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A Summary of the Theory of the Refraction of Thin Approximately Axial Pencils through a series of Media bounded by Coaxial Spherical Surfaces, with application to a Photographic Triplet, etc.

By Professor CHRYSTAL.

The optical theory referred to in the title of this communication is now fully half a century old*; and has, moreover, been well expounded in the standard English treatises of Pendlebury and Heath. Still, notwithstanding its elegance and simplicity, and its great practical importance as giving the first approximation to the theory of the great majority of the optical instruments in ordinary use, its filtration into the strata of popular knowledge has been remarkably slow. It seems, therefore, to be worth while to offer a brief summary of its leading principles, freed as much as possible from the detailed calculations which become necessary when the constants of the optical system have to be deduced from the data of construction, and to indicate methods for experimental verification. In giving this summary, I shall omit the demonstrations of some of the propositions, which can be found by those who desire them in the well known *Treatise on Geometrical Optics*, by Heath.

The whole theory may be made to depend on two elementary propositions regarding the refraction of a thin pencil at a single spherical surface, viz., the *Law of Conjugate Focal Planes* and *Helmholtz's Law of Magnification.* These laws may be stated as follows:—

Let P (Fig. 1) be any point; C the centre of the refracting surface considered, then

LAW OF CONJUGATE FOCAL PLANES.

Every pencil, all of whose rays diverge from (or converge to) a point lying in a small area through P perpendicular to the axis PC, will after refraction diverge from (or converge to) a point lying in a small area also perpendicular to the axis PC, provided we consider only rays whose inclinations to PC are small. The point P′, where the second plane meets the axis, is called the conjugate focus to P, and is in direct projective correspondence with P; that is to say, if x and

* As to its first beginnings much older: these date back to Harris's *Treatise of Optics.* London: 1775.

x' denote the distances of P and P′, both measured from a point O in the axis in the same direction (which we take to be that in which the light is proceeding, always in our diagrams from left to right), then

$$Axx' + Bx + Cx' + D = 0 \quad - \quad - \quad - \quad (1),$$

*where A, B, C, D are constants depending on the radius of the refracting surface and on the indices of refraction of the media of which it is the boundary.**

The fact that the correspondence is direct, *i.e.*, that if P moves then P′ always moves in the same direction, imposes a certain restriction upon A, B, C, D, viz., that BC − AD must always be negative, as we shall see presently.

The Law of Conjugate Focal Planes is accurate to a first approximation only, viz., we must suppose that the square of the distance of the point of incidence on the spherical surface from the axis is negligible in comparison with x, x', or the radius of the surface. To this degree of approximation the points corresponding to the points in any plane area (the object) perpendicular to the axis at P generate another similar plane area (the image) perpendicular to the axis at P′.

The residual phenomena which arise when we proceed to higher orders of approximation are included under the various heads of Spherical Aberration, Astigmatism, Distortion, and Curvature of the Image, with which we have nothing to do at present.

Since the object and image are similar, all that is necessary in order to determine the one when the other is given is to know the ratio of similarity, *i.e.*, the ratio of the distance between any two points in the one to the distance between the corresponding points in the other. Regarding this ratio, we have the following

Law of Helmholtz.

If β and β' be the linear dimensions of an object at P and its image at P′, P and P′ being axial points, and α and α' the inclinations to the axis of an incident ray through P and the corresponding emergent ray through P′, then

$$\mu\beta\tan\alpha = \mu'\beta'\tan\alpha' \quad - \quad - \quad - \quad - \quad (2);$$

* See Heath, § 41-46. The theory is here stated throughout for refraction only; the case of reflection may be included by putting $\mu = -1$ for every refracting surface.

*where μ and μ' are the refractive indices of the media in the order of the passage of the rays.**

These laws can at once be extended to any number of coaxial spherical surfaces. For, if P_1' be the conjugate of P' in the second surface, P_2' the conjugate of P_1' in the third surface, and so on, then, since each of these points is in direct projective correspondence with the one immediately preceding, the last of all, which is the image point corresponding to P after refraction by the whole system, will be in direct projective correspondence with P.

(Analytically, this involves merely the repeated application of the easily verified proposition that, if

$$\mathrm{A}xx' + \mathrm{B}x + \mathrm{C}x' + \mathrm{D} = 0,$$
$$\mathrm{A}'x'x'' + \mathrm{B}'x' + \mathrm{C}'x'' + \mathrm{D}' = 0,$$

where A, B, C, D are constants, and $\mathrm{BC} - \mathrm{AD}$, $\mathrm{B'C'} - \mathrm{A'D'}$ both negative, then

$$\mathrm{A}''xx'' + \mathrm{B}''x + \mathrm{C}''x'' + \mathrm{D}'' = 0,$$

where A″, B″, C″, D″ are constants depending on A, B, C, D · A′, B′, C′, D′ and such that $\mathrm{B''C''} - \mathrm{A''D''}$ is negative.)

It follows therefore that, *if P and P′ be conjugate foci with respect to any system of coaxial spherical surfaces, and x and x' denote the distances of P and P′ from any fixed point O, then*

$$\mathrm{A}xx' + \mathrm{B}x + \mathrm{C}x' + \mathrm{D} = 0 \qquad - \quad - \quad - \quad (3),$$

where A, B, C, D are constants depending on the radii and mutual distances of the refracting surfaces and on the refractive indices of the media which separate them.

Since the successive pairs of images are similar to each other, the final image will be similar to the original object at P. Also, *if α' be the inclination to the axis of the system of an emergent ray through the final image point P′ corresponding to an incident ray through P whose inclination to the axis is α, and β and β' the linear dimensions of the original object and of the final image at P′, then,* by successive applications of the Law of Helmholtz, *we have*

$$\mu\beta\tan\alpha = \mu'\beta'\tan\alpha' \qquad - \quad - \quad - \quad - \quad (4),$$

where μ' is the index of refraction of the final medium.

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* See Heath, § 50.

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Principal Foci, Moment, Double Points, and Stationary Points of a System.

