2. It suffices to prove that the functor $\mathsf{ShvCat}^{\mathbb{H}} : (\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})^{\mathrm{op}} \to \mathsf{Cat}_\infty$ satisfies smooth descent. For $S \in \mathsf{Aff}^{<\infty}_{\mathrm{lfp}}$, let $f : U \to S$ be a smooth cover and $U_\bullet$ the corresponding Čech resolution. Denote by $f_n : U_n \to S$ the structure maps. We are to show that the natural functor

$$\alpha : \mathbb{H}(S)\text{-}\mathbf{mod} \longrightarrow \lim_{[n]\in\mathbf{\Delta}} \mathbb{H}(U_n)\text{-}\mathbf{mod}, \quad \mathcal{C} \rightsquigarrow \{\mathbb{H}_{U_n\to S} \otimes_{\mathbb{H}(S)} \mathcal{C}\}_{n\in\mathbf{\Delta}}$$

is an equivalence.

Note that α admits a left adjoint, α^L, which sends

$$\{\mathcal{C}_n\}_{n\in\mathbf{\Delta}} \rightsquigarrow \operatorname*{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \Big(\mathbb{H}_{S\leftarrow U_n} \underset{\mathbb{H}(U_n)}{\otimes} \mathcal{C}_n\Big),$$

where we have used the left dualizability of the $\mathbb{H}_{U_n\to S}$. We will show that α and α^L are both fully faithful.

For α, it suffices to verify that the natural functor $\alpha^L \circ \alpha(\mathbb{H}(S)) \to \mathbb{H}(S)$ is an equivalence. Such functor is readily rewritten as

$$\varepsilon : \operatorname*{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \Big(\mathbb{H}_{S\leftarrow U_n} \underset{\mathbb{H}(U_n)}{\otimes} \mathbb{H}_{U_n\to S}\Big) \longrightarrow \mathbb{H}(S).$$

By Lemma 5.2.6, our claim is exactly the content of Lemma 6.1.4 applied to $\mathcal{Y} = S \times S$.

Next, we prove α^L is fully faithful: it suffices to check that the natural functor

$$\mathbb{H}_{U\to S} \otimes_{\mathbb{H}(S)} \operatorname*{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \Big(\mathbb{H}_{S\leftarrow U_n} \underset{\mathbb{H}(U_n)}{\otimes} \mathcal{C}_n\Big) \longrightarrow \mathcal{C}_0$$

is an equivalence. Using base-change for $\mathbb{H}$, this reduces to proving that

$$\operatorname*{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathbb{H}_{U\leftarrow U\times_S U_n\to U} \longrightarrow \mathbb{H}(U)$$

is an equivalence. This is again an instance of Lemma 6.1.4. □

6.2 Localization and global sections

Let $\mathcal{Y} \in \mathsf{Stk}^{<\infty}_{\mathrm{lfp}}$. In this section we equip $\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$ with a canonical object that we denote by $\mathbb{H}_{/\mathcal{Y}}$. We then use such object to define a fundamental adjunction and the notion of $\mathbb{H}$-affineness.

6.2.1 For $S \in \mathsf{Aff}^{<\infty}_{\mathrm{lfp}}$ mapping to $\mathcal{Y}$, consider the left $\mathbb{H}(S)$-module

$$\mathbb{H}_{S\to\mathcal{Y}} := \mathbb{H}^{\mathrm{geom}}_{S\to\mathcal{Y}} = \mathrm{IndCoh}_0((S\times\mathcal{Y})^\wedge_{\mathcal{Y}}).$$

Let $U \to \mathcal{Y}$ be an affine atlas with induced Čech complex $U_\bullet$. By [Ber17b], there is a natural left $\mathbb{H}(S)$-linear equivalence

$$\mathrm{IndCoh}_0((S\times\mathcal{Y})^\wedge_S) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathrm{QCoh}(U_n) \simeq \mathrm{IndCoh}_0((S\times U_n)^\wedge_{S\times_{\mathcal{Y}}U_n}) \simeq \mathbb{H}_{S\leftarrow S\times_{\mathcal{Y}}U\to U} \tag{6.7}$$

from which we obtain a left $\mathbb{H}(S)$-linear equivalence

$$\mathbb{H}_{S\to\mathcal{Y}} \simeq \lim_{U\in(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y},\mathrm{smooth}}} \mathbb{H}_{S\leftarrow S\times_{\mathcal{Y}}U\to U}, \tag{6.8}$$

where the limit on the right-hand side is formed using the $(!,0)$-pullbacks. We now show that the same category $\mathbb{H}_{S\to\mathcal{Y}}$ can be expressed as a colimit.

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Lemma 6.2.2. *Let S, $\mathcal{Y}$, $U_\bullet$ be as above. Then the natural functor*

$$\operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{IndCoh}_0((S\times U_n)^\wedge_{S\times_{\mathcal{Y}}U_n}) \longrightarrow \mathrm{IndCoh}_0((S\times\mathcal{Y})^\wedge_S)$$

given by the $(*,0)$-pushforward functors is an equivalence.

