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ON SOME GEOMETRIC INVARIANTS ASSOCIATED TO THE SPACE OF FLAT CONNECTIONS ON AN OPEN SPACE

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ABSTRACT. A geometric invariant is associated to the parabolic moduli space on a marked surface and is related to the symplectic structure of the moduli space.

0. **Introduction.** The space of equivalence classes of representations of the fundamental group of a compact oriented surface of genus at least two, in a Lie group G has a natural symplectic structure. This representation space can be identified with the space of flat connections modulo gauge-equivalence on the trivial G-bundle on M. In [Gu], a new geometric invariant was associated to the space of flat connections and related to the symplectic structure on the representation space. In this paper we prove an analogous result for marked surfaces, *i.e.*, compact oriented surface with finitely many punctures.

At the outset we briefly describe the compact case [Gu]. Let $G = \mathrm{SU}(2)$ and $E \longrightarrow M$ be the trivial G-bundle over a compact oriented surface M. Let $\mathcal{C}$ (resp. $\mathcal{C}^{\mathrm{irr}}$) be the space of all (resp. irreducible) connections and $\mathcal{F}$ (resp. $\mathcal{F}^{\mathrm{irr}}$) be subspace of all (resp. irreducible) flat connections on this G-bundle. We equip $\mathcal{C}$ with the Fréchet topology and the subspace topology on $\mathcal{F}$.

Given a loop $\sigma: S^1 \longrightarrow \mathcal{F}$, we can extend σ to the closed unit disc $\tilde{\sigma}: D^2 \longrightarrow \mathcal{C}$, since $\mathcal{C}$ is contractible. On the trivial G-bundle $E \times D^2 \longrightarrow M \times D^2$, we define a "tautological" connection $\vartheta^{\tilde{\sigma}}$ as follows

$$\vartheta^{\tilde{\sigma}}|_{(e,t)} = \tilde{\sigma}(t) \ \forall (e,t) \in E \times D^2.$$

Let $K(\vartheta^{\tilde{\sigma}})$ be the curvature form of $\vartheta^{\tilde{\sigma}}$. Evaluation of the second Chern polynomial on this curvature form $K(\vartheta^{\tilde{\sigma}})$ gives a closed 4-form on $M \times D^2$, which when integrated along D^2 yields a 2-form on M. This 2-form is closed since $\dim M = 2$ and thus defines an element in $H^2(M, R) \approx R$. In Lemma 1.3, we will show that this class is independent of the extension of σ.

We thus have a map

$$\chi: L(\mathcal{F}) \longrightarrow H^2(M, R) \approx R$$

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where $L(\mathcal{F})$ is the loop space of $\mathcal{F}$.

It is seen that χ induces a map

$$\chi: L(\mathcal{F}^{\mathrm{irr}}/\mathcal{G}) \longrightarrow R/Z$$

where $\mathcal{G}$ is the gauge-group of the G-bundle $E \longrightarrow M$.

We identify $\mathcal{F}/\mathcal{G}$ with the representation space $\mathrm{Hom}\big(\pi_1(M), G\big)/G$. When the genus ≥ 3, $\mathcal{F}^{\mathrm{irr}}/\mathcal{G}$ is simply connected and therefore for a loop σ in $\mathcal{F}^{\mathrm{irr}}/\mathcal{G}$, we can find a surface S in $\mathcal{F}^{\mathrm{irr}}/\mathcal{G}$ which bounds the loop σ. Since the symplectic form ω has integral periods, $\overline{\int_S}\omega \in R/Z$ is independent of S. The main result proved in [Gu] is that $\chi(\sigma) = \overline{\int_S}\omega$ (after suitable normalization).