If A do not vanish, a condition which we shall indicate when necessary by calling the system *Non-telescopic*, then the equation (3) can always be thrown into the form

$$(x-g)(x'-g') = -\gamma^2 \quad - \quad - \quad - \quad - \quad (5),$$

where $g = -C/A$, $g' = -B/A$, and $\gamma^2 = -(BC-AD)/A^2$, so that g, g', γ are all real finite constants, and γ may be taken to be either positive or negative. The constant γ^2 we shall call the *Moment of the System.*

We see at once from (5) that, when $x=\infty$, $x'=g'$, and, when $x=g$, $x'=\infty$. The two points F′ and F thus determined we call the Principal Foci of the System. The planes through F and F′ perpendicular to the axis we call the *Principal Focal Planes.* The optical property of these points is that any nearly axial pencil of parallel incident rays finally converges to or diverges from a point in a plane perpendicular to the axis through F′, and that any incident pencil converging to or diverging from a point in a plane perpendicular to the axis through F finally emerges as a parallel pencil.

When it is necessary to distinguish between these points, we may speak of F′ as the Principal Focus for incident rays, and F as the Principal Focus for emergent rays.

If we denote (see Fig. 2) the distances of the two conjugate foci P and P′ from F and F′ respectively by u and u', the former distance being reckoned positive when measured in a direction opposite to the passage of the light through the system, and the latter positive when measured in that direction, then we have the important relation

$$uu' = \gamma^2 \quad - \quad - \quad - \quad - \quad - \quad (6).$$

If p and p' denote the distances from F and F′ of any pair of conjugate foci P and P′ for incident and emergent rays respectively, and v and v' the distances of any other pair of conjugate foci Q and Q′ for incident and emergent rays from P and P′ respectively, the conventions as to sign for p and p' and for v and v' being the same as for u and u', then from (6) we deduce at once

$$p/v + p'/v' = 1 \quad - \quad - \quad - \quad - \quad (6^*).$$

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Returning to the equation (5), and taking the principal focus for emergent rays, F, for origin, if we denote the distance FF′ by ∂, ∂ being positive or negative according as F′ is right or left of F (the passage of the light being supposed from left to right as usual), then (5) may be written

$$x(x' - \partial) = -\gamma^2 \quad - \quad - \quad - \quad - \quad (7).$$

Let now y denote the distance between any incident focus P and its conjugate P′, then we have from (7)

$$y = -(x^2 - \partial x + \gamma^2)/x \quad - \quad - \quad - \quad - \quad (8).$$

The points for which y vanishes, *i.e.*, the points whose conjugates are coincident with themselves, are given by

$$x^2 - \partial x + \gamma^2 = 0 \quad - \quad - \quad - \quad - \quad - \quad (9)$$

These points we may call the *Double Points of the Optical System:* the image of any object (actual or virtual) placed at a double point has the same position as the object, although it is not in general of the same size.

The double points are real, coincident, or imaginary according as $\partial^2 > = < 4\gamma^2$. *We name the optical system Hyperbolic, Parabolic, or Elliptic accordingly.*

When the double points are real they lie right or left of F according as ∂ *is positive or negative.*

Farther, since the sum of the distances of the double points from F is ∂, we see that *the double points are symmetrically situated with respect to F and F′.*

Since $dy/dx = -1 + \gamma^2/x^2$, we see that the *stationary values* of y correspond to $x = \pm\gamma$, the corresponding values of x' being $\partial \mp \gamma$.

Elliptic Systems.

If we trace the graphs of (8) for the Hyperbolic and Elliptic systems, we see at once that in the case of Elliptic systems both the stationary values of y are scalar minima, while in Hyperbolic systems the one is a scalar minimum the other a scalar maximum, the incident point corresponding to the maximum being on the same side of F as F′. We may call the conjugate pairs whose existence we have just established the *stationary points of the system*, and the corresponding distances the *stationary distances.*

In every system there are two real pairs of stationary points. Each pair is symmetrically situated with respect to F and F′, the incident points of the two pairs lie at a distance γ right and left of F respectively. In Elliptic or Parabolic systems the stationary distances are both minima. In Hyperbolic systems one of the stationary distances is a maximum the other a minimum; the incident point corresponding to the maximum always lies on the same side of F as F′.

Magnification, Principal Points, Principal Focal Lengths, and Nodal Points.

In order to deduce a rule for calculating the dimensions of the image of any given object we must combine the Law of Conjugate Focal Planes with the Law of Helmholtz.

Let Pp (Fig. 3) be a linear object of length β in the plane of the paper, P′p' its image of length β' also in the plane of the paper.

It is obvious that PF and F′P′ are incident and emergent parts of an axial ray. Let pF and Q′p' be incident and emergent parts of another ray, Q′ being a point in the Principal Focal Plane for incident rays. Since pF and PF converge to a point in the principal focal plane for emergent rays, Q′p' is parallel to F′P′. If we now join Q′P′, and take this for the emergent part of a third ray, then, since Q′ is in the principal focal plane for incident rays, the incident part of this third ray will be PQ parallel to pF.

Now, by the Law of Helmholtz,

$$\mu\beta\tan\mathrm{QPF} = \mu'\beta'\tan\mathrm{Q'P'F'},$$

that is,

$$\mu\beta\tan\mathrm{PF}p = \mu'\beta'\tan\mathrm{P'Q'}p'.$$

Hence, u and u' having the same meanings as in (6),

$$\mu\beta^2/u = \mu'\beta'^2/u'.$$

Hence

$$\frac{\beta'^2}{\beta^2} = \frac{\mu}{\mu'}\cdot\frac{u'}{u},$$

$$= \frac{\mu}{\mu'}\,\frac{u'^2}{\gamma^2},$$

$$= \frac{\mu}{\mu'}\,\frac{\gamma^2}{u^2},$$

by (6). Hence we have

$$\frac{\beta'}{\beta} = \pm\sqrt{\left(\frac{\mu}{\mu'}\cdot\frac{u'}{u}\right)} = \mp\sqrt{\left(\frac{\mu}{\mu'}\right)}\frac{u'}{\gamma} = \mp\sqrt{\left(\frac{\mu}{\mu'}\right)}\cdot\frac{\gamma}{u} \qquad (10),$$

the upper or lower signs to be taken according as the image is erect or inverted or *vice versa*. By means of this relation the magnification of an object placed at any given point may be calculated.