Proof. Under the equivalence (6.7), the left-hand side becomes

$$\operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}}\Big(\mathrm{IndCoh}_0((S\times\mathcal{Y})^\wedge_S)\underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes}\mathrm{QCoh}(U_n)\Big)\simeq \mathrm{IndCoh}_0((S\times\mathcal{Y})^\wedge_S)\underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes}\Big(\operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}}\mathrm{QCoh}(U_n)\Big),$$

where the colimit on the right-hand side is taken with respect to the $*$-pushforward functors. It suffices to recall again that the obvious functor

$$\operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}}\mathrm{QCoh}(U_n)\longrightarrow\mathrm{QCoh}(\mathcal{Y})$$

is a $\mathrm{QCoh}(\mathcal{Y})$-linear equivalence; see [Gai15b, Proposition 6.2.7]. □

Lemma 6.2.3. *The collection $\{\mathbb{H}_{S\to\mathcal{Y}}\}_{S\in(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y}}}$ assembles to an object of $\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$ that we shall denote by $\mathbb{H}_{/\mathcal{Y}}$.*

Proof. We need to prove that, for $S'\to S$ a map in $\mathsf{Aff}^{<\infty}_{\mathrm{lfp}}$, the canonical arrow

$$\mathbb{H}_{S'\to S}\underset{\mathbb{H}(S)}{\otimes}\mathbb{H}_{S\to\mathcal{Y}}\longrightarrow\mathbb{H}_{S'\to\mathcal{Y}}$$

is an equivalence. We use the canonical left $\mathbb{H}(S)$-linear equivalence

$$\mathbb{H}_{S\to\mathcal{Y}}:=\mathrm{IndCoh}_0((S\times\mathcal{Y})^\wedge_{\mathcal{Y}})\simeq\lim_{U\in(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y},\mathrm{smooth}}}\mathbb{H}_{S\leftarrow S\times_{\mathcal{Y}}U\to U},$$

discussed above. Since the left leg of each correspondence above is smooth, base-change for $\mathbb{H}$ can be applied to yield

$$\mathbb{H}_{S'\to S}\underset{\mathbb{H}(S)}{\otimes}\mathbb{H}_{S\to\mathcal{Y}}\simeq\mathbb{H}_{S'\to S}\underset{\mathbb{H}(S)}{\otimes}\lim_{U\in(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y},\mathrm{smooth}}}\mathbb{H}_{S\leftarrow S\times_{\mathcal{Y}}U\to U}\simeq\lim_{U\in(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y},\mathrm{smooth}}}\mathbb{H}_{S'\leftarrow S'\times_{\mathcal{Y}}U\to U}.$$

The latter is $\mathbb{H}_{S'\to\mathcal{Y}}$, as desired. □

6.2.4 Set $\mathbb{H}(\mathcal{Y}):=\mathbb{H}^{\mathrm{geom}}(\mathcal{Y})$. Recall that the left $\mathbb{H}(S)$-module category $\mathbb{H}_{S\to\mathcal{Y}}:=\mathbb{H}^{\mathrm{geom}}_{S\to\mathcal{Y}}$ is actually an $(\mathbb{H}(S),\mathbb{H}(\mathcal{Y}))$-bimodule, where both actions are given by convolution. Since $\mathbb{H}_{S\to\mathcal{Y}}$ is dualizable as a DG category and the monoidal DG categories $\mathbb{H}(S)$ and $\mathbb{H}(\mathcal{Y})$ are both *very rigid*, Corollary 2.1.7 implies that $\mathbb{H}_{S\to\mathcal{Y}}$ is ambidextrous.

By Lemma 6.2.2 and the ambidexterity of $\mathbb{H}$, its (right, as well as left) dual is easily seen to be the obvious $(\mathbb{H}(\mathcal{Y}),\mathbb{H}(S))$-bimodule

$$\mathbb{H}_{\mathcal{Y}\leftarrow S}:=\mathbb{H}^{\mathrm{geom}}_{\mathcal{Y}\leftarrow S}:=\mathrm{IndCoh}_0((\mathcal{Y}\times S)^\wedge_S).$$

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6.2.5 We can now introduce the fundamental adjunction

$$\mathbf{Loc}^{\mathbb{H}}_{\mathcal{Y}} : \mathbb{H}(\mathcal{Y})\text{-}\mathbf{mod} \rightleftarrows \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y}) : \mathbf{\Gamma}^{\mathbb{H}}_{\mathcal{Y}}. \tag{6.9}$$

The left adjoint sends $\mathcal{C} \in \mathbb{H}(\mathcal{Y})$-**mod** to the $\mathbb{H}$-sheaf of categories represented by

$$\Big\{\mathbb{H}_{S\to\mathcal{Y}} \underset{\mathbb{H}(\mathcal{Y})}{\otimes} \mathcal{C}\Big\}_{S\in(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y}}}.$$

This makes sense in view of Lemma 6.2.3. The right adjoint sends $\mathcal{C} = \{\mathcal{C}_S\}_S \in \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$ to the $\mathbb{H}(\mathcal{Y})$-module

$$\mathbf{\Gamma}^{\mathbb{H}}_{\mathcal{Y}}(\mathcal{C}) = \lim_{S\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}} \mathbb{H}_{\mathcal{Y}\leftarrow S} \underset{\mathbb{H}(S)}{\otimes} \mathcal{C}_S, \tag{6.10}$$

where we have used Theorem 6.1.2.