After suitable modifications we prove an analogous result for marked surfaces. More specially, we consider the marked surface $X = M - \{p_1, \ldots, p_n\}$. The space of equivalent classes of representations of $\pi_1(X)$ also admits a symplectic structure when some boundary conditions are imposed *viz.* fixing the conjugacy classes of the holonomies around the punctures. This representation space $\mathcal{P}$ (also called the parabolic moduli space) can be identified with the gauge-equivalent classes of flat connections $\mathcal{F}/\mathcal{G}$ on the trivial G-bundle on X whose holonomies around the punctures lie in preassigned conjugacy classes of G. Imitating the constructions of the compact case for the marked case with suitable modifications, it is seen that we get a map

$$\chi: L(\mathcal{F}^{\mathrm{irr}}/\mathcal{G}) \longrightarrow R/Z.$$

Unlike the compact case, the symplectic form ω in the marked case is not in general integral. However, if we assume that the fixed conjugacy classes are of finite order, *i.e.*, the eigenvalues of the elements of the classes are roots of unity, then in [BR] it was shown that this ω is rational. More specifically, if q is the l.c.m. of the order of the eigenvalues, then $q\omega$ is integral. The main result of this paper analogous to the compact case is Theorem: $q\chi(\sigma) = \overline{\int_s}q\omega$ where $\partial S = \sigma$.

1. **A function on** $L(\mathcal{F})$. Let X be a compact oriented 2-dimensional manifold of genus g and $I = \{p_1, \ldots, p_n\} \subset X$ be a finite set of points. Define $X := X - I$ to be punctured surface. SU(2) is the Lie group of 2×2 unitary matrices, and su(2) is its Lie algebra.

Let $D_0 := \{z \in C \mid 0 < |z| \leq 1\}$ be the punctured disc. We fix disjoint punctured discs $D_{0,i}$, $1 \leq i \leq n$ around p_i, *i.e.*, $D_{0,i} \cup p_i$ is a neighborhood of $p_i \in X$ with $D_{0,i}$ being diffeomorphic to D_0. The bounding circle of $D_{0,i}$ is denoted by S_i. Let

$$p: E := X \times \mathrm{SU}(2) \longrightarrow X$$

be the trivial SU(2)-bundle on X. We fix n conjugacy classes in SU(2), *i.e.*, orbits of the conjugate action of SU(2) on itself, and denote them by $C_1, \ldots, C_n$. Fix once and for all a flat connection δ on E such that the holonomy of δ along S_i lies in C_i. The space of all connections on E is denoted by $\mathcal{C}^*$, and $\mathcal{F}^* \subset \mathcal{C}^*$ is the space of flat connections whose

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holonomy along S_i lies in $\mathcal{C}_i$. Let $\mathcal{C} \subset \mathcal{C}^*$ be the subspace of all connections which coincides with δ on all $D_{0,i}$. In other words

$$\mathcal{C} := \{a \in \mathcal{C}^* \mid \alpha|_{D_{0,i}} - \delta|_{D_{0,i}} = 0,\ 1 \leq i \leq n\}.$$

Define $\mathcal{F} := \mathcal{F}^* \cap \mathcal{C}$. Note that $\mathcal{C}$ is an affine space for the vector subspace $\Lambda^1_0\big(X, \mathrm{su}(2)\big) \subset \Lambda^1\big(X, \mathrm{su}(2)\big)$, consisting of all su(2)-valued, 1-forms which vanish on each $D_{0,i}$. Given a smooth map $\sigma \colon S^1 \longrightarrow \mathcal{F}$, (*i.e.*, the composition of σ with the inclusion $\mathcal{F} \hookrightarrow \mathcal{C}$ being smooth), since $\mathcal{C}$ is affine, we can extend σ to $\tilde{\sigma} \colon D^2 \longrightarrow \mathcal{C}$, where D^2 denotes the closed unit disc. Given such a $\tilde{\sigma}$, the trivial SU(2)-bundle $E \times D^2 \longrightarrow X \times D^2$ admits a canonical connection, denoted by ϑ, which is defined as follows

$$\vartheta|_{(e,z)} = \tilde{\sigma}(z)|_e,\ \forall (e,z) \in E \times D^2. \tag{1.1}$$

