Note on two intrinsically related plane curves.

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The tangent at a point P to a given plane curve intersects another given curve in Q and makes with the tangent at Q to the latter curve a variable angle ψ. It is required to connect the curvatures at P and Q with the length PQ and the angle ψ.

Figure 1.

Employing the usual notation and using capital letters for corresponding elements of the outer curve,

$$\mathrm{PT} = \mathrm{TP}' = \tfrac{1}{2}\delta s\,;$$

$$\mathrm{Q'N} = \tfrac{1}{2}\delta s + (\mathrm{PQ} + \delta\mathrm{PQ}) - (\mathrm{PQ} - \tfrac{1}{2}\delta s)$$

$$= \delta s + \delta\mathrm{PQ}\,;$$

$$\cot\psi = (\delta s + \delta\mathrm{PQ})/\mathrm{PQ}\,.\,\delta\phi\,;$$

$$\therefore\quad \mathrm{PQ}\cot\psi = \frac{d}{d\phi}(s + \mathrm{PQ}). \qquad \text{(A)}$$

Again $$\delta\mathrm{S} = \mathrm{QQ}' = (\mathrm{PQ}\,.\,\delta\phi)\operatorname{cosec}\psi\,;$$

$$\therefore\quad \mathrm{PQ}\operatorname{cosec}\psi = \frac{d\mathrm{S}}{d\phi}. \qquad \text{(B)}$$

Lastly $$\Phi = \phi + \psi. \qquad \text{(C)}$$

These formulæ are sufficient, and will frequently be found useful.

Illustrations.

(1) Let $\psi = \frac{\pi}{2}$, so that QP is always normal at Q to the locus of Q.

(A) gives (when $\cot\psi = 0$) $\frac{d}{d\phi}(s + \mathrm{PQ}) = 0$, $\quad\therefore\ s + \mathrm{PQ} = \text{constant}$.

(B) and (C) give $\mathrm{PQ} = \dfrac{d\mathrm{S}}{d\Phi}$.

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(2) If PQ is constant, $\cot\psi = \rho/\text{PQ}$, and the normal at Q passes through the centre of curvature at P.

Thus, if on the tangent at each point of a curve a constant length measured from the point of contact be taken, then the normal to the locus of the points so found passes through the centre of curvature of the proposed curve. (Bertrand.) *

(3) If PQ and ψ are both constant,

$$\text{R} = \frac{d\text{S}}{d\Phi} = \frac{d\text{S}}{d\phi} = \text{PQ}\operatorname{cosec}\psi\,;$$

$$\therefore\quad \operatorname{cosec}\psi = \text{R}/\text{PQ}\,,$$

and the normal at P passes through the centre of curvature at Q.

Thus, if through each point of a curve a line of given length be drawn making a constant angle with the normal, the normal to the curve locus of the extremities of this line passes through the centre of curvature of the proposed curve. †

(4) *Curve of Pursuit.* If Q describes a straight line with constant velocity v and the velocity of P in the direction of PQ is constant and equal to v/e, then P describes a curve of pursuit.

$$\frac{d\text{S}}{ds} = \frac{d\text{S}}{dt}\Big/\frac{ds}{dt} = e\,;\quad \frac{d\text{S}}{d\phi} = e\rho \text{ and } \frac{d\text{S}}{d\phi} = \text{PQ}\operatorname{cosec}\psi\,;$$

$$\therefore\quad \rho = \frac{\text{PQ}}{e\sin\psi} = \frac{\text{PQ}^2}{e(\text{perp. from P on locus of Q})}\,. \ddagger$$

(5) The tangent at a point P on a given curve cuts a given straight line AB in Q. Prove that when PQ is a maximum or a minimum the line through Q perpendicular to AB passes through the centre of curvature at P.

Here Φ is constant, and $d(\text{PQ}) = 0$. Thus $\text{PQ}\cot\psi = \rho$, as in (2).

* *Williamson's Diff. Calc.*, Chapter XVII., p. 296, No. 27.

† *Bertrand, Diff. Calc.*, quoted by Williamson as above.

‡ *Tait and Steele*, Chapter I., Art. 30 and Ex. 20.

(6) Let P, P′ be two points on an ellipse and Q a point on a confocal ellipse (Fig. 2).

$$\frac{d}{d\phi}(s+\text{PQ})+\frac{d}{d\phi}(s'+\text{P}'\text{Q})=\frac{d}{d\phi}(s+\text{PQ})+\frac{d}{d\phi'}(s'+\text{P}'\text{Q})\cdot\frac{d\phi}{d\phi}$$

$$=\text{PQ}\cot\psi+\text{P}'\text{Q}\cot\psi\frac{d\phi}{d\phi}$$

$$=0\,;$$

$$\text{for}\quad \frac{dS}{d\phi}=\text{PQ}\operatorname{cosec}\psi \text{ and } -\frac{dS}{d\phi'}=\text{P}'\text{Q}\operatorname{cosec}\psi,$$

$$\therefore\quad \frac{d\phi'}{d\phi}=-\frac{\text{PQ}}{\text{P}'\text{Q}}.$$

Integrating, $s+\text{PQ}+s'+\text{P}'\text{Q}=\text{constant}.$

This is Dr Graves' Theorem.

(7) Through each point of a given curve lines are drawn making a constant angle with the normal at that point; show that the normal to their envelope passes through the centre of curvature of the corresponding point on the given curve, and that the projection of the centre of curvature at the evolute of the given curve on this normal is the centre of the curvature of the envelope. *

Here ψ is constant, and $\text{R}=\frac{dS}{d\Phi}=\frac{dS}{d\phi}=\text{PQ}\operatorname{cosec}\psi$, as in (3).

$$\frac{d\text{R}}{d\Phi}=\frac{d\text{R}}{d\phi}=\operatorname{cosec}\psi\frac{d\text{PQ}}{d\phi}=\operatorname{cosec}\psi(\text{PQ}\cot\psi-\rho)$$

$$\therefore\quad \text{R}\cos\psi-\frac{d\text{R}}{d\Phi}\sin\psi=\rho.\quad \text{(Fig 3.)}$$

(8) A traveller in a railway carriage moving on a curve in a flat country is seated with his face towards the engine and looks out on the inside of the curve. Show that the country beyond a certain point disappears, and that the country nearer than this point comes into view, also, that the point moves with a velocity $\frac{d\rho}{dt}\sin\alpha+v\cos\alpha$,

* *Tripos*, 1881.

where ρ is the radius of curvature of the curve, v the velocity of the train and α the angle which the plane from the traveller's eye to the vertical edge nearest the engine of the window which bounds his view, makes with the direction of the train's motion.*

Here ψ is constant, (equal to α).

$$R = PQ\operatorname{cosec}\psi, \text{ as in (3)};$$

$$\text{or} \quad \frac{dS}{d\phi} = \sec\psi\left\{\frac{ds}{d\phi} + \frac{d(PQ)}{d\phi}\right\}$$

$$= \sec\psi\left\{\frac{ds}{d\phi} + \frac{d(PQ\operatorname{cosec}\psi)}{d\phi}\sin\psi\right\};$$

or, multiplying by $\frac{d\phi}{dt}$

$$V = \sec\psi\left\{v + \frac{dR}{dt}\sin\psi\right\}.$$

This is for motion *opposite* to that considered in the question. Changing the signs of V and v,

$$-V = \sec\psi\left(-v + \frac{dR}{dt}\sin\psi\right)$$

$$\therefore \quad v = \frac{dR}{dt}\sin\psi + V\cos\psi.$$

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* *Tripos*, 1873 (?) (Dr Pirie).