There are two pairs of conjugate foci for which the magnification is unity corresponding to

$$u = \sqrt{(\mu/\mu')}.\gamma, \qquad u' = \sqrt{(\mu'/\mu)}.\gamma \quad - \quad - \quad (11);$$

$$u = -\sqrt{(\mu/\mu')}.\gamma, \qquad u' = -\sqrt{(\mu'/\mu)}.\gamma \quad - \quad - \quad (12).$$

For one of these pairs the image is equal to the object and erect; for the other equal and inverted. Which is which depends on the absolute sign of γ, a quantity whose square alone has hitherto been defined; so that we cannot distinguish between the two pairs without farther examination of the special system. We know, however, that there are always two real pairs of the kind described. The pair for which the image is erect are called the *principal points* of the system (H, H′); the other pair we may call the *anti-principal points* (K, K′).

The distances of the principal points of incidence and emergence from the principal foci of incidence and emergence respectively are called the principal focal lengths of the system. If we denote these by f and f', we have

$$f = \sqrt{(\mu/\mu')}.\gamma, \qquad f' = \sqrt{(\mu'/\mu)}\gamma \quad - \quad - \quad (13);$$

and we observe that

$$ff' = \gamma^2 \quad - \quad - \quad - \quad - \quad (14);$$

so that (6) and (10) may now be written

$$uu' = ff' \quad - \quad - \quad - \quad - \quad (15);$$

$$\frac{\beta'}{\beta} = \pm\frac{f}{f'}\sqrt{\left(\frac{u'}{u}\right)} = \mp\frac{u'}{f'} = \mp\frac{f}{u} \quad - \quad - \quad (16).$$

Farther, if we take the two principal points H and H as points of reference, (6*) becomes

$$f/v + f'/v' = 1 \quad - \quad - \quad - \quad - \quad (17).$$

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It will be observed that in the general case which we are now considering the principal points coincide neither with the stationary points nor with the double points. The magnification for the former is

$$\pm \sqrt{(\mu/\mu')};$$

for the latter

$$\pm \sqrt{(\mu/\mu')}\{\partial/2\gamma \pm \sqrt{((\partial/2\gamma)^2 - 1)}\}.$$

The line joining the points where any incident ray and its corresponding emergent ray meet the principal planes of incidence and emergence respectively is parallel to the axis.

A similar proposition holds for the anti-principal planes, with the variation that the joining line passes through the point bisecting KK'.

To prove the first of these propositions we have only to remark that, since the principal planes are conjugate, the points Q and Q′ (Fig. 4) in which any ray meets them are conjugate foci, and so also are the principal points H, H′ themselves. Hence H′Q′ is the image of HQ; and therefore by the fundamental property of the principal points H′Q′ = HQ, both being like directed. Hence QQ′ is parallel to HH′. A similar proof establishes the second proposition.

By means of the principal planes we can readily construct in a variety of ways the emergent ray corresponding to any given incident ray.

Let the incident ray meet the principal focal plane of emergence in R (Fig. 5), and the principal plane of incidence in S. Draw SS′ and RT′ parallel to the axis meeting the principal plane of emergence in S′ and T′; then S′P′ parallel to T′F′ is the emergent ray corresponding to PS, as is at once obvious if we notice that RS, RT form a pencil diverging from a point in the principal focal plane of emergence. P′ is of course the conjugate of P.

We can also construct the conjugate of any point P not lying on the axis as follows:—

Let PR (Fig. 6) parallel to the axis meet the principal plane of emergence in R′, and PF meet the principal plane of incidence in S. Then the parallel to the axis through S meets R′F′ in P′ the conjugate of P.

Similar constructions can be effected by means of the anti-principal planes, if we replace the parallels to the axis by lines drawn through the middle point of FF′.

By means of Helmholtz's Law and the relations of (13) and (16) we find

$$\frac{\tan\alpha'}{\tan\alpha} = \mp\frac{u}{f'} \quad - \quad - \quad - \quad - \quad (18),$$

where α and α' are the inclinations to the axis of any incident ray passing through P and the corresponding emergent ray passing through the conjugate focus P′.

We see from (18) that *in every optical system there are two real pairs of axial conjugates, given by*

$$\begin{aligned} u &= -f', \; u' = -f, \\ u &= f', \; u' = f \end{aligned} \quad - \quad - \quad - \quad - \quad (19).$$

which have the property that the corresponding incident and emergent rays passing through them make equal angles with the axis. For one of these pairs, which are called the *nodal points*, the incident and emergent rays are parallel : for the other pair, which may be called the *antinodal points*, the incident and emergent rays are equally but oppositely inclined to the axis.

By considering a figure in any special case it is easy to see that if N and N′ (Fig. 7) be the nodal points, H and H′ the principal points, then N and H on the one hand and N′ and H′ on the other must always lie on the same side of F and F′ respectively.

From (16) we see that the magnification corresponding to the nodal points is $+f/f' = u/u'$.

By means of the nodal planes we can construct very neatly the position of the conjugate of any point P and also the emergent ray corresponding to any incident ray passing through P.

Let any ray through P (Fig. 8) meet the principal focal plane of emergence in Q, join PN and QN. Draw N′Q′ parallel to PQ to meet the principal focal plane of incidence in Q′, then Q′P′ parallel to QN and N′P′ parallel to PN will meet in P′ the conjugate of P; and Q′P′ will be the emergent ray corresponding to PQ. (See Heath, §71.)

Non-telescopic Systems in which the Initial and Final Media are the same.

In the special case where the initial and final media are the same the above general theory undergoes considerable simplification.

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Since $\mu = \mu'$, we have from (13) $f = f'' = \gamma$ - - - (13′).