In other words, the restriction of ϑ to $E \times \{z\} \longrightarrow X \times \{z\}$ is the connection form $\tilde{\sigma}(z)$ itself. Let $K(\vartheta)$ be the curvature of ϑ. Let $K(\vartheta)$ be the curvature form of ϑ and C_2 be the second Chern polynomial on su(2). The specific formula for C_2 is $C_2(A) = \frac{1}{8\pi^2}\,\mathrm{trace}(A^2)$ for $A \in \mathrm{su}(2)$. Evaluation of C_2 on $K(\vartheta)$ gives a closed 4-form $\overline{C_2\big(K(\vartheta)\big)}$ on $E \times D^2$ which projects to the closed 4-form $C_2\big(K(\vartheta)\big)$ on $X \times D^2$. Integrating $C_2\big(K(\vartheta)\big)$ along D^2 yields a closed form on X.

LEMMA 1.2. *The 2-form $\int_{D^2} C_2\big(K(\vartheta)\big)$ on X is compactly supported.*

PROOF. The image of $\tilde{\sigma}$ lies in $\mathcal{C}$, and the fixed connection δ is flat. Hence the connection ϑ is flat on each $D_{0,i} \times D^2$. So the 2-form $\int_{D^2} C_2\big(K(\vartheta)\big)$ is supported on $X - \cup_i \mathring{D}_{0,i}$ where $\mathring{D}_{0,i}$ denotes the interior of $D_{0,i}$, thus proving the Lemma. ∎

So the form $\int_{D^2} C_2\big(K(\vartheta)\big)$ defines an element of the compactly supported cohomology $H^2_c(X, R)$. Note that since X is oriented, $H^2_c(X, R) = R$.

LEMMA 1.3. *The element in $H^2_c(X, R)$ represented by $\int_{D^2} C_2\big(K(\vartheta)\big)$ depends only on σ and does not depend on the extension on $\tilde{\sigma}$.*

PROOF. Let $\tilde{\sigma}$ and $\tilde{\sigma}'$ be two extensions of σ. ϑ and ϑ' denote the corresponding canonical connections on $X \times D^2$, and the respective curvature forms are denoted by $K(\vartheta)$ and $K(\vartheta')$. First we want to show that

$$\int_{D^2} C_2\big(K(\vartheta)\big) - \int_{D^2} C_2\big(K(\vartheta')\big)$$

is an exterior derivative of a compactly supported 1-form on E. To prove that, recall that there is a *secondary Chern-Simons form* T on $E \times D^2$ such that $dT = C_2$ (Section 3 of [CS]). Similarly, let T' be the secondary Chern-Simons for the second Chern form of ϑ'. Therefore

$$\int_{D^2} C_2\big(K(\vartheta)\big) - \int_{D^2} C_2\big(K(\vartheta')\big) = \int_{D^2} d(T - T').$$

As in Lemma 2.3 of [GK], by Stokes' Theorem for integration along fibers, we have

$$\int_{D^2} d(T - T') = \int_{S^1} (T|_{E \times S^1} - T'|_{E \times S^1}) + d\int_{D^2} (T - T').$$

But $\vartheta = \vartheta'$ on $X \times S^1$. So on $E \times S^1$ we have $T = T'$, and hence the first integral on the right-hand side vanishes. On the other hand, since the connection $\tilde{\sigma}((z))|_{D_{0,i}}$ does not depend on $z \in D^2$, the construction of secondary Chern-Simons class (*cf.* [CS]) implies that the form $\int_{D^2}(T - T')$ is supported on $p^{-1}(X - \cup_i \mathring{D}_{0,i})$. The form $\int_{D^2} C_2\big(K(\vartheta)\big)$ (similarly $\int_{D^2} C_2\big(K(\vartheta')\big)$ is also supported on $p^{-1}(X - \cup_i \mathring{D}_{0,i})$). In other words $\int_{D^2} C_2\big(K(\vartheta)\big)$ and $\int_{D^2} C_2\big(K(\vartheta')\big)$ represent the same element in $H^2_c(E, R)$. Since the bundle E is trivial, $\int_{D^2} C_2\big(K(\vartheta)\big)$ and $\int_{D^2} C_2\big(K(\vartheta')\big)$ represent the same element in $H^2_c(X, R)$. ∎

