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Semi-simple classes in a variety satisfying an Andrunakievich Lemma

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It is shown that in a variety of (not necessarily associative) algebras which satisfies a variant of Andrunakievich's Lemma, a class C containing no solvable algebras is the semi-simple class corresponding to some supernilpotent radical class if and only if C is hereditary and is closed under extensions and subdirect products. Semi-simple classes in general are not characterized by these properties. If the variety satisfies the further condition that some proper power of every ideal is an ideal, then analogous results hold for the semi-simple classes corresponding to radical classes containing no solvable algebras. In particular, for algebras over a field in the latter situation, all semi-simple classes are characterized by the three closure properties mentioned.

Introduction

In any universal class of algebras (in which subdirect products can be formed) semi-simple classes are closed under subdirect products and extensions. In a 1965 paper of Anderson, Divinsky, and Suliński [3], it was shown that for associative or alternative algebras, semi-simple classes are also hereditary. Recently, Sands [17] and van Leeuwen, Roos, and Wiegandt [12] have proved the converse. A class of algebras in general which satisfies the three mentioned closure conditions is called a *coradical class* (the term seems to have been introduced by Ryabukhin [16]),

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as this set of properties is essentially dual to a set of properties which has long been known to characterize radical classes [1]. For associative and alternative algebras, then, the concepts "coradical class" and "semi-simple class" coincide.

In the universal class of all (not necessarily associative) algebras, on the other hand, hereditary semi-simple classes are quite uncommon, being determined solely by the additive structure of their members [9]; in the special case of algebras over a field, there are no non-trivial ones at all [6], [9]. On the other hand, there are coradical classes which are not semi-simple classes in this universal class [11].

It is our purpose in this paper to take varieties of algebras satisfying one or both of a pair of conditions (to be described shortly) satisfied by the variety of associative algebras, but not by all varieties of algebras, use these as universal classes, and examine the connections between semi-simple and coradical classes in the resulting radical theory.

The first of these conditions is that the variety satisfy a variant of the Andrunakievich Lemma. This lemma [5] for associative rings, asserts that if I is an ideal of J, J is an ideal of A and I^* is the ideal of A generated by I, then $(I^*/I)^3 = 0$. We shall call a variety V of algebras an *Andrunakievich variety of index* n if $(I^*/I)^{(n)} = 0$ for all $I \triangleleft J \triangleleft A \in V$ and if n is the smallest such integer. Here, for an algebra B, we define $B^{(0)} = B$, $B^{(k+1)} = B^{(k)}B^{(k)}$, and call B *solvable* if $B^{(k)} = 0$ for some k. A variety U is called an *s-variety*, where s is an integer greater than 1, if for every ideal I of every algebra $A \in U$, I^s is also an ideal, where I^s is the linear span of the set of products of s elements of I.

It turns out that in an Andrunakievich variety, a class with no solvable members is semi-simple if and only if it is coradical, though in general coradical classes need not be semi-simple and semi-simple classes need not be coradical. In an s-variety, semi-simple classes need not be coradical, while the status of the converse implication is not known. In an Andrunakievich variety which is also an s-variety, the properties "coradical" and "semi-simple" are equivalent for classes which contain all the solvable algebras in the variety or none. In particular, in such a variety of algebras over a field, coradical classes are always semi-simple, and conversely.

1. Preliminaries

We shall work throughout with algebras over a commutative associative ring with identity, on occasion specializing to rings (Z-algebras) or algebras over a field. The symbol $\triangleleft$ indicates an ideal. A subalgebra B of an algebra A is *accessible* if there exists a finite chain

$$B \triangleleft I_1 \triangleleft \dots \triangleleft I_n \triangleleft A .$$

If $I \triangleleft J \triangleleft A$, we denote by I^* the ideal of A generated by I (clarifying if there is any ambiguity). All classes of algebras considered are assumed to include 0.