Hence *the stationary points and the nodal points coincide with the principal points.* It is therefore sufficient in any such system to confine our attention merely to the four points F, F′, H, H′. Since these are now symmetrically disposed about the middle point of FF′, we may speak of this point as the *centre of the system* and call the system *symmetrical.*

The fundamental formulæ (4), (15), (16), (17), (18) now become

$$\beta \tan\alpha = \beta' \tan\alpha' \qquad - \quad - \quad (4');$$

$$uu' = f^2 \qquad - \quad - \quad (15');$$

$$\frac{\beta'}{\beta} = \pm\sqrt{\left(\frac{u'}{u}\right)} = \mp\frac{u'}{f''} = \mp\frac{f}{u} \qquad - \quad - \quad (16');$$

$$\frac{1}{v} + \frac{1}{v'} = \frac{1}{f} \qquad - \quad - \quad - \quad - \quad (17');$$

$$\frac{\tan\alpha'}{\tan\alpha} = \mp\frac{u}{f} \qquad - \quad - \quad - \quad - \quad (18').$$

Classification of Symmetrical Optical Systems.

Since symmetrical systems are of great importance, it seems to be worth while to classify the fundamentally distinct kinds that can arise and to indicate how typical models of them can be constructed for the purposes of laboratory instruction.

We may suppose one of the four determining points F, F′, H, H′ kept fixed. There arise therefore as many distinct cases as there are distinct topological arrangements of the three remaining points relative to F and to each other. Bearing in mind the symmetry of the system, we thus get the following eight distinct cases :—

F H H′F′, F H′H F′, H′F F′H, H′F′F H ;
F′H′H F, F′H H′F, H F′F H′, H F F′H

To remove the latent ambiguity arising from the indeterminate sign of γ in the general theory we divide the eight systems into two classes, viz., *Inverting Systems*, which give an inverted image of a distant object; and *Erecting Systems*, which give an erect image of a distant object. By considering the construction given above for the image of a non-axial point, it is easily seen that a system is Inverting or Erecting according as the direction of FH is the same as or opposite to the direction of the passage of the light through the system. If, therefore, we suppose, as hitherto, the light to pass from left to right, we see at once that the first four systems above set down are inverting and the rest erecting.

The following table indicates farther the nature of the systems as to the reality of the double points and the nature of the stationary distances HH′ between the principal points and KK′ between the anti-principal points.

Inverting Systems.	Erecting Systems.	Nature as to Double points.	H H′	K K′
F H H′F′	F′H′H F	Hyperbolic	Maximum	Minimum
F H′H F′	F′H H′F	Elliptic	Minimum	Minimum
H′F F′H	H F′F H′	Elliptic	Minimum	Minimum
H′F′F H	H F F′H′	Elliptic or Hyperbolic	Minimum	Minimum or Maximum

There are of course transition cases such for example as a system for which $FF'=0$, $FH'=0$, $FH=0$, or $HH'=0$. The last corresponds to the "thin lens" of the older theory, which still occupies the attention of the writer of elementary text books exclusively. We see however that a thin lens is not representative of optical systems in general; and it is not easy by means of experiments with lenses of moderate thickness to bring home to the beginner in optics the characteristic properties of a general system, because in such lenses the distance between the principal points barely exceeds the errors of such measurements as can be made without special refinements which are out of place in elementary instruction. The construction of thick lenses of special kinds would meet the difficulty, but would be troublesome and somewhat costly. On the other hand a doublet of thin lenses can always be constructed so as to have the same fundamental points as any given system.

If we take FHH′F′ as the standard case, and construct a doublet of two thin lenses whose focal lengths are f' and f'', the distance between the lenses being c (positive), then by specialising the formulæ given by Heath, § 60, we find

$$\left.\begin{aligned} HH' &= -c^2/(f+f'-c)\,; \\ FF' &= (2ff'-c^2)/(f+f'-c)\,; \\ FH &= (ff')/(f+f'-c)\,; \\ FH' &= (ff'-c^2)/(f+f'-c). \\ KK' &= (4ff'-c^2)/(f+f'-c) \end{aligned}\right\} \quad (20)$$

The following table, therefore, gives the conditions of construction for our eight types :—

Inverting	1	F H H′F′	$c>f+f'$	$ff'<0$	
	2	F H′H F′	$c<f+f'$	$ff'>0$	$ff'>c^2$
	3	H′F F′H	$c<f+f'$	$ff'>0$	$c^2>ff'>\frac{1}{2}c^2$
	4	H′F′F H	$c<f+f'$	$ff'>0$	$ff'<\frac{1}{2}c^2$
Erecting	5	F′H′H F	$c<f+f'$	$ff'<0$	
	6	F′H H′F	$c>f+f'$	$ff'>0$	$ff'>c^2$
	7	H F′F H′	$c>f+f'$	$ff'>0$	$c^2>ff'>\frac{1}{2}c^2$
	8	H F F′H′	$c>f+f'$	$ff'>0$	$ff'<\frac{1}{2}c^2$

(21.)

The only systems for which lenses of negative focal length are absolutely required are (1) and (5). All the others, except (7), can be constructed with two identical lenses of equal positive focal length placed at the proper distance apart. Perhaps the simplest way to work out a complete set of models is to take advantage of the fact that (5) can be derived from (1) merely by altering c; and that (2), (3), (4) can be converted into (6), (7), (8) respectively by changing the signs of f and f'.

The following table gives a convenient set of models. As it is convenient to know the positions of the anti-principal points when the combinations have to be measured experimentally, I have indicated them by the letters KK′ in the table :—