Let $L(\mathcal{F})$ be the loop space of $\mathcal{F}$. Lemma 1.3 gives a map

$$\chi : L(\mathcal{F}) \longrightarrow H^2_c(E, R) = R. \tag{1.4}$$

It is easy to check that the function χ is invariant under reparametrization of a loop. And also for two loops σ and σ' with same base point, $\chi(\sigma \circ \sigma') = \chi(\sigma) + \chi(\sigma')$. In other words χ is a homomorphism of groupoids.

2. **A symplectic structure.** We continue with the notations of the previous section. The group of SU(2) equivariant automorphisms of E, also known as the gauge group, is denoted by $\mathcal{G}^*$, and $\mathcal{G} \subset \mathcal{G}^*$ is the subgroup consisting of all automorphisms which are identity on all $D_{0,i}$. Let $\mathcal{P}$ be the space of equivalence classes of parabolic representations of $\pi_1(X)$ in SU(2), *i.e.*, all those representations of $\pi_1(x)$ in SU(2) which map S_i into C_i. Taking holonomy of a connection along a loop, we have an identification of $\mathcal{F}^* / \mathcal{G}^*$ with $\mathcal{P}$. The inclusion $\mathcal{F} \hookrightarrow \mathcal{F}^*$ induces a map

$$f : \mathcal{F} / \mathcal{G} \longrightarrow \mathcal{F}^* / \mathcal{G}^*.$$

It is easy to see that f is injective, and moreover, since any two connections on $D_{0,i}$ are gauge equivalent if and only if their holonomies around S_i are conjugate to each other, f is in fact onto.

The real algebraic variety structure on SU(2) combines with the fact that the group $\pi_1(X)$ is finitely generated to give a real algebraic variety structure to $\mathcal{P}$. The equivalence classes of irreducible representations $\mathcal{P}^{\mathrm{ir}}$ in $\mathcal{P}$ form a smooth Zariski open subset. Let $\mathcal{P}^{\mathrm{re}} := \mathcal{P} - \mathcal{P}^{\mathrm{ir}}$ be the equivalence classes of reducible representations.

For $A \in \mathcal{F}$, let $\mathrm{su}(2)_A$ be the local system on X given by the induced flat connection on the vector bundle $\mathrm{Ad}(E)$. We have the obvious homomorphism.

$$\lambda : H^1_c\big(X, \mathrm{su}(2)_A\big) \longrightarrow H^1\big(X, \mathrm{su}(2)_A\big).$$

The tangent space $T_A(\mathcal{P})$ can be identified with the image of λ [BG]. The Killing form $\alpha\beta \longmapsto \mathrm{trace}(\alpha\beta)$ on su(2) induces the following pairing on the image of λ

$$\phi \otimes \psi \longmapsto \int_X \mathrm{trace}(\phi \wedge \psi).$$

It was proved in [BG] that this 2-form on $\mathcal{P}$ is closed and non-degenerate; in other words, it gives a symplectic structure. This symplectic form will be denoted by ω_0.

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It is easy to check that if the co-dimension of $\mathcal{P}^{\text{re}}$ in $\mathcal{P}$ is at least three, then $\mathcal{P}^{\text{ir}}$ is simply connected. This co-dimension condition is satisfied, for example, when the genus $g > 2$ or when the set $\{C_1, \ldots, C_n\}$ contains enough regular orbits (an orbit is called regular if the eigenvalues of its elements are distinct; hence the only non-regular orbits are $\pm I$). Henceforth, we will assume that the initial data is such that the variety $\mathcal{P}^{\text{ir}}$ is simply connected.