The universal classes we shall use will all be Andrunakievich varieties or s-varieties. The possibility of using slightly weaker conditions, by allowing the n and the s in the definitions of these variety properties (see Introduction) to vary with the algebras under consideration, naturally suggests itself. However, the ostensibly weaker conditions are in fact equivalent to the ones we have used. Before proceeding, we establish these equivalences.

PROPOSITION 1.1. *Let V be a variety of algebras such that for every $I \triangleleft J \triangleleft A \in V$ there is a positive integer n such that $(I^*/I)^{(n)} = 0$. Then V is an* Andrunakievich *variety.*

Proof. For $I \triangleleft J \triangleleft A \in V$, let $m(I, J, A) = \min\{n \mid (I^*/I)^{(n)} = 0\}$ and let $F = \{m(I, J, A) \mid I \triangleleft J \triangleleft A \in U\}$. Suppose F is unbounded. Then there is a strictly increasing infinite sequence $m_1, m_2, \dots$ of positive integers, together with sequences

$$I_1, I_2, \dots; J_1, J_2, \dots; A_1, A_2, \dots$$

of algebras, with $I_i \triangleleft J_i \triangleleft A_i$ and $m_i = m(I_i, J_i, A_i)$ for each i. For simplicity, let $\hat{I}_j = \prod_{i \neq j} I_i$, $\hat{J}_j = \prod_{i \neq j} J_i$, $\hat{A}_j = \prod_{i \neq j} A_i$ for each j.

Now for each j, we have

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$$I_j \oplus \hat{I}_j \triangleleft J_j \oplus \hat{J}_j \triangleleft \prod A_i ,$$

so for some integer k we have $[(I_j \oplus \hat{I}_j)^*/(I_j \oplus \hat{I}_j)]^{(k)} = 0$. Then $I_j^* \oplus \hat{I}_j^* \triangleleft \prod A_i$, where $\hat{I}_j^*$ is the ideal of $\hat{A}_j$ generated by $\hat{I}_j$, so that $(I_j \oplus \hat{I}_j)^* \subseteq I_j^* \oplus \hat{I}_j^*$. On the other hand, if $I_j \oplus \hat{I}_j \subseteq M \triangleleft \prod A_i$, then $M \supseteq A_jI_j,\ I_jA_j,\ (A_jI_j)A_j$, and so on, so $I_j^* \subseteq M$. Similarly $\hat{I}_j^* \subseteq M$. Therefore $(I_j \oplus \hat{I}_j)^* = I_j^* \oplus \hat{I}_j^*$, and we have $[I_j^* \oplus \hat{I}_j^*/I_j \oplus \hat{I}_j]^{(k)} = 0$, and thus $(I_j^*/I_j)^{(k)} = 0$. But then $k > m_i$ for each i - an impossibility. Thus F has a largest element, m .

If now $I \triangleleft J \triangleleft A \in V$, we have $(I^*/I)^{(n)} = 0$ for some $n \leq m$, whence $(I^*/I)^{(m)} = 0$ and V is an Andrunakievich variety of index less than or equal to m . //

PROPOSITION 1.2. *Let V be a variety of algebras such that for every $I \triangleleft A \in V$ there is an integer $n > 1$ such that $I^n \triangleleft A$. Then V is an s-variety.*

Proof. For $I \triangleleft A \in V$, let $n(I, A) = \min\{k \mid k > 1, I^k \triangleleft A\}$, and let $E = \{n(I, A) \mid I \triangleleft A \in V\}$. If E is unbounded, there is an infinite strictly increasing sequence $n_1, n_2, \ldots$ of integers greater than 1 , together with sequences $A_1, A_2, \ldots; I_1, I_2, \ldots$ of V-algebras such that $I_i \triangleleft A_i$ and $n_i = n(I_i, A_i)$ for each i . Let $\hat{I}_j = \prod_{i \neq j} I_i$ for each j . Then $I_j \oplus \hat{I}_j \triangleleft \prod A_i$, so there is a positive integer $m > 1$ such that

$$I_j^m \oplus \hat{I}_j^m = (I_j \oplus \hat{I}_j)^m \triangleleft \prod A_i .$$

It follows that $I_j^m \triangleleft A_j$, so that $m \geq n_j$, for each j . This is impossible, so E is bounded.