			f	f'	c	H H′	F F′	F H	F H′	K K′
Inverting	1	K F H H′F′K′	f	$-\cdot 5f$	$\cdot 6f$	$+3\cdot 6f$	$+13\cdot 6f$	$+5f$	$+8\cdot 6f$	$+23\cdot 6f$
	2	K F H′H F′K′	f	f	$\cdot 6f$	$-\cdot 26f$	$+1\cdot 17f$	$+\cdot 71f$	$+\cdot 46f$	$+2\cdot 6f$
	3	K H′F F′H K′	f	f	$1\cdot 2f$	$-1\cdot 80f$	$+\cdot 70f$	$+1\cdot 25f$	$-\cdot 55f$	$+3\cdot 20f$
	4	H′K F′F K′H	f	f	$1\cdot 7f$	$-9\cdot 63f$	$-2\cdot 96f$	$+3\cdot 33f$	$-6\cdot 30f$	$+3\cdot 70f$
Erecting	5	K′F′H′H F K	f	$-\cdot 5f$	$\cdot 3f$	$-4\cdot 5f$	$-9\cdot 5f$	$-2\cdot 5f$	$-7f$	$-14\cdot 5f$
	6	K′F′H H′F K	$-f$	$-f$	$\cdot 6f$	$+\cdot 14f$	$-\cdot 63f$	$-\cdot 38f$	$-\cdot 25f$	$-1\cdot 38f$
	7	K′H F′F H′K	$-f$	$-f$	$1\cdot 2f$	$+\cdot 45f$	$-\cdot 17f$	$-\cdot 31f$	$+\cdot 14f$	$-\cdot 80f$
	8	H K′F F′K H′	$-f$	$-f$	$1\cdot 7f$	$+\cdot 78f$	$+\cdot 24f$	$-\cdot 27f$	$+\cdot 51f$	$-\cdot 30f$

(22)

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With six lenses which can be bought for a few shillings, and a couple of draw-tubes for adjusting them at different distances apart, apparatus can be constructed by means of this table for illustrating the various typical systems and for exercising students in measuring their constants.

Telescopic Systems.

Hitherto we have supposed that the constant A in the equation

$$Axx' + Bx + Cx' + D = 0 \qquad - \quad - \quad - \quad (3)$$

does not vanish. We shall now briefly consider systems for which $A = 0$. In this case the equation (3) reduces to

$$Bx + Cx' + D = 0 \qquad - \quad - \quad - \quad - \quad (23)$$

We may pass over the cases where either $B = 0$ or $C = 0$, which are of no practical interest. They correspond to systems having an infinitely small focal length.

A system in which $A = 0$, $B \neq 0$, $C \neq 0$ we shall call a *Telescopic System.*

The special case where $B = -C$ is worthy of separate consideration. The equation (23) in this case takes the form

$$x' = x - d \qquad - \quad - \quad - \quad - \quad (24),$$

where d is a constant positive or negative according to the nature of the system. The meaning of (24) is that the conjugate focus P′ corresponding to any given point P always lies at a fixed distance d from P in the direction of the passage of the light or in the contrary direction according as d is positive or negative. In particular, we see that to a point at infinity corresponds a point at infinity. Hence *any incident pencil of parallel rays emerges as a parallel pencil. It is obvious in fact, from the more general equation* (23), *that this is a property of any Telescopic System.*

Let Pp (Fig. 9) be any linear object in the plane of the paper perpendicular to the axis of the system and meeting it in P, and let P′p' be the image of Pp. To the incident ray pQ parallel to the axis will correspond the emergent ray p'Q′ also parallel to the axis. Let PQ, P′Q′ be any incident and corresponding emergent rays through P and P′, then Q and Q′ are conjugate foci. Hence, if QM and Q′M′ be perpendicular to the axis, it follows from (24) that

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MM$'=d=$PP$'$. Hence PM$'=$PM. Now, by Helmholtz's Law, we have

$$\mu\beta\tan\alpha = \mu'\beta'\tan\alpha'$$

Hence $$\mu\beta^2/\text{PM} = \mu'\beta'^2/\text{P}'\text{M}'\,;$$

and therefore, since P$'$M$'=$PM,

$$\beta'/\beta = \pm\sqrt{(\mu/\mu')} \qquad - \quad - \quad - \quad (25).$$

In systems of the present kind, therefore, the image is always shifted through a constant distance depending on the nature of the system; the magnification is constant and depends merely on the initial and final media; and the image may be erect or inverted according to the nature of the system.

If the initial and final media be the same, the image is equal to the object. A plane-parallel plate of glass is an example, and lenses can be constructed having the same property, as may easily be seen by working out the characteristic equation (3) for a pair of coaxial spherical surfaces and then applying the conditions A$=0$, B$+$C$=0$.

Next suppose B$+$C$\neq 0$. Then, by shifting, the origin through a distance $h=-$D/(B$+$C), the equation (23) may be reduced to the form

$$x' = kx \qquad - \quad - \quad - \quad - \quad (26),$$

where $k=-$B/C. The definite point O which is now the origin we may call *the centre of the telescopic system.*

We see from (26) that the law of conjugate foci reduces now to the statement that *the distance of the image point from the central plane is proportional to the distance of the object point from the same plane. In particular, infinity corresponds to infinity, and the centre to itself.*

Since image and object move always in the same direction, the constant k must be positive.

Let now Pp (Fig. 10) and P$'p'$ be a linear object and its image as before. To the ray Op will correspond the emergent Op', since O corresponds to itself. Also, if PQ and P$'$Q$'$ be parallel to Op and Op' respectively, then PQ and P$'$Q$'$ will be a pair of corresponding rays, since the incident parallel pencil Op, PQ must emerge parallel. Applying Helmholtz's law, we therefore have

$$\mu\beta\tan\alpha = \mu'\beta'\tan\alpha'\,;$$
$$\mu\beta^2/x = \mu'\beta'^2/x.'$$

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Hence $$\beta'/\beta \quad = \pm \sqrt{(\mu k/\mu')} \quad - \quad - \quad - \quad (27);$$
$$\tan\alpha'/\tan\alpha = \pm \sqrt{(\mu/k\mu')} \quad - \quad - \quad - \quad (28).$$

From (27) and (28) we see that the *magnification of the image is constant; and so also is the ratio of the tangents of the inclinations to the axis of any incident and corresponding emergent rays. The image may be erect or inverted according to circumstances.*

We may call the telescopic system Erecting or Inverting according as the image of an infinitely distant point is erect or inverted.