We say that $\mathcal{Y}$ is $\mathbb{H}$-*affine* if the adjoint functors (6.9) are mutually inverse equivalences.

Remark 6.2.6. Note that $\mathbf{\Gamma}^{\mathbb{H}}_{\mathcal{Y}}(\mathcal{C})$ can be computed as

$$\mathcal{H}om_{\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})}(\mathbb{H}_{/\mathcal{Y}}, \mathcal{C}),$$

where $\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$ is regarded as an $(\infty,2)$-category and $\mathcal{H}om$ denotes the $(\infty,1)$-category of 1-arrows in an $(\infty,2)$-category.

6.3 Pushforwards and the Beck–Chevalley conditions

For any arrow $f : \mathcal{Y} \to \mathcal{Z}$ in $\mathsf{Stk}^{<\infty}_{\mathrm{lfp}}$, the functor $f^{*,\mathbb{H}}$ commutes with colimits, whence it admits a right adjoint, denoted by $f_{*,\mathbb{H}}$. Moreover, since $\mathbb{H}$ satisfies the left Beck–Chevalley condition, $f^{*,\mathbb{H}}$ commutes with limits as well, whence it also admits a left adjoint, denoted by $f_{!,\mathbb{H}}$.

In this section we give formulas for these pushforward functors and discuss base-change for $\mathsf{ShvCat}^{\mathbb{H}}$.

6.3.1 Let $f : \mathcal{Y} \to \mathcal{Z}$ be an arrow in $\mathsf{Stk}^{<\infty}_{\mathrm{lfp}}$. For $\mathcal{C} \in \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$, we will compute the $\mathbb{H}$-sheaf of categories $f_{*,\mathbb{H}}(\mathcal{C})$. By Theorem 6.1.2, it suffices to specify the value of $f_{*,\mathbb{H}}(\mathcal{C})$ on affine schemes $U \in \mathsf{Aff}^{<\infty}_{\mathrm{lfp}}$ mapping *smoothly* to $\mathcal{Z}$. For each such $\phi_{U\to\mathcal{Y}} : U \to \mathcal{Y}$, consider the $\mathbb{H}(U)$-module

$$\mathcal{E}_U := \lim_{V\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/U\times_{\mathcal{Z}}\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}} \mathbb{H}_{U\leftarrow V} \underset{\mathbb{H}(V)}{\otimes} \mathcal{C}_V.$$

The limit is well defined thanks to the left Beck–Chevalley condition, that is, exploiting the $(\infty,2)$-functor $\mathbb{H}^{\text{L-BC}}$ of § 5.2.14. Next, using the right Beck–Chevalley condition, one readily checks that the natural functor

$$\mathbb{H}_{U'\to U} \underset{\mathbb{H}(U)}{\otimes} \mathcal{E}_U \longrightarrow \mathcal{E}_{U'}$$

is an equivalence for any smooth map $U' \to U$ in Aff. This guarantees that $\{\mathcal{E}_U\}_{U\in(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Z},\mathrm{smooth}}}$ is a well-defined object of $\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Z})$. We leave it to the reader to verify that such object in the required pushforward $f_{*,\mathbb{H}}(\mathcal{C})$.

6.3.2 Similarly, the !-pushforward of $\mathcal{C}$ is written as

$$f_{!,\mathbb{H}}(\mathcal{C}) \simeq \{\mathcal{D}_U\}_{U\in(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Z},\mathrm{smooth}}},$$

where $\mathcal{D}_U$ is defined, using the $(\infty,2)$-functor $\mathbb{H}^{\text{R-BC}}$, as

$$\mathcal{D}_U := \operatorname*{colim}_{V\in(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/U\times_{\mathcal{Z}}\mathcal{Y},\mathrm{smooth}}} \mathbb{H}_{U\leftarrow V} \underset{\mathbb{H}(V)}{\otimes} \mathcal{C}_V.$$