Let s be the largest element of E . Take any $K \triangleleft B \in V$ and choose $L \triangleleft C \in V$ with $n(L, C) = s$. We have $K \oplus L \triangleleft B \oplus C$ and if r is any integer greater than 1 such that $(K \oplus L)^r \triangleleft B \oplus C$, then

$L^r \triangleleft C$, so $r \geq s$; in particular, taking $r = n(K\oplus L, B\oplus C)$, we get $n(K\oplus L, B\oplus C) = s$. But then $(K \oplus L)^s \triangleleft B \oplus C$, whence $K^s \triangleleft B$. Since B and K are arbitrarily selected, this means that V is an s-variety. //

2. Examples

We present, in this section, some examples of Andrunakievich varieties and s-varieties of algebras, including some varieties with both properties; our principal results involve the latter.

As is well known, the variety of associative algebras is an Andrunakievich variety of index 2 [*5*] and a 2-variety. The alternative algebras form a 2-variety, but it is not known (see [*10*]) whether or not they form an Andrunakievich variety. The (-1, 1)-algebras, those algebras satisfying the identities

$$yx^2 = (yx)x \ , \quad (x, y, z) + (y, z, x) + (z, x, y) = 0 \ ,$$

where $(x, y, z) = (xy)z - x(yz)$, form a 2-variety, while the Jordan algebras form a 3-variety, but not a 2-variety. Further information on s-varieties can be obtained from [*2*], [*4*], [*15*], [*18*], [*19*].

We shall call an algebra 4-*permutable* if it satisfies the identities

$$f_1(x_1, x_2, x_3, x_4) = f_2(x_1, x_2, x_3, x_4) = \ldots = f_{120}(x_1, x_2, x_3, x_4) \ ,$$

where the $f_i(x_1, x_2, x_3, x_4)$ represent the 120 possible products, with order and bracketing varying, of x_1, x_2, x_3, x_4 . All commutative associative algebras are 4-permutable, as are all nilpotent algebras of index less than or equal to 4 . The latter algebras need not even be power-associative, as the following example shows.

EXAMPLE 2.1. Let V be the algebra defined on a free module with basis $\{u, v, w, t\}$ by the multiplication table

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	u	v	w	t
u	w	0	0	0
v	w	t	0	0
w	t	0	0	0
t	0	0	0	0

Let $\langle w, t\rangle$ denote the subalgebra generated by $\{w, t\}$, and so on. Then $V^2 \subseteq \langle w, t\rangle$, so $V^2V^2 \subseteq \langle w, t\rangle^2 = 0$. Also, $VV^2 \subseteq V\langle w, t\rangle = 0$ and $V^2V \subseteq \langle w, t\rangle V \subseteq \langle t\rangle$, so $V^3 \subseteq \langle t\rangle$, and finally, $VV^3 \subseteq V\langle t\rangle = 0$, and $V^3V \subseteq \langle t\rangle V = 0$. Thus $V^4 = 0$ and V is 4-permutable. Now $u^2 = w$, so $uu^2 = uw = 0$, while $u^2u = wu = t$, whence V is not power-associative.

PROPOSITION 2.2. *The 4-permutable algebras form an* Andrunakievich *variety of index* 1 .

Proof. For a pair of subsets S, T of an algebra, we shall find it useful in what follows to denote by $Y(S, T)$ the submodule generated by all products st, ts with $s \in S$ and $t \in T$; that is, $Y(S, T) = ST + TS$.

Let A be 4-permutable, $I \triangleleft J \triangleleft A$. Then

$$I + Y(I, A) + Y\big(Y(I, A), A\big) \subseteq I^* .$$

By 4-permutability, we have

$$Y\big(Y\big(Y(I, A), A\big), A\big) = Y\big(Y(I, A), A^2\big) \subseteq Y\big(Y(I, A), A\big) ,$$

whence $I + Y(I, A) + Y\big(Y(I, A), A\big) \triangleleft A$, so that $I + Y(I, A) + Y\big(Y(I, A), A\big) = I^*$. We now examine I^*I^* .