Taking the simple case where the astronomical telescope consists of a field glass of focal length f and an eye piece of focal length g placed at a distance apart equal to $f+g$, if we take the common principal focus for origin, the equation (23) is, if we neglect the thickness of the lenses, and suppose the field-glass turned towards the incident rays,

$$-g^2x + f^2x' - 2fg/(f+g) = 0 \quad - \quad - \quad - \quad (29).$$

Hence we have

$$h \quad = 2fg/(f-g) \quad - \quad - \quad - \quad - \quad (30);$$
$$h - g = g(f+g)/(f-g) \quad - \quad - \quad - \quad (31),$$

The centre therefore lies at a distance from the eye lens somewhat exceeding its focal length.

From (29) we see that $k = g^2/f^2$. Hence, since, under ordinary circumstances, $\mu = \mu'$, we have

$$\beta'/\beta \quad = \pm g/f \quad - \quad - \quad - \quad (32);$$
$$\tan\alpha'/\tan\alpha = \pm f/g \quad - \quad - \quad - \quad (33):$$

the latter ratio, for reasons into which it is not necessary to enter here, is commonly called the *magnifying power of the telescope.*

Modification of a Symmetrical Photographic Doublet by the introduction of a Thin Lens between its Elements.

For the purposes of landscape photography it is essential to have a series of lenses of widely different focal length. The best possible arrangement would of course be to have a specially constructed lens for each of the focal lengths required; but such a battery of lenses is expensive, and, if doublets are used, it is heavy to carry. Moreover, while the requirements of pictorial perspective absolutely demand variability in the focal length of the photographer's lens, for

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many purposes the utmost refinement in definition and flatness of field is not necessary, or, it may be not desirable.

For general photographic purposes the handiest lens is a symmetrical doublet of the rapid rectilinear or euryscope type; and it has long been known that the focal length of such a combination can be varied within wide limits without destroying its efficiency as a photographic instrument by inserting between its elements a thin lens of positive or negative focal length. As an example of the foregoing general theory, we propose to calculate the effect of such a lens in shifting the principal points and in altering the focal length of the doublet.

We shall suppose the thickness of the inserted lens to be negligible; *i.e.*, we shall take its principal points to be coincident with the middle point of the substance of the lens; and in the first instance we shall suppose the inserted lens to be actinically achromatised, so that it has no "focal difference," in other words that its principal foci for the rays of maximum visual and maximum chemical intensity coincide.

Let f (Fig. 11) be the focal length of each of the components of the doublet, F and ϕ the focal lengths of the doublet itself and of the adjuster respectively, we take ϕ to be positive as usual when the adjuster is of positive focal length.

Let L (Fig. 11) be the position of the adjuster; C the central point of the doublet; H, H′, K, K′ the principal points of the components of the doublet; P, P′, Q′, Q successive congugate foci with respect to the three lenses:

$$\begin{aligned} &\mathrm{CL} = d, \text{ positive when L is right of C;} \\ &\mathrm{CH'} = \mathrm{CK'} = h', \ \mathrm{CH} = \mathrm{CK} = h; \\ &\mathrm{CP} = x, \ \mathrm{CP'} = x' \text{ measured to left;} \\ &\mathrm{CQ} = y, \ \mathrm{CQ'} = y' \text{ measured to right.} \end{aligned}$$

Then we have, by (17′) above,

$$1/(x-h) - 1/(x'-h') = 1/f \qquad (34);$$

$$1/(y-h) - 1/(y'-h') = 1/f \qquad (35);$$

$$1/(x'+d) + 1/(y'-d) = 1/\phi \qquad (36).$$

From (34) and (35) we get

$$\left.\begin{aligned} x'-d &= \{(h'-h)f + hh' + d(f+h) + (f-h'-d)x\}/(f+h-x); \\ y'-d &= \{(h'-h)f + hh' - d(f+h) + (f-h'+d)y\}/(f+h-y). \end{aligned}\right\} \quad (37).$$

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If we put $(h'-h)f+hh'=k^2, f+h=l, f-h'=m,$ - - } (38)
so that $k^2=(2f-l-m)f+(f-m)(l-f)=f^2-lm,$ - -

then we may replace h, h', f by k, l, m, or by l, m, f; where it will be observed that l and m are the distances from the central point of the doublet of the outer and inner principal focal points of either of its elements.

If now we substitute the expressions just found for $x'+d$ and $y'-d$ in (36), we get

$$\phi(l-x)\{k^2-dl+(m+d)y\}+\phi(l-y)\{k^2+dl+(m-d)x\}$$
$$=\{k^2+dl+(m-d)x\}\{k^2-dl+(m+d)y\}.$$

Hence the characteristic equation for the triplet is

$$Axy+By+Cx+D=0 \quad - \quad - \quad - \quad (39),$$

where

$$\left.\begin{aligned} A &= 2m\phi+m^2-d^2, \\ B &= (k^2+dl)(m+d)+(k^2-lm)\phi, \\ C &= (k^2-dl)(m-d)+(k^2-lm)\phi, \\ D &= k^4-d^2l^2-2k^2l\phi. \end{aligned}\right\} \quad (40).$$

For the coordinates of the principal focal points of the triplet we have

$$\left.\begin{aligned} y &= -\frac{B}{A}=\frac{(lm-k^2)\phi-(k^2+dl)(m+d)}{2m\phi+m^2-d^2}; \\ y' &= -\frac{C}{A}=\frac{(lm-k^2)\phi-(k^2-dl)(m-d)}{2m\phi+m^2-d^2} \end{aligned}\right\} \quad (41).$$

Also, if F_3 be the principal focal length of the triplet,

$$F_3^2=\frac{BC-AD}{A^2}=\left(\frac{(k^2+lm)\phi}{2m\phi+m^2-d^2}\right)^2;$$

whence

$$F_3=\frac{(k^2+lm)\phi}{2m\phi+m^2-d^2}=\frac{f^2\phi}{2m\phi+m^2-d^2} \quad - \quad - \quad (42).$$

If $\phi=\infty$, $F_3=F$; so that $F=f^2/2m$, and we may put the last equation into the form

$$F_3=\frac{F\phi}{\phi+\frac{1}{2}m-d^2/2m} \quad (43).$$

If χ, χ' be the coordinates of the principal points of the triplet, we have

$$\left.\begin{aligned} \chi &= g - F_3 = -\frac{2k^2\phi + (k^2 + dl)(m+d)}{2m\phi + m^2 - d^2}; \\ \chi' &= g' - F_3 = \frac{2k^2\phi + (k^2 - dl)(m-d)}{2m\phi + m^2 - d^2} \end{aligned}\right\} \quad (44).$$

Case where the Adjuster is Central.