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6.3.3 It is then *tautological* to verify that the $\mathsf{ShvCat}^{\mathbb{H}}$ has the right Beck–Chevalley condition with respect to bounded arrows, that is, the assignment

$$[\mathcal{X} \xleftarrow{h} \mathcal{W} \xrightarrow{v} \mathcal{Y}] \rightsquigarrow v_{*,\mathbb{H}} \circ h^{*,\mathbb{H}}$$

upgrades to an $(\infty, 2)$-functor

$$\mathsf{ShvCat}^{\mathbb{H}}_{*,*} : \mathsf{Corr}(\mathsf{Stk}_{\mathrm{type}})^{\mathrm{bdd},2-\mathrm{op}}_{\mathrm{bdd;all}} \longrightarrow \mathsf{Cat}_\infty, \tag{6.11}$$

with Cat_∞ being regarded here as an $(\infty, 2)$-category. Symmetrically, the assignment

$$[\mathcal{X} \xleftarrow{h} \mathcal{W} \xrightarrow{v} \mathcal{Y}] \rightsquigarrow v_{!,\mathbb{H}} \circ h^{*,\mathbb{H}}$$

upgrades to an $(\infty, 2)$-functor

$$\mathsf{ShvCat}^{\mathbb{H}}_{!,*} : \mathsf{Corr}(\mathsf{Stk}_{\mathrm{type}})^{\mathrm{bdd}}_{\mathrm{all;bdd}} \longrightarrow \mathsf{Cat}_\infty. \tag{6.12}$$

Remark 6.3.4. Combining the two functors together, we deduce that we have base-change isomorphisms

$$g^{*,\mathbb{H}} \circ f_{*,\mathbb{H}} \simeq F_{*,\mathbb{H}} \circ G^{*,\mathbb{H}}, \quad g^{*,\mathbb{H}} \circ f_{!,\mathbb{H}} \simeq F_{!,\mathbb{H}} \circ G^{*,\mathbb{H}}$$

as soon as at least one between f and g is bounded.

Remark 6.3.5. We will show later that !- and $*$-pushforwards of $\mathbb{H}$-sheaves of categories are naturally identified; see Corollary 6.5.5.

6.4 Extension/restriction of coefficients

In this section we relate $\mathbb{H}$-sheaves of categories with the more familiar quasi-coherent sheaves of categories developed in [Gai15b]. The latter are the ones obtained from the coefficient system $\mathbb{Q}$.

6.4.1 The relation between $\mathsf{ShvCat}^{\mathbb{H}}$ and $\mathsf{ShvCat}^{\mathbb{Q}}$ is induced by the map $\mathbb{Q} \to \mathbb{H}$ of coefficient systems on $\mathsf{Aff}^{<\infty}_{\mathrm{lfp}}$. Specifically, $\mathbb{Q} \to \mathbb{H}$ induces a natural transformation

$$\mathsf{oblv}^{\mathbb{Q}\to\mathbb{H}} : \mathsf{ShvCat}^{\mathbb{H}} \Longrightarrow \mathsf{ShvCat}^{\mathbb{Q}}$$

between functors out of $(\mathsf{Stk}^{<\infty}_{\mathrm{lfp}})^{\mathrm{op}}$. In other words, this means that $\mathsf{oblv}^{\mathbb{Q}\to\mathbb{H}}$ is compatible with the pullback functors.

Lemma 6.4.2. *For $\mathcal{Y} \in \mathsf{Stk}^{<\infty}_{\mathrm{lfp}}$, the functor $\mathsf{oblv}^{\mathbb{Q}\to\mathbb{H}}_{\mathcal{Y}} : \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y}) \to \mathsf{ShvCat}^{\mathbb{Q}}(\mathcal{Y})$ is conservative and admits a left adjoint, which we will call $\mathsf{ind}^{\mathbb{Q}\to\mathbb{H}}_{\mathcal{Y}}$.*

Proof. Conservativeness is obvious. The existence of the left adjoint is clear thanks to the fact that $\mathsf{oblv}^{\mathbb{Q}\to\mathbb{H}}_{\mathcal{Y}}$ commutes with limits. □

6.4.3 The functor

$$\mathsf{ind}^{\mathbb{Q}\to\mathbb{H}}_{\mathcal{Y}} : \mathsf{ShvCat}^{\mathbb{Q}}(\mathcal{Y}) \longrightarrow \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$$

is really easy to describe explicitly. Namely,

$$\mathsf{ind}^{\mathbb{Q}\to\mathbb{H}}_{\mathcal{Y}}(\mathcal{C}) \simeq \underset{S\in(\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y},\mathrm{smooth}}}{\mathrm{colim}} (\phi_{S\to\mathcal{Y}})_{!,\mathbb{H}}\big(\mathbb{H}(S) \otimes_{\mathrm{QCoh}(S)} \mathcal{C}_S\big).$$

LEMMA 6.4.4. *The induction functor* $\mathsf{ind}_{\mathcal{Y}}^{\mathbb{Q}\to\mathbb{H}} : \mathsf{ShvCat}^{\mathbb{Q}}(\mathcal{Y}) \to \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})$ *sends* $\mathbb{Q}_{/\mathcal{Y}}$ *to* $\mathbb{H}_{/\mathcal{Y}}$.