We call a symplectic structure *integral* if the cohomology class it represents is integral. The symplectic form ω_0 is in general not integral. But if the eigenvalues of the elements of all C_i are roots of unity (the eigenvalues being conjugate invariant depends only on the class), and if q is the l.c.m. of the order of the eigenvalues, then the $q\omega_0$ is integral [BR]. Henceforth, we will assume that the eigenvalues of C_i are roots of unity. The integral symplectic form $g\omega_0$ will be denoted by ω.

Given a loop $\sigma: S^1 \longrightarrow \mathcal{P}^{\text{ir}}$, using simply connectedness of $\mathcal{P}^{\text{ir}}$, the map σ can be extended to $\tilde{\sigma}: D^2 \longrightarrow \mathcal{P}^{\text{ir}}$. Integrating the pull-back of ω on D^2, we get a real number, which of course depends on the extension $\tilde{\sigma}$. But if $\tilde{\sigma}'$ is another extension of σ, then, since ω is integral, we have

$$\int_{D^2} \tilde{\sigma}^* \omega = \Big(\int_{D^2} \tilde{\sigma}'^* \omega\Big) \bmod Z.$$

Let $L(\mathcal{P}^{\text{ir}})$ be the loop space of $\mathcal{P}^{\text{ir}}$. So the above discussion gives a function which we also call

$$\begin{array}{rcl} \omega: L(\mathcal{P}^{\text{ir}}) & \longrightarrow & R/Z \\ \sigma & \longmapsto & (\int_{D^2} \tilde{\sigma}^* \omega) \bmod Z. \end{array} \tag{2.1}$$

3. **A function on $L(\mathcal{F}/\mathcal{G})$.** From now onwards we assume that the punctured surface X is equipped with a complex structure. We also assume that $2g - 2 + n > 0$. This would imply that X is uniformized by the upper half plane; in other words, X inherits the Poincaré metric.

Let $\Lambda^i\big(X, \mathrm{su}(2)\big)$ be the space of su(2) valued i-forms on X, and $\Lambda^i_c\big(X, \mathrm{su}(2)\big)$ be the subspace consisting of all those forms which vanish on every $D_{0,i}$.

We will now briefly describe the Coulomb connection on the space of connections; see [NR] for details. A connection on E is said to be *irreducible* if the only elements of the gauge group which preserve the connection are $\pm I$. A connection ∇ is irreducible if and only if the 0-th cohomology of the complex

$$C_\nabla: \Lambda^0\big(X, \mathrm{su}(2)\big) \xrightarrow{d_\nabla} \Lambda^1\big(X, \mathrm{su}(2)\big) \xrightarrow{d_\nabla} \Lambda^2\big(X, \mathrm{su}(2)\big)$$

vanishes (Theorem 3.1 of [FU]). Let $\mathcal{C}^{\text{ir}} \subset \mathcal{C}$ be the subspace consisting of all those connections which are also irreducible. For $\nabla \in \mathcal{C}^{\text{ir}}$, the tangent space $T_\nabla \mathcal{C}^{\text{ir}}$ can be identified with the $\Lambda^1_c\big(X, \mathrm{su}(2)\big)$. Let d^*_∇ be the adjoint (with respect to the Poincaré metric) of d_∇, and

$$\Delta_\nabla := d^*_\nabla d_\nabla : \Lambda^0_c\big(X, \mathrm{su}(2)\big) \longrightarrow \Lambda^0_c\big(X, \mathrm{su}(2)\big)$$

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be the Laplacian.