Firstly, $II^* \subseteq I$, because $I^* \subseteq J$. Next, by 4-permutability,

$$Y(I, A)Y(J, A) \subseteq Y\big(I, Y\big(Y(J, A), A\big)\big) \subseteq Y(I, J) \subseteq I .$$

Hence

$$\begin{aligned} Y(I, A)I^* &= Y(I, A)\big(I+Y(I, A)+Y\big(Y(I, A), A\big)\big) \\ &\subseteq I + Y(I, A)Y(I, A) + Y(I, A)Y(J, A) \\ &\subseteq I + Y(I, A)Y(J, A) \subseteq I . \end{aligned}$$

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Finally, by 4-permutability,

$$Y\big(Y(I, A), A\big)I^* \subseteq Y\big(I, Y\big(Y(I^*, A), A\big)\big) \subseteq Y(I, J) \subseteq I .$$

Thus $(I^*)^{(1)} = (I^*)^2 \subseteq I$. //

PROPOSITION 2.3. *The 4-permutable algebras form a 3-variety.*

Proof. If $I \triangleleft A$, then by 4-permutability,

$$AI^3 + I^3A \subseteq (AI)I^2 + I^2(IA) \subseteq I^3 ,$$

so $I^3 \triangleleft A$. //

The product $U \circ V$ of two varieties U and V is the class

$$\{A \mid \exists I \triangleleft A \text{ with } I \in U \text{ and } A/I \in V\} ,$$

and this is a variety [13]. The next result gives a method of building new Andrunakievich varieties. Let Z denote the variety of zero-algebras, the algebras in which all products are zero.

PROPOSITION 2.4. *Let V be an* Andrunakievich *variety of index n. Then $Z \circ V$ is an* Andrunakievich *variety of index less than or equal to $n + 2$.*

Proof. If $I \triangleleft J \triangleleft A \in Z \circ V$, it may be assumed without loss of generality that $J = I^*$. Now A has an ideal P such that $P^2 = 0$ and $A/P \in V$; furthermore, we have

$$(P+I)/P \triangleleft (P+J)/P \triangleleft A/P ,$$

where $(P+J)/P = (P+I^*)/P = [(P+I)/P]^*$, so that $[(P+I^*)/(P+I)]^{(n)} = 0$; that is, $(P+I^*)^{(n)} \subseteq P + I$. Then

$$(P+I^*)^{(n+1)} = \big[(P+I^*)^{(n)}\big]^2 \subseteq (P+I)^2 \subseteq PI + IP + I^2 ,$$

so

$$(P+I^*)^{(n+2)} = \big[(P+I^*)^{(n+1)}\big]^2 \subseteq \big[(PI+IP)+I^2\big]\big[(PI+IP)+I^2\big] \subseteq I ,$$

since $PI + IP \subseteq P \cap I^*$ and $P^2 = 0$. Finally,

$$(I^*)^{(n+2)} \subseteq (P+I^*)^{(n+2)} \subseteq I . \qquad //$$

COROLLARY 2.5. *Let* A *be the class of associative algebras. Then*

$Z \circ A$ *is an* Andrunakievich *variety*. //

The algebras over the field of two elements in the variety $\mathcal{D}$ defined by the identities

$$x(yz) = (xy)(xz) , \quad (xy)z = (xz)(yz) ,$$

are called *autodistributive* (see [7]). It was shown in [8] (Corollary 2.5) that every algebra $A \in \mathcal{D}$ has a unique decomposition $A = N \oplus B$, where $N^3 = 0$ and $b^2 = b$ for each $b \in B$. Let $\mathcal{D}_1$ denote the variety

$\{A \in \mathcal{D} \mid a^2 = a$ for every $a \in A\}$.