Proof. The above formula and § 6.3.2 yield

$$\begin{aligned}\mathsf{ind}_{\mathcal{Y}}^{\mathbb{Q}\to\mathbb{H}}(\mathbb{Q}_{/\mathcal{Y}}) &\simeq \underset{S\in(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/\mathcal{Y}}}{\mathrm{colim}} (\phi_{S\to\mathcal{Y}})_{!,\mathbb{H}}(\mathbb{H}(S)) \\ &\simeq \underset{S\in(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/\mathcal{Y},\mathrm{smooth}}}{\mathrm{colim}} \left\{ \underset{V\in(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/U\times_{\mathcal{Y}}S,\mathrm{smooth}}}{\mathrm{colim}} \mathbb{H}_{U\leftarrow V} \underset{\mathbb{H}(V)}{\otimes} \mathbb{H}_{V\to S} \right\}_U .\end{aligned}$$

We now apply Lemma 6.1.4 twice. Firstly,

$$\underset{V\in(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/U\times_{\mathcal{Y}}S,\mathrm{smooth}}}{\mathrm{colim}} \mathbb{H}_{U\leftarrow V} \underset{\mathbb{H}(V)}{\otimes} \mathbb{H}_{V\to S} = \underset{V\in(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/U\times_{\mathcal{Y}}S,\mathrm{smooth}}}{\mathrm{colim}} \mathrm{IndCoh}_0((U\times S)^{\wedge}_{V})$$

is equivalent to $\mathrm{IndCoh}_0((U\times S)^{\wedge}_{U\times_{\mathcal{Y}}S})$. Secondly,

$$\underset{S\in(\mathsf{Aff}_{\mathrm{lfp}}^{<\infty})_{/\mathcal{Y},\mathrm{smooth}}}{\mathrm{colim}} \mathrm{IndCoh}_0((U\times S)^{\wedge}_{U\times_{\mathcal{Y}}S}) \simeq \mathrm{IndCoh}_0((U\times\mathcal{Y})^{\wedge}_{U}) =: \mathbb{H}_{U\to\mathcal{Y}}.$$

This concludes the computation. □

6.5 $\mathbb{H}$-affineness

In this section we prove our main theorem, the $\mathbb{H}$-affineness of algebraic stacks, and deduce that $\mathbb{H}^{\mathrm{geom}}$ is a strict $(\infty,2)$-functor.

THEOREM 6.5.1. *Any* $\mathcal{Y}\in\mathsf{Stk}_{\mathrm{lfp}}^{<\infty}$ *is* $\mathbb{H}$*-affine, that is, the adjunction*

$$\mathbf{Loc}_{\mathcal{Y}}^{\mathbb{H}} : \mathbb{H}(\mathcal{Y})\text{-}\mathbf{mod} \rightleftarrows \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y}) : \mathbf{\Gamma}_{\mathcal{Y}}^{\mathbb{H}}. \tag{6.13}$$

is an equivalence of ∞*-categories.*

Proof. Our strategy is to reduce to the known $\mathbb{Q}$-affineness of such stacks (see [Gai15b, Theorem 2.2.6]) using the adjunction

$$\mathsf{ind}_{\mathcal{Y}}^{\mathbb{Q}\to\mathbb{H}} : \mathsf{ShvCat}^{\mathbb{Q}}(\mathcal{Y}) \rightleftarrows \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y}) : \mathsf{oblv}_{\mathcal{Y}}^{\mathbb{Q}\to\mathbb{H}}.$$

Step 1. For a monoidal functor $f:\mathcal{A}\to\mathcal{B}$, we denote by $\mathsf{ind}[f] : \mathcal{A}\text{-}\mathbf{mod} \rightleftarrows \mathcal{B}\text{-}\mathbf{mod} : \mathsf{oblv}[f]$ the standard adjunction. Let $\delta_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) \to \mathbb{H}(\mathcal{Y})$ be the usual monoidal functor.

By Lemma 6.4.4, the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod} & \xrightarrow{\mathsf{ind}[\delta_{\mathcal{Y}}]} & \mathbb{H}(\mathcal{Y})\text{-}\mathbf{mod} \\ \downarrow{\scriptstyle \mathbf{Loc}_{\mathcal{Y}}} & & \downarrow{\scriptstyle \mathbf{Loc}_{\mathcal{Y}}^{\mathbb{H}}} \\ \mathsf{ShvCat}(\mathcal{Y}) & \xrightarrow{\mathsf{ind}_{\mathcal{Y}}^{\mathbb{Q}\to\mathbb{H}}} & \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y}) \end{array} \tag{6.14}$$

is commutative. It follows that the square

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod} & \xleftarrow{\mathsf{oblv}[\delta_{\mathcal{Y}}]} & \mathbb{H}(\mathcal{Y})\text{-}\mathbf{mod} \\ \uparrow{\scriptstyle \mathbf{\Gamma}_{\mathcal{Y}}^{\mathbb{Q}}} & & \uparrow{\scriptstyle \mathbf{\Gamma}_{\mathcal{Y}}^{\mathbb{H}}} \\ \mathsf{ShvCat}(\mathcal{Y}) & \xleftarrow{\mathsf{oblv}_{\mathcal{Y}}^{\mathbb{Q}\to\mathbb{H}}} & \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y}) \end{array} \tag{6.15}$$

is commutative too.