Let $q: \mathcal{C}^{\mathrm{ir}} \longrightarrow \mathcal{C}^{\mathrm{ir}}/\mathcal{G}$ be the principal $\mathcal{G}$ bundle. The Lie algebra of $\mathcal{G}$ is $\Lambda^0_c\big(X, \mathrm{su}(2)\big)$, and the kernel of the operator $\Lambda^1_c\big(X, \mathrm{su}(2)\big) \xrightarrow{d^*_\nabla} \Lambda^0\big(X, \mathrm{su}(2)\big)$ gives an equivariant splitting of the projections $T_\nabla \mathcal{C}^{\mathrm{ir}} \xrightarrow{dq} T_\nabla(\mathcal{C}^{\mathrm{ir}}/\mathcal{G})$. In other words, this defines a connection on the principal $\mathcal{G}$-bundle. It can be checked that the Lie-algebra valued 1-form on the total space $\mathcal{C}^{\mathrm{ir}}$ which defines this connection if given by the operator

$$\Delta_\nabla^{-1} \circ d^*_\nabla : \Lambda^1_c\big(X, \mathrm{su}(2)\big) \longrightarrow \Lambda^0_c\big(X, \mathrm{su}(2)\big).$$

We will call this connection the *Coulomb connection*.

Let $\mathcal{F}^{\mathrm{ir}} := \mathcal{F} \cap \mathcal{C}^{\mathrm{ir}}$. For the rest of this section, using this Coulomb connection and the function χ constructed in Section 2, we will construct a function on $L(\mathcal{F}^{\mathrm{ir}}/\mathcal{G})$ with values in R/Z.

Let $\beta: S^1 \longrightarrow \mathcal{G}$ be a loop. Of course, since E is trivial, $\mathcal{G}$ is the space of all smooth maps from X to SU(2) which are the identity on all $D_{0,i}$. In other words, β can be identified with a map $\beta: M \times S^1 \longrightarrow \mathrm{SU}(2)$ (recall that M is the compactification of X). If the induced map $Z = H^3\big(\mathrm{SU}(2), Z\big) \xrightarrow{\beta^*} H^3(X \times S^1, Z) = Z$ is multiplication by d, then d is called the *degree* of β. For $\nabla \in \mathcal{F}^{\mathrm{ir}}$, using the action of $\mathcal{G}$ on $\mathcal{F}^{\mathrm{ir}}$, the loop β gives a loop in $\mathcal{F}^{\mathrm{ir}}$ based at ∇, which we denote by $\bar\beta$. Now $\chi(\bar\beta) \in R$, where χ is the function defined in Section 2.

PROPOSITION 3.1. *Under the above notation, $4\pi^2\chi(\bar\beta)$ coincides with d, the degree of β.*

PROOF. Let

$$\theta := \begin{pmatrix} i\theta_1 & \theta_2 + i\theta_3 \\ -\theta_2 + i\theta_3 & -i\theta_1 \end{pmatrix}$$

be the Maurer-Cartan form on SU(2). It is easy to see that the unique invariant (normalized) volume form on SU(2) is given by $\theta_1 \wedge \theta_2 \wedge \theta_3$. So

$$\int_{X \times S^1} \beta^*(\theta_1 \wedge \theta_2 \wedge \theta_3) = d.$$

Now for any $t \in S^1$, the connection $\bar\beta(t)$ can be expressed in the form

$$\bar\beta(t) := \begin{pmatrix} i\omega_1(t) & \omega_2(t) + i\omega_3(t) \\ -\omega_2(t) + i\omega_3(t) & -i\omega_1(t) \end{pmatrix}$$

where $\omega_i(t)$ are 1-forms on X, and are smooth as function of t. Since $\bar\beta(t)$ is flat, we have

$$d\bar\beta(t) = -\frac{1}{2}[\bar\beta(t), \bar\beta(t)] = \bar\beta(t) \wedge \bar\beta(t).$$

Recall the definition of χ; we need to extend $\bar\beta$ to the disc. For $z = re^{i\alpha} \in D^2$, define

$$\tilde\beta(z) := r\bar\beta(e^{i\alpha}) + (1 - r)\delta.$$

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Clearly $\tilde{\beta}$ maps D^2 to $\mathcal{C}$, and coincides with $\bar{\beta}$ on S^1. Let ϑ be the connection, given by $\tilde{\beta}$ on the bundle $E \times D^2 \longrightarrow X \times D^2$, and $K(\vartheta)$ be the curvature. After a straightforward calculation (such a calculation is done explicitly, for example, in [Gu]), we get $C_2\big(K(\vartheta)\big) = 1/8\pi^2 \operatorname{trace}\big(K(\vartheta)\big)$ cohomologous to

$$\frac{1}{4\pi^2}\int_{S^1}\Big((\frac{d}{dt}\omega_1)\omega_1 + (\frac{d}{dt}\omega_2)\omega_2 + (\frac{d}{dt}\omega_3)\omega_3\Big)\,dt.$$