PROPOSITION 2.6. $\mathcal{D}_1$ *is an* Andrunakievich *variety of index* 0.

Proof. If $I \triangleleft J \triangleleft A \in \mathcal{D}_1$, then for $a \in A$, $i \in I$, we have $ai = ai^2 = (ai)(ai) = [(ai)a][(ai)i] \in JI \subseteq I$, so $AI \subseteq I$; similarly $IA \subseteq I$, so $I = J = I^*$. //

COROLLARY 2.7. $\mathcal{D}$ *is an* Andrunakievich *variety of index less than or equal to* 4.

Proof. By Proposition 2.4, $Z \circ (Z \circ \mathcal{D}_1)$ is an Andrunakievich variety of index less than or equal to $0 + 2 + 2 = 4$. If $A \in \mathcal{D}$, then, as noted, $A = N \oplus B$, where $N^3 = 0$ and $B \in \mathcal{D}_1$. Since $N^3 = 0$, we have $(N^2)^2 = 0$; also $N^2 \triangleleft A$. Now $A/N^2 \cong (N/N^2) \oplus B \in Z \circ \mathcal{D}_1$, so $A \in Z \circ (Z \circ \mathcal{D}_1)$. As a subvariety of $Z \circ (Z \circ \mathcal{D}_1)$, $\mathcal{D}$ is an Andrunakievich variety of index less than or equal to 4. //

PROPOSITION 2.8. $\mathcal{D}$ *is a 2-variety*.

Proof. If $I \triangleleft A \in \mathcal{D}$, then $I = I_1 \oplus I_2$, $A = A_1 \oplus A_2$, where $I_1^3 = 0$, $A_1^3 = 0$, and $I_2, A_2 \in \mathcal{D}_1$. In particular, I_1 and A_1 are associative, so $I_1^2 \triangleleft A_1$. Also $I_2^2 = I_2 \triangleleft A_2$, so $I^2 = I_1^2 \oplus I_2^2 \triangleleft A$. //

3. Examples

We begin by looking at semi-simple classes in an Andrunakievich variety.

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THEOREM 3.1. *Let* V *be an* Andrunakievich *variety of algebras,* C *a coradical class in* V *containing no solvable algebras. Then* C *is a semi-simple class in* V .

Proof. Consider the upper radical class R defined by C . Take any $A \in V$ and let $K = \cap\{I \triangleleft A \mid A/I \in C\}$. Since C is closed under subdirect products, we have $A/K \in C$, and thus $R(A/K) = 0$. Suppose $K \notin R$. Then K has an ideal T such that $0 \neq K/T \in C$. Now $T \triangleleft K \triangleleft A$, so for some integer m , we have $(T^*/T)^{(m)} = 0$. But $T^*/T \triangleleft K/T \in C$, so T^*/T is in C , as well as being solvable. Thus $T^*/T = 0$; that is, $T \triangleleft A$. Hence there is an exact sequence

$$0 \to K/T \to A/T \to A/K \to 0$$

whose end terms are in C . But then $A/T \in C$, so $K \subseteq T$, contradicting our assumption. We conclude that $K \in R$. Since $R(A/K) = 0$, it follows that $K = R(A)$. In particular, $R(A) = 0$ if and only if $K = 0$; that is, $A \in C$, so C is a semi-simple class. //

THEOREM 3.2. *Let* R *be a radical class, containing all solvable algebras, in an* Andrunakievich *variety* V . *Then* R *has a hereditary semi-simple class.*

Proof. If $R(A) = 0$ and $I \triangleleft A$, then $R(I) \triangleleft I \triangleleft A$, so $R(I)^*/R(I)$ is solvable, and thus belongs to R . Since $R(I) \in R$, $R(I)^*$ is an R-ideal of A , so $0 = R(I)^* = R(I)$. //

These two theorems have as a joint consequence:

COROLLARY 3.3. *Let* C *be a subclass, containing no solvable algebras, of an* Andrunakievich *variety* V . *Then* C *is a semi-simple class if and only if it is a coradical class.* //

These results cannot be extended to cover arbitrary coradical and semi-simple classes, however. As before, in the next theorem Z is the variety of zero-algebras. A the variety of associative algebras.