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Step 2. By changing the vertical arrows with their left adjoints, we obtain a lax commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod} & \xleftarrow{\mathsf{oblv}[\delta_{\mathcal{Y}}]} & \mathbb{H}(\mathcal{Y})\text{-}\mathbf{mod} \\ \Big\downarrow{\scriptstyle \mathbf{Loc}_{\mathcal{Y}}} & \searrow & \Big\downarrow{\scriptstyle \mathbf{Loc}^{\mathbb{H}}_{\mathcal{Y}}} \\ \mathsf{ShvCat}(\mathcal{Y}) & \xleftarrow{\mathsf{oblv}^{\mathbb{Q}\to\mathbb{H}}_{\mathcal{Y}}} & \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y}) \end{array} \tag{6.16}$$

However, this diagram is genuinely commutative thanks to the canonical $(\mathrm{QCoh}(S), \mathbb{H}(\mathcal{Y}))$-linear equivalence

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbb{H}(\mathcal{Y}) \xrightarrow{\simeq} \mathbb{H}_{S\to\mathcal{Y}}.$$

Step 3. We are now ready to prove the theorem by checking that the two compositions $\mathbf{Loc}^{\mathbb{H}}_{\mathcal{Y}} \circ \mathbf{\Gamma}^{\mathbb{H}}_{\mathcal{Y}}$ and $\mathbf{\Gamma}^{\mathbb{H}}_{\mathcal{Y}} \circ \mathbf{Loc}^{\mathbb{H}}_{\mathcal{Y}}$ are isomorphic to the corresponding identity functors. This is easily done by using the commutative diagrams (6.15) and (6.16), the conservativity of the functors

$$\mathsf{oblv}^{\mathbb{Q}\to\mathbb{H}}_{\mathcal{Y}} : \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y}) \longrightarrow \mathsf{ShvCat}^{\mathbb{Q}}(\mathcal{Y}), \quad \mathsf{oblv}[\delta_{\mathcal{Y}}] : \mathbb{H}(\mathcal{Y})\text{-}\mathbf{mod} \longrightarrow \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod},$$

and the $\mathbb{Q}$-affineness of $\mathcal{Y}$. □

6.5.2 Combining the $(\infty, 2)$-functor

$$\mathsf{ShvCat}^{\mathbb{H}}_{*,*} : \mathsf{Corr}(\mathsf{Stk}^{<\infty}_{\mathrm{lfp}})^{\mathrm{bdd},2-\mathrm{op}}_{\mathrm{bdd;all}} \longrightarrow \mathsf{Cat}_{\infty}$$

of (6.11) with Theorem 6.5.1, we obtain another *strict* $(\infty, 2)$-functor,

$$\mathbb{H}^{\mathrm{cat}} : \mathsf{Corr}(\mathsf{Stk}^{<\infty}_{\mathrm{lfp}})^{\mathrm{bdd},2-\mathrm{op}}_{\mathrm{bdd;all}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat}), \tag{6.17}$$

defined by

$$\mathcal{X} \rightsquigarrow \mathbb{H}^{\mathrm{cat}}(\mathcal{X}) := \mathbb{H}(\mathcal{X}),$$
$$[\mathcal{X} \xleftarrow{v} \mathcal{W} \xrightarrow{h} \mathcal{Y}] \rightsquigarrow (\mathbb{H}^{\mathrm{cat}})_{\mathcal{X}\leftarrow\mathcal{W}\to\mathcal{Y}} := \mathbf{\Gamma}^{\mathbb{H}}_{\mathcal{Y}} \circ (h_{*,\mathbb{H}} \circ v^{*,\mathbb{H}}) \circ \mathbf{Loc}^{\mathbb{H}}_{\mathcal{Y}}(\mathbb{H}(\mathcal{Y})).$$

Theorem 6.5.3. *The lax $(\infty, 2)$-functor*

$$\mathbb{H}^{\mathrm{geom}} : \mathsf{Corr}\big(\mathsf{Stk}^{<\infty}_{\mathrm{lfp}}\big)^{\mathrm{schem\&bdd\&proper}}_{\mathrm{bdd;all}} \longrightarrow \mathsf{ALG}^{\mathrm{bimod}}(\mathsf{DGCat})$$

of § 3.2 is naturally equivalent to the restriction of $\mathbb{H}^{\mathrm{cat}}$ to $\mathsf{Corr}\big(\mathsf{Stk}^{<\infty}_{\mathrm{lfp}}\big)^{\mathrm{schem\&bdd\&proper}}_{\mathrm{bdd;all}}$. Hence, $\mathbb{H}^{\mathrm{geom}}$ is strict.

Henceforth, we will denote both $(\infty, 2)$-functors simply by $\mathbb{H}$.