Then $d = 0$; and we have $F_3 = F\phi/(\phi + \frac{1}{2}m)$ - - - - (45);

whence $\phi = \frac{1}{2}mF_3/(F - F_3)$ - - - - (46):

these formulæ give the focal length of the triplet corresponding to a given adjuster and the focal length of the adjuster required to produce a triplet of given focal length.

The coordinates of the principal points now become

$$\chi = \chi' = -k^2/m = l - f'^2/m = f + h - 2F \qquad (47):$$

Since these are the values of χ and χ' given by the general formulæ (44) when $\phi = \infty$, we see that a central adjuster leaves the principal points of the doublet unaltered.

Non-Central Adjuster.

Returning to the general expression for F_3, we have

$$F_3 = \frac{F\phi}{\phi + \frac{1}{2}m} \bigg/ \left\{1 - \frac{d^2}{2m(\phi + \frac{1}{2}m)}\right\}.$$

Taking the case of the Voigtländer's Euryscope with which I experimented, we have $F = 9{\cdot}85''$, $f = 18{\cdot}06''$, $h' = 1{\cdot}66''$, $h = 1{\cdot}82''$, $m = 16{\cdot}4''$. The utmost possible value of d is $1''$: so that, if we take the extreme case where $\phi = -16$ (which about doubles the focal length of the Euryscope), we find $d^2/2m(\phi + \frac{1}{2}m)$ numerically less than 1/255. We thus arrive at the remarkable conclusion that, so far as the focal length of the triplet is concerned, the position of the internal adjuster is practically a matter of indifference.

The effect on the principal points under the circumstances just supposed is that the distance between them is practically unaltered, while each is shifted towards the inserted lens through a distance λ which is given approximately by

$$\lambda = \mathrm{F}d/(\phi + \tfrac{1}{2}m) \qquad - \quad - \quad - \quad (48);$$

or

$$\lambda = d(\mathrm{F} - \mathrm{F}_3)/\tfrac{1}{2}m \qquad - \quad - \quad - \quad (49)$$

The result just stated may be put into another form. Reverting to the expression for F_3 when $d = 0$, we have

$$\frac{1}{\mathrm{F}_3} = \frac{1}{\mathrm{F}} + \frac{m}{2\mathrm{F}\phi} = \frac{1}{\mathrm{F}} + \frac{f - h'}{2\mathrm{F}\phi}\cdot$$

Now

$$\frac{1}{\mathrm{F}} = \frac{2}{f} - \frac{2h'}{f^2}\cdot$$

Hence

$$\frac{1}{f} = \frac{1}{2\mathrm{F}} + \frac{h'}{f^2}\cdot$$

A first approximation to the value of f is $f = 2\mathrm{F}$

For a second approximation

$$\frac{1}{f} = \frac{1}{2\mathrm{F}} + \frac{h'}{4\mathrm{F}^2};$$

and for a third approximation

$$\frac{1}{f} = \frac{1}{2\mathrm{F}}\left\{1 + \frac{h'}{2\mathrm{F}} + \frac{h'^2}{2\mathrm{F}^2}\right\}$$

Hence, to a third approximation, we get

$$f - h' = 2\mathrm{F}\left\{1 - \frac{h'}{\mathrm{F}} - \frac{h'^2}{4\mathrm{F}^2}\right\}$$

To the same degree of approximation

$$\frac{1}{\mathrm{F}_3} = \frac{1}{\mathrm{F}} + \frac{1}{\phi} - \frac{h'}{\phi\mathrm{F}} - \frac{h'^2}{4\mathrm{F}^2\phi}$$

If, therefore, $h^2/4\mathrm{F}^2\phi$ be negligible, we get

$$\frac{1}{\mathrm{F}_3} = \frac{1}{\mathrm{F}} + \frac{1}{\phi} - \frac{h'}{\mathrm{F}\phi}\cdot \qquad (50).$$

In other words, the triplet behaves, *qua* focal length, as if the doublet were replaced by a thin lens of its own focal length placed at its centre and the adjuster were placed at the inner principal point of one of its components. This agrees with what we have

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already said regarding the indifference of the position of the adjuster so far as the focal length of the triplet is concerned.

When a negative adjuster is used to lengthen the focus of the doublet, there is a practical advantage in placing it as near the back component as possible, because the effect of this is to shift the principal points forward (λ being negative) so that less camera extension is required. Thus, for example, my camera scarcely allows me to use the back lens ($f = 18{\cdot}06''$) of my Voigtländer's Euryscope as a single landscape lens, whereas I can readily use the Euryscope adjusted to a focus of 20″ by inserting an adjuster for which $\phi = -16$ near the back lens.

In practice the simplest method for obtaining data for the construction of adjusters of a given symmetrical doublet is to measure the focal length of the doublet itself; this gives F; then to measure the distance behind the central point of the doublet (usually the place where the diaphragm is put) of the inner principal focal point of its front element; this gives m. The formulæ (45) (46) will then give F, or ϕ as may be required. If there is any reason to doubt the accuracy of the approximation when the adjuster is non-central, the more accurate formula (43) may be used.

The following table gives the results of several (visual) experiments made to test the foregoing results. The doublet was the Euryscope above mentioned; and the formula used for calculating F_3 was $F_3 = 9{\cdot}85\phi/(\phi + 8{\cdot}2)$.

ϕ	F_3 Obs.	F_3 Calc.
$+37{\cdot}60$	$8{\cdot}00$	$8{\cdot}08$
$-24{\cdot}97$	$14{\cdot}74$	$14{\cdot}67$
$-19{\cdot}38$	$17{\cdot}00$	$17{\cdot}08$
$-15{\cdot}89$	$20{\cdot}39$	$20{\cdot}35$

As the experiments were roughly made, without special appliances for centering the lenses or measuring distances, the agreement between observation and calculation is all that could be expected.