So from the definition of χ, we have

$$\chi(\bar{\beta}) = \frac{1}{4\pi^2}\int_{S^1}\Big((\frac{d}{dt}\omega_1)\omega_1 + (\frac{d}{dt}\omega_2)\omega_2 + (\frac{d}{dt}\omega_3)\omega_3\Big)\,dt. \tag{3.3}$$

It is easy to check that the two 2-forms

$$\int_{S^1}\Big((\frac{d}{dt}\omega_1)\omega_1 + (\frac{d}{dt}\omega_2)\omega_2 + (\frac{d}{dt}\omega_3)\omega_3\Big)\,dt \text{ and } \bar{\beta}^*\theta_1 \wedge \bar{\beta}^*\theta_2 \wedge \bar{\beta}^*\theta_3$$

differ by an exact form. In the light of (3.2) and (3.3), this proves the proposition. ∎

Given a loop $\sigma\colon S^1 \longrightarrow \mathcal{F}^{\mathrm{ir}}/\mathcal{G}$ and a base point $\nabla \in \mathcal{F}^{\mathrm{ir}}$, using the Coulomb connection on $\mathcal{F}^{\mathrm{ir}} \longrightarrow \mathcal{F}^{\mathrm{ir}}/\mathcal{G}$, the loop σ can be lifted horizontally to a path $\bar{\sigma}\colon [0,1] \longrightarrow \mathcal{F}^{\mathrm{ir}}$ with $\bar{\sigma}(0) = \nabla$. The group $\mathcal{G}$ is path connected, and hence there is a path γ in $q^{-1}\big(q(\nabla)\big)$ connecting $\bar{\sigma}(1)$ and $\bar{\sigma}(0)$. So $\gamma \circ \bar{\sigma}$ is a loop in $\mathcal{F}^{\mathrm{ir}}$, which we denote by σ_γ. So depending on the path γ, we have $\chi(\sigma_\gamma) \in R$. If μ is any other path in $q^{-1}\big(q(\nabla)\big)$ connecting $\bar{\sigma}(1)$ and $\bar{\sigma}(0)$, then the corresponding number $\chi(\sigma_\mu)$ of course need not coincide with $\chi(\sigma_\gamma)$. But we noted earlier that the function χ is a homomorphism of groupoids. In other words

$$\chi(\sigma_\gamma) = \chi(\gamma \circ \mu^{-1}) + \chi(\sigma_\mu).$$

Now using (3.1) we have the following result.

THEOREM 3.4. *The map $\sigma \longmapsto 4\pi^2\chi(\sigma_\mu)$ gives a map $\bar{\chi}\colon L(\mathcal{F}^{\mathrm{ir}}/\mathcal{G}) \longrightarrow R/Z$.*

4. **Relation between χ and ω.** The space $\mathcal{F}^{\mathrm{ir}}/\mathcal{G}$, by taking holonomy, can naturally be identified with $\mathcal{P}^{\mathrm{ir}}$ (defined in Section 3). We want the two functions on $L(\mathcal{F}^{\mathrm{ir}}/\mathcal{G}) = L(\mathcal{P}^{\mathrm{ir}})$ namely, $q\chi$ and ω (defined in (2.1)), to coincide.