Proof. By Remark 6.2.6, the DG category underlying $\mathbb{H}^{\mathrm{cat}}_{\mathcal{X}\leftarrow\mathcal{W}\to\mathcal{Y}}$ is computed as follows:

$$\begin{aligned} \mathbb{H}^{\mathrm{cat}}_{\mathcal{X}\leftarrow\mathcal{W}\to\mathcal{Y}} &\simeq \mathcal{H}om_{\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y})}\big(\mathbb{H}_{/\mathcal{Y}}, h_{*,\mathbb{H}} \circ v^{*,\mathbb{H}}(\mathbb{H}_{/\mathcal{X}})\big) \\ &\simeq \mathcal{H}om_{\mathsf{ShvCat}^{\mathbb{H}}(\mathcal{W})}\big(h^{*,\mathbb{H}}(\mathbb{H}_{/\mathcal{X}}), v^{*,\mathbb{H}}(\mathbb{H}_{/\mathcal{Y}})\big) \\ &\simeq \lim_{U\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{W},\mathrm{smooth}})^{\mathrm{op}}} \mathcal{H}om_{\mathbb{H}(U)}\big(\mathbb{H}_{U\to\mathcal{X}}, \mathbb{H}_{U\to\mathcal{Y}}\big) \end{aligned}$$

$$\simeq \lim_{U\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{W},\mathrm{smooth}})^{\mathrm{op}}} \mathbb{H}_{\mathcal{X}\leftarrow U} \underset{\mathbb{H}(U)}{\otimes} \mathbb{H}_{U\to\mathcal{Y}}$$

$$\simeq \lim_{U\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{W},\mathrm{smooth}})^{\mathrm{op}}} \lim_{S\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{X},\mathrm{smooth}})^{\mathrm{op}}} \lim_{T\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}} \times \mathbb{H}_{S\leftarrow S\times_{\mathcal{X}}U\to U} \underset{\mathbb{H}(U)}{\otimes} \mathbb{H}_{U\leftarrow U\times_{\mathcal{Y}}T\to T}.$$

By base-change for $\mathbb{H}$, we have

$$\mathbb{H}_{S\leftarrow S\times_{\mathcal{X}}U\to U} \underset{\mathbb{H}(U)}{\otimes} \mathbb{H}_{U\leftarrow U\times_{\mathcal{Y}}T\to T} \simeq \mathbb{H}_{S\leftarrow S\times_{\mathcal{X}}U\times_{\mathcal{Y}}T\to T} \simeq \mathrm{IndCoh}_0\big((S\times T)^{\wedge}_{S\times_{\mathcal{X}}U\times_{\mathcal{Y}}T}\big).$$

By taking the limit, we obtain

$$\mathbb{H}^{\mathrm{cat}}_{\mathcal{X}\leftarrow\mathcal{W}\to\mathcal{Y}} \simeq \mathrm{IndCoh}_0((\mathcal{X}\times\mathcal{Y})^{\wedge}_{\mathcal{W}}) =: \mathbb{H}^{\mathrm{geom}}_{\mathcal{X}\leftarrow\mathcal{W}\to\mathcal{Y}},$$

as desired. □

Corollary 6.5.4. *For $f:\mathcal{Y}\to\mathcal{Z}$ in $\mathsf{Stk}^{<\infty}_{\mathrm{lfp}}$. Then the functors $f_{*,\mathbb{H}}$ and $f^{*,\mathbb{H}}$ correspond under $\mathbb{H}$-affineness to the functors of $\mathbb{H}_{\mathcal{Z}\leftarrow\mathcal{Y}}\otimes_{\mathbb{H}(\mathcal{Y})}-$ and $\mathbb{H}_{\mathcal{Y}\to\mathcal{Z}}\otimes_{\mathbb{H}(\mathcal{Z})}-$, respectively.*

Proof. Let $\mathcal{C}\in\mathbb{H}(\mathcal{Y})$-**mod**. We need to exhibit a natural equivalence

$$\mathbf{\Gamma}^{\mathbb{H}}_{\mathcal{Z}}\circ f_{*,\mathbb{H}}\circ\mathbf{Loc}^{\mathbb{H}}_{\mathcal{Y}}(\mathcal{C}) \simeq \mathbb{H}_{\mathcal{Z}\leftarrow\mathcal{Y}} \underset{\mathbb{H}(\mathcal{Y})}{\otimes} \mathcal{C}.$$

This easily reduces to the case $\mathcal{C}=\mathbb{H}(\mathcal{Y})$, where it holds true by construction. The assertion for $f^{*,\mathbb{H}}$ is proven similarly. □

Corollary 6.5.5. *Pullbacks of $\mathbb{H}$-sheaves of categories are ambidextrous: for any $f:\mathcal{Y}\to\mathcal{Z}$ in $\mathsf{Stk}^{<\infty}_{\mathrm{lfp}}$, there is a canonical equivalence $f_{!,\mathbb{H}}\simeq f_{*,\mathbb{H}}$.*