THEOREM 3.4. *Over any field, the (*Andrunakievich*) variety* $Z \circ A$ *contains*

(i) a non-semi-simple coradical class containing all solvable algebras, and

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(ii) a hereditary radical class with a non-hereditary semi-simple class containing all solvable algebras.

Proof. Let A be the algebra with basis $\{u, v, w\}$ and multiplication table

	u	v	w
u	0	0	u
v	0	v	u
w	u	u	0

Let $\langle u, v\rangle$ denote the subalgebra generated by $\{u, v\}$, and so on. Calculation shows that the ideals of A are

$$0,\ \langle u\rangle,\ \langle u, v\rangle,\ \langle u, w\rangle,\ A\ .$$

Let $C = \{R \mid R$ has no non-zero accessible idempotent subalgebras$\}$. Then C is a coradical class containing all solvable algebras. Examining the non-zero ideals of A, we see that $\langle u\rangle \in C$,

$$\langle u, v\rangle/\langle u, v\rangle^2 = \langle u, v\rangle/\langle v\rangle \cong \langle u\rangle \in C\ ,$$

$$\langle u, w\rangle/\langle u, w\rangle^2 = \langle u, w\rangle/\langle u\rangle \cong \langle w\rangle \in C\ ,$$

and

$$A/\langle u, v\rangle \cong \langle w\rangle \in C\ ,$$

so every non-zero ideal of A has a non-zero homomorphic image in C. But $\langle v\rangle \triangleleft \langle u, v\rangle \triangleleft A$ and $\langle v\rangle^2 = \langle v\rangle$, so $A \notin C$. Consequently, C is not a semi-simple class.

Now let H be the class of hereditarily idempotent algebras, the algebras with every accessible subalgebra idempotent. Then H is a hereditary radical class. Returning to our example A, we see that $\langle u\rangle^2 = 0$ and $\langle u\rangle$ is an ideal of every non-zero ideal of A, so that $H(A) = 0$. But $\langle v\rangle$ is isomorphic to the field of scalars, and $\langle v\rangle \triangleleft \langle u, v\rangle$, so $H(\langle u, v\rangle) \neq 0$ and H does not have a hereditary semi-simple class. //

If we impose a further restriction, that our universal class be an *s*-variety, as well as an Andrunakievich variety, things improve somewhat.

We precede our first theorem for such a variety with a result which we shall use subsequently, and which is of some independent interest.

PROPOSITION 3.5. *Let* V *be an* s*-variety and an* Andrunakievich *variety of index* n . *If* $I \triangleleft J \triangleleft A \in V$ *and* $I^2 = I$, *then* $I \triangleleft A$.

Proof. We have $I \triangleleft I^* \triangleleft A$, so $I = I^s \subseteq (I^*)^s \triangleleft A$. But then $I^* \subseteq (I^*)^s$, so I^* is idempotent. Hence $(I^*)^{(n)} = I^*$, so $I^*/I = (I^*/I)^{(n)} = 0$; that is, $I \triangleleft A$. //

THEOREM 3.6. *Let* V *be an* s*-variety and an* Andrunakievich *variety. If* C *is a coradical class, containing all solvable algebras, in* V , *then* C *is a semi-simple class.*