Let $f\colon \mathcal{F}^{\mathrm{ir}} \longrightarrow \mathcal{F}^{\mathrm{ir}}/\mathcal{G}$ be the projection. The pull-back $f^*\omega$ of ω (defined in Section 2) on $\mathcal{F}^{\mathrm{ir}}$ has a natural extension to $\mathcal{C}$, which is defined as follows: for $\alpha, \beta \in \Lambda^1_c\big(X, \mathrm{su}(2)\big)$, the correspondence

$$\alpha \otimes \beta \longmapsto q\int_X \alpha \wedge \beta$$

defines a 2-form on $\mathcal{C}$ which we denote by $\bar{\omega}$. Since $\mathcal{C}$ is affine and $\bar{\omega}$ is closed $\bar{\omega}$ is also exact. It is easy to see that the restriction of $\bar{\omega}$ to $\mathcal{F}^{\mathrm{ir}}$ coincides with $f^*\omega$. The space $\mathcal{C}$ also admits a natural 1-form θ which is defined as follows. First note that since E is trivial, a connection on E can be thought of as an element of $\Lambda^1\big(X, \mathrm{su}(2)\big)$. Now for $\nabla \in \mathcal{C}$ and $\alpha \in \Lambda^1\big(X, \mathrm{su}(2)\big)$, the correspondence $\alpha \longmapsto \int_X \operatorname{trace}(\alpha \wedge \nabla)$ defines a 1-form on $\mathcal{C}$, which is denoted by θ. It is easy to check that $qd\theta = \bar{\omega}$.

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As in the previous section, let σ be a loop in $\mathcal{F}^{\mathrm{ir}}/\mathcal{G}$, and σ_γ be a corresponding loop in $\mathcal{F}^{\mathrm{ir}}$. And $\tilde{\sigma}_\gamma$ be the extension of σ_γ to D^2 (as in Section 2). Let $\tilde{\sigma}$ be an extension of σ to D^2 such that $f \circ \tilde{\sigma}_\gamma = \tilde{\sigma}$. We have

$$\omega(\sigma) := \int_{D^2} \tilde{\sigma}^* \omega = \int_{D^2} \tilde{\sigma}_\gamma^* \bar{\omega} = q \int_{D^2} \tilde{\sigma}_\gamma^* \, d\theta = q \int_{S^1} \sigma_\gamma^* \theta.$$

Suppose for $t \in S^1$ the connection $\sigma_\gamma(t)$ is of the form

$$\sigma_\gamma(t) := \begin{pmatrix} i\omega_1(t) & \omega_2(t) + i\omega_3(t) \\ -\omega_2(t) + i\omega_3(t) & -i\omega_1(t) \end{pmatrix}$$

where ω_i are su(2)-valued 1-forms on X. Now

$$\begin{aligned} \int_{S^1} \sigma_\gamma^* \theta &= \int_{S^1} \operatorname{trace}\Big(\frac{d}{dt}\sigma_\gamma(t)\Big) \wedge \sigma_\gamma(t)\, dt \\ &= \int_{S^1} \Big((\frac{d}{dt}\omega_1)\omega_1 + (\frac{d}{dt}\omega_2)\omega_2 + (\frac{d}{dt}\omega_3)\omega_3 \Big)\, dt. \end{aligned} \tag{4.1}$$

Now using (3.3) which actually holds for any loop in $\mathcal{F}^{\mathrm{ir}}$, we have $\omega(\sigma) = q\chi(\sigma)$. Thus from (4.1) we have proved the following proposition.

THEOREM 4.2. *For any loop σ in $\mathcal{F}^{\mathrm{ir}}/\mathcal{G}$ the following equality holds*

$$\omega(\sigma) = q\chi(\sigma).$$

REMARK. In [BR], the authors prove the existence of a natural hermitian line bundle on the parabolic moduli space $\mathcal{P}$. Restricted to $\mathcal{P}^{\mathrm{ir}}$, this line bundle carries a natural connection whose curvature (up to a factor of i) is the standard symplectic form. It is easy to check that $\omega: L(\mathcal{P}^{\mathrm{ir}}) \longrightarrow S^1$ is then (up to a constant) the holonomy of this connection. Similar material is also treated in [DW].

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