Proof. Recall the formulas for $f_{!,\mathbb{H}}$ and $f_{*,\mathbb{H}}$ from §§ 6.3.1 and 6.3.2. By $\mathbb{H}$-affineness, it suffices to exhibit a natural equivalence $f_{!,\mathbb{H}}(\mathbb{H}_{/\mathcal{Y}})\simeq f_{*,\mathbb{H}}(\mathbb{H}_{/\mathcal{Y}})$. The latter is constructed as in Lemma 6.1.4. □

6.6 The $\mathbb{H}$-action on IndCoh

This final section contains an example of our techniques. We view $\mathrm{IndCoh}(\mathcal{Y})$ as a left module for $\mathbb{H}(\mathcal{Y})$ and compute $\mathbb{H}$-pullbacks along smooth maps, as well as $\mathbb{H}$-pushforwards along arbitrary maps.

Lemma 6.6.1. *For a smooth map $\mathcal{X}\to\mathcal{Y}$ in $\mathsf{Stk}^{<\infty}_{\mathrm{lfp}}$, the natural $\mathbb{H}(\mathcal{X})$-linear functor*

$$\mathbb{H}_{\mathcal{X}\to\mathcal{Y}} \underset{\mathbb{H}(\mathcal{Y})}{\otimes} \mathrm{IndCoh}(\mathcal{Y}) \longrightarrow \mathrm{IndCoh}(\mathcal{X})$$

is an equivalence.

Proof. This is just a consequence of the $(\mathrm{QCoh}(\mathcal{X}),\mathbb{H}(\mathcal{Y}))$-bilinear equivalence

$$\mathbb{H}_{\mathcal{X}\to\mathcal{Y}} \simeq \mathrm{QCoh}(\mathcal{X}) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbb{H}(\mathcal{Y}),$$

together with [Gai13, Proposition 4.5.3]. □

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Remark 6.6.2. The example of $\mathcal{Y} = \mathrm{pt}$ shows that we should not expect this result to be true for non-smooth maps.

Proposition 6.6.3. *For a map $f : \mathcal{Y} \to \mathcal{Z}$ in $\mathsf{Stk}^{<\infty}_{\mathrm{lfp}}$, the natural $\mathbb{H}(\mathcal{Z})$-linear functor*

$$\mathbb{H}_{\mathcal{Z}\leftarrow\mathcal{Y}} \underset{\mathbb{H}(\mathcal{Y})}{\otimes} \mathrm{IndCoh}(\mathcal{Y}) \longrightarrow \mathrm{IndCoh}(\mathcal{Z}^{\wedge}_{\mathcal{Y}})$$

is an equivalence.

Proof. Let

$$\mathrm{IndCoh}_{/\mathcal{Y}} := \mathbf{Loc}^{\mathbb{H}}_{\mathcal{Y}}(\mathrm{IndCoh}(\mathcal{Y})) \in \mathsf{ShvCat}^{\mathbb{H}}(\mathcal{Y}).$$

Lemma 6.6.1 gives the equivalence $(\phi_{V\to\mathcal{Y}})^{*,\mathbb{H}}(\mathrm{IndCoh}_{/\mathcal{Y}}) \simeq \mathrm{IndCoh}(V)$ for any affine scheme V mapping smoothly to $\mathcal{Y}$. We then have

$$\begin{aligned}
\mathbf{\Gamma}^{\mathbb{H}}_{\mathcal{Z}} f_{*,\mathbb{H}}(\mathrm{IndCoh}_{/\mathcal{Y}}) &\simeq \lim_{V\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}} \mathbb{H}_{\mathcal{Z}\leftarrow V} \underset{\mathbb{H}(V)}{\otimes} \mathrm{IndCoh}(V) \\
&\simeq \lim_{V\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}} \lim_{U\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Z},\mathrm{smooth}})^{\mathrm{op}}} \mathbb{H}_{U\leftarrow U\times_{\mathcal{Z}}V\to V} \underset{\mathbb{H}(V)}{\otimes} \mathrm{IndCoh}(V) \\
&\simeq \lim_{V\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}} \lim_{U\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Z},\mathrm{smooth}})^{\mathrm{op}}} \mathrm{IndCoh}(U^{\wedge}_{U\times_{\mathcal{Z}}V}) \\
&\simeq \lim_{V\in((\mathsf{Aff}^{<\infty}_{\mathrm{lfp}})_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}} \mathrm{IndCoh}(\mathcal{Z}^{\wedge}_{V}) \\
&\simeq \mathrm{IndCoh}(\mathcal{Z}^{\wedge}_{\mathcal{Y}}).
\end{aligned}$$

Here we have used the self-duality of $\mathrm{IndCoh}(S)$, the rigidity of $\mathbb{H}(S)$, Proposition 4.3.2 (i.e. the special case of the assertion for affine schemes), Lemma 6.6.1 and smooth descent for IndCoh. The conclusion now follows from Corollary 6.5.4. □

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