Proof. Let R be the upper radical class defined by C . Take an algebra A and let $K = \cap\{I \triangleleft A \mid A/I \in C\}$. Then $A/K \in C$, so $R(A/K) = 0$. Also, $K^s \triangleleft A$ and $K/K^s \in C$, so from the exact sequence

$$0 \to K/K^s \to A/K^s \to A/K \to 0 ,$$

we see that $A/K^s \in C$, whence $K \subseteq K^s$ and K is idempotent. Now let $K_1 = \cap\{I \triangleleft K \mid K/I \in C\}$. Then $K_1 \triangleleft K \triangleleft A$. As above, K_1 is idempotent, so by Proposition 3.5, $K_1 \triangleleft A$. But from the exact sequence

$$0 \to K/K_1 \to A/K_1 \to A/K \to 0 ,$$

whose end terms are in C , we see that $A/K_1 \in C$. This means that $K = K_1$, so K has no non-zero homomorphic images in C ; that is, $K \in R$. Since $R(A/K) = 0$, it follows that $K = R(A)$. In particular, $R(A) = 0$ if and only if $K = 0$; that is, $A \in C$. //

THEOREM 3.7. *Let* V *be an* s*-variety and an* Andrunakievich *variety. Let* R *be a radical class, containing only idempotent algebras, in* V . *Then* R *has a hereditary semi-simple class.*

Proof. (Note that the hypothesis implies that all solvable algebras are R-semi-simple.) If $R(A) = 0$ and $I \triangleleft A$, we have $R(I) \triangleleft I \triangleleft A$, with $R(I)$ idempotent. By Proposition 3.5, $R(I) \triangleleft A$, so $R(I) = 0$. //

The following result is a consequence of Theorems 3.6 and 3.7.

COROLLARY 3.8. *Let* V *be an* s*-variety and an* Andrunakievich *variety. A subclass* C *of* V*, containing all solvable algebras, is a semi-simple class if and only if it is a coradical class.*

Proof. We need only observe that if C is a semi-simple class containing all solvable algebras, then the corresponding radical class consists of idempotent algebras. //

For algebras over a field, Corollaries 3.3 and 3.8 can be combined as follows.

THEOREM 3.9. *Let* V *be an* s*-variety and an* Andrunakievich *variety of algebras over a field. A subclass* C *of* V *is a semi-simple class in* V *if and only if it is a coradical class in* V *.*

Proof. Let K be the field, K^0 the one-dimensional K-zero-algebra. If a radical class R in V contains K^0, it contains the zeroalgebras and hence all solvable algebras. If $K^0 \notin R$, then $R(K^0) = 0$. If now $A \in R$, then $A^2 = A$, since otherwise K^0, as a non-zero homomorphic image of A, would be in R. This means that all solvable algebras are R-semi-simple. It follows that a semi-simple class in V must contain all the solvable algebras or none of them, according as it contains or does not contain K^0.

Furthermore, if a coradical class C contains a solvable algebra not equal to 0, it contains each $A^{(n)}$ for each of its members A, being hereditary, and hence it contains K^0. Closure under direct sums then ensures that C contains all zero-algebras and closure under extensions guarantees that all solvable algebras are in C.

The result now follows from Corollaries 3.3 and 3.8. //

We have seen that in an Andrunakievich variety which is not an s-variety, semi-simple classes need not be coradical, and conversely. The corresponding questions naturally arise for s-varieties without the Andrunakievich property. The most we know here is that sometimes semi-simple classes are not hereditary.

THEOREM 3.10 (Miheev [14]). *In the* 2*-varieties of* $(-1, 1)$*-rings and of* $(-1, 1)$*-algebras over a field of characteristic* 2*, the nil* *radical class has a non-hereditary semi-simple class.*

Proof. Miheev [14] has given an example of a $(-1, 1)$-algebra over an arbitrary field of characteristic 2 which is nil semi-simple but has a non-zero accessible subalgebra S with $S^2 = 0$. This algebra for the prime field provides the proof for rings. //

This investigation suggests the following problems.

(1) Do the concepts "semi-simple class" and "coradical class" coincide in a variety of rings which is both an s-variety and an Andrunakievich variety?

(2) Does "coradical" imply "semi-simple" in all s-varieties?

(3) Do all radical classes without solvable members (or, equivalently, all radical classes with only idempotent members) in an s-variety have hereditary semi-simple classes?

In connection with (3), we note that in [9] it was shown that the (radical) class of all idempotent rings always has a hereditary semi-simple class in an s-variety.

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