Chromatic Aberration Caused by a Non-Achromatic Adjuster.

Hitherto the adjusting lens has been supposed to be actinically achromatic. As a matter of fact, the lenses used in my experiments were simple lenses of crown glass. It is easy to calculate, by means of the formulæ given above, the chromatic aberration or actinic focal difference produced by the non-achromatic adjuster. The elements of the Euryscope itself are approximately corrected for actinic chromatism: We have therefore only to deal with the dispersion of the adjuster itself. Assuming the lens to be of hard crown glass, and taking the rays of maximum visual and maximum chemical intensity respectively to be D and G, we may suppose $\mu_D = 1 \cdot 5167$, $\mu_G = 1 \cdot 5280$: hence $\omega = \partial\mu/(\mu_D - 1) = \cdot 022$, say.

From the approximate formulæ for the triplet we have

$$1/F_3 = 1/F + \tfrac{1}{2}m/F\phi, \quad \lambda = d(F - F_3)/\tfrac{1}{2}m.$$

Hence*
$$\partial\left(\frac{1}{F_3}\right) = \frac{\tfrac{1}{2}m}{F}\,\partial\left(\frac{1}{\phi}\right) = \frac{\tfrac{1}{2}m\omega}{F\phi}:$$

therefore
$$\partial F_3 \qquad = -\tfrac{1}{2}mF_3\omega/\phi; \tag{51}$$
$$\partial\lambda = -d\partial F_3/\tfrac{1}{2}m = \quad dF_3\omega/\phi. \tag{52}$$

Let us suppose the adjuster placed behind the centre of the doublet as before; let u and v be the distance of object and image (corresponding to the ray D of the spectrum) from the first and second principal points of the triplet respectively; and let us calculate the longitudinal aberration of the ray G. This is caused partly by the shifting of the principal points and partly by the alteration of the focal length. The first principal point is shifted to the right through $\partial\lambda$; and u as measured from the new first principal point is increased by the same amount: v as measured from the new second principal point is increased by an amount ∂v which is given by

$$\partial\lambda/u^2 + \partial v/v^2 = \partial F_3/F_3^2,$$

that is, by (51) and (52)

$$\begin{aligned}\partial v &= \quad v^2\partial F_3/F_3^2 \quad - v^2\partial\lambda/u^2,\\ &= -\tfrac{1}{2}mv^2\omega/F_3\phi - dv^2F_3\omega/u^2\phi. \end{aligned} \tag{53}$$

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* See Heath, §182.

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To this must be added the shift, $\partial\lambda$, of the second principal point to the right. If, therefore, a denote the whole longitudinal aberration of the ray G, we have

$$a = \frac{\omega F_3 d}{\phi} - \frac{\omega v^2 \frac{1}{2} m}{F_3 \phi} - \frac{\omega v^2 F_3 d}{u^2 \phi} \quad - \quad - \quad - \quad (54).$$

If we consider only the correction for very distant objects, we may put $u = \infty$, $v = F_3$: we then get

$$a = \omega(d - \tfrac{1}{2}m)F_3/\phi \quad - \quad - \quad - \quad (55).$$

Taking $d = \cdot 6''$, $\frac{1}{2}m = 8\cdot 2''$, and $\omega = \cdot 022$, we find

$$a = -\cdot 167 F_3/\phi \quad - \quad - \quad - \quad (56),$$

by means of which we can calculate the actinic focal difference for any given adjuster. Thus, for example, when $\phi = -15\cdot 89''$ and $F_3 = 20\cdot 35''$, $a = +\cdot 22''$; that is to say, the camera must be racked out a little over $\cdot 2''$ after the view has been focussed on the ground glass.

Experimental Determination of the Constants of a Symmetrical System.

Since this paper is intended mainly for the use of laboratory students, a word or two on the experimental determination of the characteristic points of a symmetrical optical system may not be out of place. In what follows, when Object and Image are spoken of, they are assumed to be real, *i.e.*, such that rays actually pass through them. When the object is virtual, it can be generated by means of an auxiliary optical system, say a simple biconvex lens, whose optical constants are accurately known; and a virtual image can be dealt with in like manner if necessary.

The first step usually consists in determining the positions of the principal foci F and F′. This is done by centering the optical system in the axis of a telescope of moderate power focussed for infinitely distant objects. An object w is moved backwards and forwards on the axis until its image can be seen sharp on the cross wires of the telescope: w is then at one of the principal foci, say F, whose distance from any arbitrarily chosen point, O, on the axis of the system is thus determined. Reversing the system and again adjusting w we determine in like manner OF′.

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If now we determine the position P′ of any axial object P, we can find FP and F′P′ and hence f, by the equation $FP \cdot F'P' = f^2$ and it only remains to note whether the system is erecting or inverting in order to be able to lay down the principal and anti-principal points H, H′ and K, K′. In the case of a convex lens, for example, this is readily done by a method given by Gauss, which consists in focussing a microscope on an object in the axis of the lens in contact with its surface, first through the lens and then after the lens has been removed: the amount by which the draw tube has been displaced between the two focussings is the distance between P and P′, from which FP and FP′ can be found.

We may also determine the principal or the antiprincipal points directly. For this purpose two identical photographic negatives O and O′ of the same object, say a divided scale, may be used. O is placed perpendicular to the axis of the system and its image received on O′ similarly placed as a focussing screen, so that the two scales overlap. When the image of O is exactly of the same size as O′, O and O′ are in the positions H, H′ or else K, K′ respectively. The coincidence may be very accurately determined by observing O′ and the image of O by means of a microscope or other magnifier of moderate power carefully focussed on O′ beforehand.

These measurements are susceptible of considerable accuracy if monochromatic light of sufficent intensity be used, and the experimenter is provided with an optic bench fitted with telescope and low power microscope with micrometer eyepiece and micrometer displacement screw, together with arrangements for fixing and centering the systems to be measured.

The observations quoted above, which are probably accurate to about 1%, were made in my own library, with white light, the only apparatus available being a small pocket telescope, the *debris* of a toy microscope, a steel measuring tape and a couple of retort stands.

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