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ON THE MATRIX WITH ELEMENTS $1/(r+s-1)$

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A standard example of a matrix for which the computation of eigenvectors and eigenvalues is very awkward is the matrix A with $a_{rs}=1/(r+s-1)$, $1\le r\le n$, $1\le s\le n$. It is therefore of interest that A commutes with a tridiagonal matrix.

In [2] it was shown that the integral operator, H, defined by

$$Hf(x)=\int_1^n (x+y)^{-1}f(y)\,dy,$$

commutes with the differential operator $L=(d/dx)\cdot(x^2-1)(n^2-x^2)\cdot(d/dx)-2x^2$. There is a finite analogue of this result, namely that $AB=BA$, where B is the $n\times n$ matrix with zero elements except for

$$b_{r,r}=c(r-1)+c(r)+2r(r-1) \qquad 1\le r\le n$$

$$b_{r,r+1}=b_{r+1,r}=-c(r) \qquad 1\le r\le n-1,$$

where $c(r)=r^2(n^2-r^2)$.

This result can be verified by direct calculation. The matrix BA is found to be symmetric. Hence $BA=(BA)^T=A^TB^T=AB$ as required.

The matrix A is totally positive, and hence is an oscillation matrix of exponent 1. Accordingly it has simple eigenvalues $\lambda_1>\lambda_2>\cdots>\lambda_n>0$ and the eigenvector corresponding to λ_k, say u_k, has exactly $k-1$ sign changes in its coordinate sequence. (See [1], p. 100.)

The matrix B, in the terminology of [1], is a normal Jacobi matrix, and hence has simple eigenvalues $\mu_1<\mu_2<\cdots<\mu_n$, and the eigenvector, say v_k, corresponding to μ_k, has exactly $k-1$ sign changes in its coordinate sequence. (See [1], p. 80.)

As $AB=BA$, it follows that A and B have the same eigenvectors, and the sign change properties indicate that order is preserved, that is, $u_k=v_k$. It should be noted that the eigenvalues of B are arranged in increasing order of magnitude, while those of A are in decreasing order. We shall also have $A=P_1(B)$ and $B=P_2(A)$ for some polynomials, P_1 and P_2, with corresponding relations between the eigenvalues.

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